Bollettino Unione Matematica Italiana

Michel Chipot, Karen Yeressian

On some Variational Inequalities in Unbounded Domains

Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **5** (2012), n.2, p. 243–262.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2012_9_5_2_243_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2012.

Bollettino U. M. I.
(9) **V** (2012), 243-262

On some Variational Inequalities in Unbounded Domains

MICHEL CHIPOT - KAREN YERESSIAN

Dedicated to the Memory of Professor Enrico Magenes

Abstract. – *We study the variational formulation of the obstacle problem in unbounded domains when the force term might grow at infinity. We derive the appropriate variational formulation and prove existence and uniqueness of solution. We also show the rate of growth at infinity of the solution in terms of the growth rates of the obstacle and the force, and prove the exponential convergence of the solutions in approximating bounded domains to the solution in the unbounded domain.*

1. – Introduction

Let us consider $\Omega \subset \mathbb{R}^2$ as in the figure 1, that is to say

$$\Omega \subset \mathbb{R} \times (-a, a),\ a > 0. \tag{1.1}$$

Let us denote by φ a function defined on Ω and such that

$$\varphi \in H^1(\Omega),\ \varphi \leq 0 \ \text{ on } \ \partial\Omega$$

($\partial\Omega$ denotes the boundary of Ω). If

$$f \in H^{-1}(\Omega) \tag{1.2}$$

it is well known (see [KS], [C1], [R]) that there exists a unique solution u to the variational inequality

$$\begin{cases} u \in \mathcal{K} = \{v \in H_0^1(\Omega) \mid v \geq \varphi \text{ a.e. in } \Omega\}, \\ \displaystyle\int_\Omega \nabla u \cdot \nabla(v-u)dx \geq \langle f, v-u \rangle, \ \ \forall v \in \mathcal{K}. \end{cases} \tag{1.3}$$

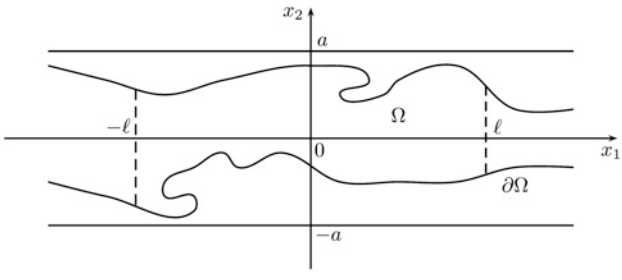

Fig. 1. – Ω.

However in the case where Ω is unbounded there are many simple distributions (functions) f for which (1.2) is not satisfied and for which the variational inequality (1.3) does not make sense. This is one goal of this note to bridge this gap. To see that (1.3) fails for many functions or distributions suppose that for some $0<b<a$ one has

$$\mathbb{R} \times (-b, b) \subset \Omega$$

and consider a function $f \in L^2(-a, a)$ such that

$$f \not\equiv 0 \text{ on } (-b, b).$$

Then there exists $\psi \in \mathcal{D}(-b, b)$ such that

$$\int_{-b}^{b} f(x_2)\psi(x_2)dx_2 \neq 0.$$

Let us introduce ρ_ε a function of x_1 only defined as follows

$$\rho_\varepsilon(x_1) = \min\big(\varepsilon, \operatorname{dist}(x_1, \mathbb{R}\backslash(-\varepsilon-\varepsilon^{-1}, \varepsilon+\varepsilon^{-1}))\big),$$

the graph of ρ_ε being depicted in the figure 2.

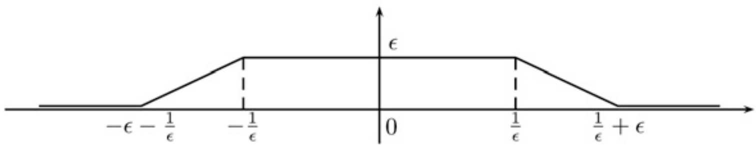

Fig. 2. – ρ_ε.

Then it is easy to see that when $\varepsilon \to 0$

$$v_\varepsilon = \rho_\varepsilon(x_1)\psi(x_2) \to 0 \text{ in } H_0^1(\Omega).$$

This is indeed an easy consequence of

$$\int_\Omega |\nabla v_\varepsilon|^2 dx = \int_\Omega \{\psi^2(x_2)(\partial_{x_1}\rho_\varepsilon)^2 + \rho_\varepsilon^2(x_1)(\partial_{x_2}\psi)^2\}dx$$

$$\leq 2\int_{\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}+\varepsilon}\int_{-b}^{b} \psi^2(x_2)dx_2dx_1 + \int_{-\frac{1}{\varepsilon}-\varepsilon}^{\frac{1}{\varepsilon}+\varepsilon}\int_{-b}^{b} \varepsilon^2(\partial_{x_2}\psi(x_2))^2 dx_2dx_1$$

$$= 2\varepsilon\int_{-b}^{b} \psi^2(x_2)dx_2 + 2(\varepsilon+\varepsilon^3)\int_{-b}^{b} (\partial_{x_2}\psi(x_2))^2 dx_2.$$

However

$$\int_{\Omega} f v_\varepsilon dx = \int_{-\frac{1}{\varepsilon}-\varepsilon}^{\frac{1}{\varepsilon}+\varepsilon} \int_{-b}^{b} f(x_2)\psi(x_2)\rho_\varepsilon(x_1)dx_1 dx_2 = \int_{-b}^{b} f(x_2)\psi(x_2)dx_2 \int_{-\frac{1}{\varepsilon}-\varepsilon}^{\frac{1}{\varepsilon}+\varepsilon} \rho_\varepsilon(x_1)dx_1$$

$$= (2+\varepsilon^2)\int_{-b}^{b} f(x_2)\psi(x_2)dx_2 \not\to 0$$

this shows that $f \notin H^{-1}(\Omega)$. This simple computation provides a lot of distributions f for which the variational inequality (1.3) does not make sense.

Under appropriate conditions, we prove existence and uniqueness of a solution to problems in unbounded domain as the limit of solutions in bounded domains. To carry this out, we use estimates in bounded domains which were considered to determine the asymptotic behavior of elliptic equations and variational inequalities in [C0], [CY] and [Y]. For the open set $\Omega \subset \mathbb{R}^n$ we will set

$$W_0^{1,\infty}(\Omega) = W^{1,\infty}(\Omega) \cap H_0^1(\Omega).$$

The rest of this paper is divided as follows. In the next section we treat an example in two dimensions to enlight the features of our method. We give next some generalizations.

2. – A Simple Case

We suppose here that Ω is an unbounded open set satisfying (1.1). For $\ell > 0$ we denote by Ω_ℓ the open set defined by

$$\Omega_\ell = \big((-\ell,\ell)\times(-a,a)\big)\cap\Omega.$$

We set

$$V_\ell = \Big\{ v \in H^1(\Omega_\ell) \;\Big|\; v = 0 \text{ on } \partial\Omega_\ell \cap \partial\Omega \Big\}.$$

It is well established (see [DL], [C2]) that V_ℓ is a Hilbert space when equipped with the norm

$$|v|_{V_\ell} = \|\nabla v\|_{2,\Omega_\ell} = \left(\int_{\Omega_\ell} |\nabla v(x)|^2 dx\right)^{\frac{1}{2}} \tag{2.1}$$

($|\ |_{2,O}$ denotes the usual $L^2(O)$-norm).

We denote by V_ℓ^* the strong dual of V_ℓ equipped with the usual dual norm, which we denote by $|\ |_{V_\ell^*}$.

We denote by φ a function defined on Ω such that

$$\varphi \in H^1_{loc}(\Omega),\ \varphi \le 0 \text{ on } \partial\Omega.$$

Then for $f \in V_\ell^*$ there exists a unique u_ℓ solution to

$$\begin{cases} u_\ell \in \mathcal{K}_\ell = \Big\{ v \in V_\ell \ \Big| \ v(x) \ge \varphi(x) \text{ a.e. in } \Omega \Big\}, \\ \displaystyle\int_{\Omega_\ell} \nabla u_\ell \cdot \nabla(v - u_\ell) dx \ge \langle f, v - u_\ell \rangle, \ \ \forall v \in \mathcal{K}_\ell, \end{cases} \tag{2.2}$$

(see [KS], [C2], [LS]).

We would like now to prove the following:

THEOREM 1. – *Suppose that $f \in V_\ell^*$ and if $\varphi^+ = \max(\varphi, 0)$ denotes the positive part of φ*

$$|\varphi^+|_{V_\ell},\ |f|_{V_\ell^*} = O(\ell^\gamma) \tag{2.3}$$

for some $\gamma > 0$. Then there exists a unique u_∞ solution to

$$\begin{cases} u_\infty \in \mathcal{K}_\infty = \big\{ v \in H^1_{loc}(\Omega),\ v = 0 \ \ on \ \ \partial\Omega,\ v(x) \ge \varphi(x) \ \ a.e. \ \ x \in \Omega \big\}, \\ \displaystyle\int_{\Omega_\ell} \nabla u_\infty \cdot \nabla((v - u_\infty)\rho(x_1)) dx \ge \langle f, (v - u_\infty)\rho \rangle_{V_\ell}, \\ \forall v \in \mathcal{K}_\infty,\ \forall \ell > 0, \forall \rho \in W_0^{1,\infty}(-\ell, \ell),\ \rho \ge 0, \\ \|\nabla u_\infty\|_{2,\Omega_\ell} = O(\ell^\gamma), \end{cases} \tag{2.4}$$

moreover one has

$$\|\nabla(u_\ell - u_\infty)\|_{2,\Omega_{\frac{\ell}{2}}} = O(e^{-\beta\ell}) \tag{2.5}$$

for some positive β.

REMARK 1. – Replacing possibly ρ by $\dfrac{\rho}{|\rho|_\infty}$ in (2.4) one can always assume that

$$0 \le \rho \le 1. \tag{2.6}$$

In the case when $f \in H^{-1}(\Omega)$ the solution of (2.4) coincides with the solution of the natural variational inequality

$$\begin{cases} u_\infty \in \mathcal{K}'_\infty = \Big\{ v \in H^1_0(\Omega) \ \Big| \ v(x) \ge \varphi(x) \text{ a.e. } x \in \Omega \Big\}, \\ \displaystyle\int_{\Omega} \nabla u_\infty \cdot \nabla(v - u_\infty) dx \ge \langle f, v - u_\infty \rangle, \ \ \forall v \in \mathcal{K}'_\infty. \end{cases} \tag{2.7}$$

Indeed if u_∞ is solution to (2.7) then one can for $w \in \mathcal{K}_\infty$ consider as test function in (2.7)

$$v = u_\infty + (w - u_\infty)\rho(x_1)$$

with $\rho \in W_0^{1,\infty}(-\ell, \ell)$ satisfying (2.6) to get that u_∞ satisfies (2.4) and the claim follows by uniqueness of the solution to (2.4).

One can also recover (2.7) from (2.4) by considering in (2.4) $\rho = \rho_n$ defined by

$$\rho_n(x_1) = \min\big(1, \operatorname{dist}(x_1, \mathbb{R}\backslash(-(n+1), n+1))\big)$$

then ρ_n is piecewise linear and

$$\rho_n(x) = 1 \text{ on } (-n, n),\ \rho_n(x) = 0 \text{ outside } (-(n+1), n+1).$$

Now noting that $(v - u_\infty)\rho_n \to v - u_\infty$ in $H_0^1(\Omega)$ for $v, u_\infty \in H_0^1(\Omega)$ we recover (2.7).

Proof of Theorem 1. – Step 1. Estimate of $u_\ell - u_{\ell+r}$ when $0 \leq r \leq 1$.

For $\ell_1 \leq \ell - 1$ consider

$$\rho(x_1) = \min\big(1, \operatorname{dist}(x_1, \mathbb{R}\backslash(-(\ell_1+1), \ell_1+1))\big)$$

then we have

$$u_\ell - (u_\ell - u_{\ell+r})\rho \in \mathcal{K}_\ell$$

and thus from (2.2)

$$\int_{\Omega_\ell} \nabla u_\ell \cdot \nabla(-(u_\ell - u_{\ell+r})\rho)dx \geq -\langle f, (u_\ell - u_{\ell+r})\rho\rangle. \tag{2.8}$$

Similarly

$$u_{\ell+r} + (u_\ell - u_{\ell+r})\rho \in \mathcal{K}_{\ell+r}$$

and thus (since ρ vanishes outside of Ω_ℓ)

$$\int_{\Omega_\ell} \nabla u_{\ell+r} \cdot \nabla((u_\ell - u_{\ell+r})\rho)dx \geq \langle f, (u_\ell - u_{\ell+r})\rho\rangle. \tag{2.9}$$

Adding (2.8) and (2.9) leads to

$$\int_{\Omega_{\ell_1+1}} \nabla(u_\ell - u_{\ell+r}) \cdot \nabla((u_\ell - u_{\ell+r})\rho)dx \leq 0$$

which can also be written

$$(2.10)\quad \int_{\Omega_{\ell_1+1}} |\nabla(u_\ell - u_{\ell+r})|^2 \rho dx \leq \int_{\Omega_{\ell_1+1}\setminus\Omega_{\ell_1}} \partial_{x_1}(u_\ell - u_{\ell+r})\rho'(x_1)(u_\ell - u_{\ell+r})dx$$

$$\leq \int_{\Omega_{\ell_1+1}\setminus\Omega_{\ell_1}} |\partial_{x_1}(u_\ell - u_{\ell+r})||u_\ell - u_{\ell+r}|dx$$

$$\leq \frac{\varepsilon}{2}\int_{D_{\ell_1}} (\partial_{x_1}(u_\ell - u_{\ell+r}))^2 dx + \frac{1}{2\varepsilon}\int_{D_{\ell_1}} (u_\ell - u_{\ell+r})^2 dx$$

by the Young inequality and for $D_{\ell_1} = \Omega_{\ell_1+1}\setminus\Omega_{\ell_1}$, $\varepsilon > 0$ will be fixed later.

Using the Poincaré inequality in the x_2-direction one has

$$\int_{D_{\ell_1}} (u_\ell - u_{\ell+r})^2 dx \leq \left(\frac{2a}{\pi}\right)^2 \int_{D_{\ell_1}} (\partial_{x_2}(u_\ell - u_{\ell+r}))^2 dx$$

and it follows from (2.10)

$$\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \leq \frac{\varepsilon}{2}\int_{D_{\ell_1}} (\partial_{x_1}(u_\ell - u_{\ell+r}))^2 dx + \frac{1}{2\varepsilon}\left(\frac{2a}{\pi}\right)^2 \int_{D_{\ell_1}} (\partial_{x_2}(u_\ell - u_{\ell+r}))^2 dx.$$

Choosing $\varepsilon = \dfrac{2a}{\pi}$ we get

$$\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \leq \frac{a}{\pi}\int_{D_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx$$

which leads to

$$\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \leq \frac{a}{\pi + a}\int_{\Omega_{\ell_1+1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx$$

Starting from $\ell_1 = \dfrac{\ell}{2}$ we iterate this formula $\left[\dfrac{\ell}{2}\right]$ times, to get

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \leq \left(\frac{a}{\pi + a}\right)^{[\frac{\ell}{2}]} \int_{\Omega_{\frac{\ell}{2}+[\frac{\ell}{2}]}} |\nabla(u_\ell - u_{\ell+r})|^2 dx$$

$\left(\left[\dfrac{\ell}{2}\right]\right.$ denotes the integer part of $\left.\dfrac{\ell}{2}\right)$. Since

$$\frac{\ell}{2} - 1 < \left[\frac{\ell}{2}\right] \leq \frac{\ell}{2}$$

we obtain

$$(2.11)\quad \int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \le \Big(\frac{\pi+a}{a}\Big) e^{-\frac{\ell}{2}\ln(\frac{\pi+a}{a})} \int_{\Omega_\ell} |\nabla(u_\ell - u_{\ell+r})|^2 dx$$

$$= \Big(\frac{\pi+a}{a}\Big) e^{-\alpha'\ell} \int_{\Omega_\ell} |\nabla(u_\ell - u_{\ell+r})|^2 dx \quad \text{with} \quad \alpha' = \frac{1}{2}\ln\Big(\frac{\pi+a}{a}\Big).$$

STEP 2. Estimate of u_ℓ.

Taking $v = \varphi^+$ as test function in (2.2), by the Young inequality we get

$$\int_{\Omega_\ell} |\nabla u_\ell|^2 dx \le \int_{\Omega_\ell} \nabla u_\ell \cdot \nabla\varphi^+ dx + \big|\langle f, \varphi^+ - u_\ell\rangle\big|$$

$$\le ||\nabla u_\ell||_{2,\Omega_\ell} ||\nabla\varphi^+||_{2,\Omega_\ell} + |f|_{V_\ell^*}\big\{\|\nabla\varphi^+\|_{2,\Omega_\ell} + \|\nabla u_\ell\|_{2,\Omega_\ell}\big\}$$

$$\le \frac{\varepsilon}{2}||\nabla u_\ell||^2_{2,\Omega_\ell} + \frac{1}{2\varepsilon}||\nabla\varphi^+||^2_{2,\Omega_\ell} + \frac{1}{2\varepsilon}|f|^2_{V_\ell^*} + \frac{\varepsilon}{2}\big\{\|\nabla\varphi^+\|_{2,\Omega_\ell} + \|\nabla u_\ell\|_{2,\Omega_\ell}\big\}^2$$

$$\le \frac{\varepsilon}{2}||\nabla u_\ell||^2_{2,\Omega_\ell} + \frac{1}{2\varepsilon}||\nabla\varphi^+||^2_{2,\Omega_\ell} + \frac{1}{2\varepsilon}|f|^2_{V_\ell^*} + \varepsilon\big\{\|\nabla\varphi^+\|^2_{2,\Omega_\ell} + \|\nabla u_\ell\|^2_{2,\Omega_\ell}\big\}$$

$$= \Big(\varepsilon + \frac{\varepsilon}{2}\Big)||\nabla u_\ell||^2_{2,\Omega_\ell} + \Big(\varepsilon + \frac{1}{2\varepsilon}\Big)||\nabla\varphi^+||^2_{2,\Omega_\ell} + \frac{1}{2\varepsilon}|f|^2_{V_\ell^*}.$$

From here by choosing ε small enough we deduce

$$\|\nabla u_\ell\|^2_{2,\Omega_\ell} \le C\big(||\nabla\varphi^+||^2_{2,\Omega_\ell} + |f|^2_{V_\ell^*}\big)$$

which leads (see (2.3)) to

$$(2.12)\quad \|\nabla u_\ell\|_{2,\Omega_\ell} = O(\ell^\gamma).$$

STEP 3. u_ℓ is a Cauchy sequence.

Combining (2.11) and (2.12) we get for $\ell \ge 1$

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \le \Big(\frac{a+\pi}{a}\Big) 2\Big\{\int_{\Omega_\ell} |\nabla u_\ell|^2 dx + \int_{\Omega_\ell} |\nabla u_{\ell+r}|^2 dx\Big\} e^{-\alpha'\ell}$$

$$\le C\big\{\ell^{2\gamma} + (\ell+r)^{2\gamma}\big\}e^{-\alpha'\ell} \le C\Big\{1 + \Big(\frac{\ell+r}{\ell}\Big)^{2\gamma}\Big\}\ell^{2\gamma}e^{-\alpha'\ell} \le C\{1 + 2^{2\gamma}\}\ell^{2\gamma}e^{-\alpha'\ell} \le Ce^{-\alpha\ell}$$

for any $\alpha < \alpha'$ with $C = C(a, \gamma)$.

This can be written as

$$(2.13)\quad |u_\ell - u_{\ell+r}|_{V_{\frac{\ell}{2}}} \le Ce^{-\beta\ell}$$

with $\beta = \dfrac{\alpha}{2} < \dfrac{1}{4}\ln\Big(\dfrac{a+\pi}{a}\Big)$.

From (2.13) we then deduce for any t that

$$(2.14) \quad |u_\ell - u_{\ell+t}|_{V_{\frac{\ell}{2}}} \le |u_\ell - u_{\ell+1}|_{V_{\frac{\ell}{2}}} + |u_{\ell+1} - u_{\ell+2}|_{V_{\frac{\ell}{2}}} + \ldots + |u_{\ell+[t]} - u_{\ell+t}|_{V_{\frac{\ell}{2}}}$$

$$\le Ce^{-\beta\ell} + Ce^{-\beta(\ell+1)} + \ldots + Ce^{-\beta(\ell+[t])} \le \frac{Ce^{-\beta\ell}}{1-e^{-\beta}}.$$

STEP 4. Passage to the limit.

Let $\ell_0 < \frac{\ell}{2}$. Choose $\rho \in W_0^{1,\infty}(-\ell_0, \ell_0)$, $0 \le \rho \le 1$, then for any $v \in \mathcal{K}_\infty$ we have

$$u_\ell + (v - u_\ell)\rho \in \mathcal{K}_\ell$$

and by (2.2) we get

$$(2.15) \qquad \int_{\Omega_{\ell_0}} \nabla u_\ell \cdot \nabla((v-u_\ell)\rho)dx \ge \langle f, (v-u_\ell)\rho\rangle, \quad \forall v \in \mathcal{K}_\infty.$$

We have clearly $(v - u_\ell)\rho \in V_{\ell_0}$ and by (2.14) there exists $u_\infty \in H^1(\Omega_{\ell_0})$ such that

$$u_\ell \to u_\infty \text{ in } H^1(\Omega_{\ell_0})$$

when $\ell \to \infty$. Passing to the limit in (2.15) we get

$$\int_{\Omega_{\ell_0}} \nabla u_\infty \cdot \nabla((v-u_\infty)\rho)dx \ge \langle f, (v-u_\infty)\rho\rangle, \quad \forall v \in \mathcal{K}_\infty.$$

Moreover, it is clear that $u_\infty = 0$ on $\partial\Omega$, $u_\infty \ge \varphi$ a.e. in Ω and we have obtained the two first lines of (2.4).

STEP 5. Estimate of $\|\nabla u_\infty\|_{2,\Omega_\ell}$ and proof of (2.5).

By (2.14) written for 2ℓ we have

$$|u_{2\ell} - u_{2\ell+t}|_{V_\ell} \le C'e^{-2\beta\ell}.$$

Letting $t \to \infty$ it comes

$$|u_{2\ell} - u_\infty|_{V_\ell} \le C'e^{-2\beta\ell}$$

and thus (2.5) holds. Moreover

$$|u_\infty|_{V_\ell} \le C'e^{-2\beta\ell} + |u_{2\ell}|_{V_\ell} \le C'e^{-2\beta\ell} + |u_{2\ell}|_{V_{2\ell}} \le C'e^{-2\beta\ell} + C(2\ell)^\gamma = O(\ell^\gamma)$$

(by (2.12), (2.1)).

STEP 6. Uniqueness of u_∞.

This is the only point remaining to prove. Let u_∞, u'_∞ be two solutions to (2.4). In the inequalities satisfied by u_∞, u'_∞, by taking respectively $v = u'_\infty, u_\infty$ we obtain for $\rho \in W_0^{1,\infty}(-\ell,\ell)$ such that $\rho \geq 0$

$$\int_{\Omega_\ell} \nabla u_\infty \cdot \nabla((u'_\infty - u_\infty)\rho)dx \geq \langle f, (u'_\infty - u_\infty)\rho\rangle,$$

$$\int_{\Omega_\ell} \nabla u'_\infty \cdot \nabla((u_\infty - u'_\infty)\rho)dx \geq \langle f, (u_\infty - u'_\infty)\rho\rangle.$$

Adding we get

$$\int_{\Omega_\ell} \nabla(u'_\infty - u_\infty) \cdot \nabla((u'_\infty - u_\infty)\rho)dx \leq 0, \ \ \forall \rho \in W_0^{1,\infty}(-\ell,\ell),\ \rho \geq 0.$$

Considering again the function ρ introduced in the step 1 of this proof and proceeding as in (2.10)-(2.11) we obtain

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u'_\infty - u_\infty)|^2 dx \leq Ce^{-\alpha'\ell} \int_{\Omega_\ell} |\nabla(u_\infty - u'_\infty)|^2 dx \leq C\ell^{2\gamma}e^{-\alpha'\ell}$$

for some constants C. The equality $u_\infty = u'_\infty$ follows by letting $\ell \to \infty$. This completes the proof of the theorem. □

REMARK 2. – Without imposing some growth condition on u_∞, uniqueness might fail. Indeed, take for instance

$$f \equiv 0,\ \varphi \equiv 0$$

then $u_\infty = 0$ is clearly solution to (2.4).

Now if

$$\Omega = \mathbb{R} \times (-a, a)$$

the function

$$u'_\infty = e^{\frac{\pi}{2a}x_1} \cos\left(\frac{\pi}{2a}x_2\right)$$

is also solution since this is a nonnegative harmonic function in the strip $\mathbb{R} \times (-a, a)$ vanishing on the lateral boundary (compare with [CM]).

3. – Some Generalization

We are going to consider a special class of domains in $\mathbb{R}^n$. For that, for $x \in \mathbb{R}^n$ we split the coordinates into two parts i.e. we set

$$x = (X_1, X_2)$$

where $X_1 = (x_1, \ldots, x_p)$ denotes the p first coordinates and $X_2 = (x_{p+1}, \ldots, x_n)$ the $n-p$ last ones. If ω is a bounded open subset of $\mathbb{R}^{n-p}$ we consider a domain Ω such that

$$\Omega \subset \mathbb{R}^p \times \omega$$

and we denote for $\ell > 0$ by Ω_ℓ the set

$$\Omega_\ell = (B_p(0, \ell) \times \omega) \cap \Omega$$

here $B_p(0, \ell)$ denotes the euclidean ball with center at 0 and radius ℓ. In the framework of section 2, $p = n - p = 1$, $\omega = (-a, a)$.

We denote by $A = A(x)$ a $n \times n$ matrix satisfying for some positive constants λ and Λ

$$A(x)\zeta \cdot \zeta \geq \lambda |\zeta|^2 \text{ a.e. } x \in \Omega, \forall \zeta \in \mathbb{R}^n, \tag{3.1}$$

$$|A(x)\zeta| \leq \Lambda |\zeta| \text{ a.e. } x \in \Omega, \forall \zeta \in \mathbb{R}^n. \tag{3.2}$$

As in section 2 we define V_ℓ by

$$V_\ell = \left\{ v \in H^1(\Omega_\ell) \;\middle|\; v = 0 \text{ on } \partial\Omega_\ell \cap \partial\Omega \right\}$$

and we assume this Hilbert space is normed by

$$|v|_{V_\ell} = \|\nabla v\|_{2,\Omega_\ell} = \Bigg(\int_{\Omega_\ell} |\nabla v|^2 dx \Bigg)^{\frac{1}{2}}.$$

As before also let

$$\varphi \in H^1_{loc}(\Omega), \ \varphi \leq 0 \text{ on } \partial\Omega. \tag{3.3}$$

For $f \in V_\ell^*$ we denote by u_ℓ the solution to

$$\begin{cases} u_\ell \in \mathcal{K}_\ell = \left\{ v \in V_\ell \;\middle|\; v(x) \geq \varphi(x) \text{ a.e. in } \Omega_\ell \right\}, \\ \displaystyle\int_{\Omega_\ell} A(x)\nabla u_\ell \cdot \nabla(v - u_\ell)dx \geq \langle f, v - u_\ell \rangle, \ \ \forall v \in \mathcal{K}_\ell, \end{cases} \tag{3.4}$$

($\langle , \rangle$ denotes the V_ℓ^*, V_ℓ duality). Then we have:

THEOREM 2. – *We suppose that*

$$|\varphi^+|_{V_\ell}, \ |f|_{V_\ell^*} = O(e^{\delta\ell}) \tag{3.5}$$

when $\ell \to \infty$. Then there exists δ_0 such that for $\delta < \delta_0$ the problem

$$(3.6)\quad \begin{cases} u_\infty \in \mathcal{K}_\infty = \Big\{ v \in H^1_{loc}(\Omega) \ \big| \ v = 0 \text{ on } \partial\Omega,\, v(x) \geq \varphi(x) \text{ a.e. } x \in \Omega \Big\}, \\ \int_{\Omega_\ell} A(x)\nabla u_\infty \cdot \nabla((v - u_\infty)\rho)dx \geq \langle f, (v - u_\infty)\rho \rangle, \quad \forall v \in \mathcal{K}_\infty, \\ \forall \ell > 0,\, \forall \rho \in W_0^{1,\infty}(B_p(0,\ell)),\, \rho \geq 0, \\ \|\nabla u_\infty\|_{2,\Omega_\ell} = O(e^{\delta\ell}) \end{cases}$$

possesses a unique solution. Moreover there exists a $\beta > 0$ *such that*

$$(3.7)\qquad \|\nabla(u_\ell - u_\infty)\|_{2,\Omega_{\frac{\ell}{2}}} = O(e^{-\beta\ell})$$

when $\ell \to \infty$.

PROOF. – One can develop the same arguments as in the proof of theorem 1, however we will proceed differently.

STEP 1. Estimate of $u_\ell - u_{\ell+r}$ when $0 \leq r \leq 1$.

We use here another technique. Set

$$(3.8)\qquad \rho(x) = \left(e^{-\alpha|X_1|} - e^{-\alpha\ell}\right)^+$$

where α is a positive constant that we will fix later on, $|X_1|$ denotes the euclidean norm of X_1, recall that $x = (X_1, X_2)$. Then it is clear that

$$u_\ell - (u_\ell - u_{\ell+r})\rho \in \mathcal{K}_\ell$$

and by (3.4) we get

$$(3.9)\qquad \int_{\Omega_\ell} A(x)\nabla u_\ell \cdot \nabla(-(u_\ell - u_{\ell+r})\rho)dx \geq \langle f, -(u_\ell - u_{\ell+r})\rho \rangle.$$

Similarly

$$u_{\ell+r} + (u_\ell - u_{\ell+r})\rho \in \mathcal{K}_{\ell+r}$$

and since ρ vanishes out of Ω_ℓ we get from (3.4) where ℓ is replaced by $\ell + r$

$$(3.10)\qquad \int_{\Omega_\ell} A(x)\nabla u_{\ell+r} \cdot \nabla((u_\ell - u_{\ell+r})\rho)dx \geq \langle f, (u_\ell - u_{\ell+r})\rho \rangle.$$

Adding (3.9) and (3.10) leads to

$$(3.11)\qquad \int_{\Omega_\ell} A(x)\nabla(u_\ell - u_{\ell+r}) \cdot \nabla((u_\ell - u_{\ell+r})\rho)dx \leq 0$$

which can be written as

$$\int_{\Omega_\ell} \big(A(x)\nabla(u_\ell - u_{\ell+r}) \cdot \nabla(u_\ell - u_{\ell+r})\big)\rho dx \le -\int_{\Omega_\ell} \big(A(x)\nabla(u_\ell - u_{\ell+r}) \cdot \nabla\rho(X_1)\big)(u_\ell - u_{\ell+r})dx.$$

We remark that

$$\nabla\rho = \left(-\alpha e^{-\alpha|X_1|}\frac{X_1}{|X_1|}, 0\right)$$

where the 0 above is the 0 in $\mathbb{R}^{n-p}$ and thus we derive using (3.1), (3.2)

$$\lambda\int_{\Omega_\ell} |\nabla(u_\ell - u_{\ell+r})|^2\rho dx \le \Lambda\int_{\Omega_\ell} \alpha|\nabla(u_\ell - u_{\ell+r})||u_\ell - u_{\ell+r}|e^{-\alpha|X_1|}dx \le \frac{1}{2}\alpha\Lambda\int_{\Omega_\ell}\big\{|\nabla(u_\ell - u_{\ell+r})|^2 + |u_\ell - u_{\ell+r}|^2\big\}e^{-\alpha|X_1|}dx. \tag{3.12}$$

Using the Poincaré inequality one has for almost any $X_1 \in B_p(0,\ell)$

$$\int_\omega (u_\ell - u_{\ell+r})^2 e^{-\alpha|X_1|}dX_2 \le c_p^2(\omega)\int_\omega |\nabla_{X_2}(u_\ell - u_{\ell+r})|^2 e^{-\alpha|X_1|}dX_2$$

where $\nabla_{X_2} = \partial_{x_{p+1}}, \dots, \partial_{x_n}$ and $c_p(\omega)$ is the Poincaré constant in the section ω. Integrating in X_1 leads to

$$\int_{\Omega_\ell} (u_\ell - u_{\ell+r})^2 e^{-\alpha|X_1|}dx \le c_p^2(\omega)\int_{\Omega_\ell} |\nabla_{X_2}(u_\ell - u_{\ell+r})|^2 e^{-\alpha|X_1|}dx \le c_p^2(\omega)\int_{\Omega_\ell} |\nabla(u_\ell - u_{\ell+r})|^2 e^{-\alpha|X_1|}dx.$$

Going back to (3.12) and using (3.8) we get

$$\lambda\int_{\Omega_\ell} |\nabla(u_\ell - u_{\ell+r})|^2(e^{-\alpha|X_1|} - e^{-\alpha\ell})dx \le \frac{1}{2}\alpha\Lambda(1 + c_p^2(\omega))\int_{\Omega_\ell} |\nabla(u_\ell - u_{\ell+r})|^2 e^{-\alpha|X_1|}dx$$

which can be written as

$$\left\{\lambda - \frac{1}{2}\alpha\Lambda(1 + c_p^2(\omega))\right\}\int_{\Omega_\ell} |\nabla(u_\ell - u_{\ell+r})|^2 e^{-\alpha|X_1|}dx \le \lambda e^{-\alpha\ell}\int_{\Omega_\ell} |\nabla(u_\ell - u_{\ell+r})|^2 dx.$$

One choose then α such that

$$\frac{1}{2}\alpha\Lambda(1 + c_p^2(\omega)) < \lambda \tag{3.13}$$

to get

$$\int_{\Omega_\ell} |\nabla(u_\ell - u_{\ell+r})|^2 e^{-\alpha|X_1|} dx \le C e^{-\alpha\ell} \int_{\Omega_\ell} |\nabla(u_\ell - u_{\ell+r})|^2 dx$$

where we have set

$$C = \frac{\lambda}{\lambda - \frac{1}{2}\alpha\Lambda(1 + c_p^2(\omega))}.$$

Integrating only on $\Omega_{\frac{\ell}{2}}$ in the left hand side we obtain easily

$$e^{-\frac{\alpha\ell}{2}} \int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \le C e^{-\alpha\ell} \int_{\Omega_\ell} |\nabla(u_\ell - u_{\ell+r})|^2 dx$$

which is

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \le C e^{-\frac{\alpha\ell}{2}} \int_{\Omega_\ell} |\nabla(u_\ell - u_{\ell+r})|^2 dx \tag{3.14}$$

for any α satisfying (3.13).

STEP 2. Estimate of u_ℓ. Due to (3.3) we notice that $\varphi^+ \in \mathcal{K}_\ell$ and by (3.4) we have

$$\int_{\Omega_\ell} A(x)\nabla u_\ell \cdot \nabla(\varphi^+ - u_\ell) dx \ge \langle f, \varphi^+ - u_\ell \rangle.$$

It follows then using the Young inequality

$$\begin{aligned}
\lambda \int_{\Omega_\ell} |\nabla u_\ell|^2 dx &\le \int_{\Omega_\ell} A(x)\nabla u_\ell \cdot \nabla u_\ell dx \\
&\le \Lambda ||\nabla u_\ell||_{2,\Omega_\ell} ||\nabla\varphi^+||_{2,\Omega_\ell} + |f|_{V_\ell^*} \big\{ ||\nabla u_\ell||_{2,\Omega_\ell} + ||\nabla\varphi^+||_{2,\Omega_\ell} \big\} \\
&\le \frac{\Lambda\varepsilon}{2} ||\nabla u_\ell||_{2,\Omega_\ell}^2 + \frac{\Lambda}{2\varepsilon} ||\nabla\varphi^+||_{2,\Omega_\ell}^2 + \frac{1}{2\varepsilon} |f|_{V_\ell^*}^2 \\
&\quad + \frac{\varepsilon}{2} \big\{ \|\nabla u_\ell\|_{2,\Omega_\ell} + \|\nabla\varphi^+\|_{2,\Omega_\ell} \big\}^2 \\
&\le \frac{\Lambda\varepsilon}{2} \|\nabla u_\ell\|_{2,\Omega_\ell}^2 + \frac{\Lambda}{2\varepsilon} \|\nabla\varphi^+\|_{2,\Omega_\ell}^2 + \frac{1}{2\varepsilon} |f|_{V_\ell^*}^2 \\
&\quad + \varepsilon \big\{ \|\nabla u_\ell\|_{2,\Omega_\ell}^2 + \|\nabla\varphi^+\|_{2,\Omega_\ell}^2 \big\} \\
&= \left(\varepsilon + \frac{\Lambda\varepsilon}{2}\right) \|\nabla u_\ell\|_{2,\Omega_\ell}^2 + \left(\varepsilon + \frac{\Lambda}{2\varepsilon}\right) \|\nabla\varphi^+\|_{2,\Omega_\ell}^2 + \frac{1}{2\varepsilon} |f|_{V_\ell^*}^2
\end{aligned}$$

and thus for small enough ε by (3.5) we obtain

$$\|\nabla u_\ell\|^2_{2,\Omega_\ell} \le C\big\{\|\nabla\varphi^+\|^2_{2,\Omega_\ell} + |f|^2_{V^*_\ell}\big\} \le Ce^{2\delta\ell}. \tag{3.15}$$

STEP 3. u_ℓ is a Cauchy sequence.

The proof goes like in theorem 1. Indeed combining (3.14) and (3.15) one gets

$$\begin{aligned}\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_{\ell+r})|^2 dx &\le Ce^{-\frac{\alpha\ell}{2}}\int_{\Omega_\ell} |\nabla(u_\ell - u_{\ell+r})|^2 dx\\ &\le 2Ce^{-\frac{\alpha\ell}{2}}\big\{\|\nabla u_\ell\|^2_{2,\Omega_\ell} + \|\nabla u_{\ell+r}\|^2_{2,\Omega_{\ell+r}}\big\}\\ &\le 2Ce^{-\frac{\alpha\ell}{2}}\big\{e^{2\delta\ell} + e^{2\delta(\ell+r)}\big\}\\ &\le 2C\{1+e^{2\delta r}\}e^{(-\frac{\alpha}{2}+2\delta)\ell} \le C'e^{-2\beta\ell}\end{aligned}$$

for some positive β provided δ is chosen such that

$$\delta < \frac{\alpha}{4}.$$

Then one shows as in the proof of theorem 1 that for $t \ge 0$ we have

$$|u_\ell - u_{\ell+t}|_{V_{\frac{\ell}{2}}} \le Ce^{-\beta\ell}. \tag{3.16}$$

STEP 4. Passage to the limit.

This is almost identical as in the proof of the theorem 1.

STEP 5. Estimate of $|u_\infty|_{V_\ell}$.

From (3.16) we get by letting $t \to \infty$

$$|u_\ell - u_\infty|_{V_{\frac{\ell}{2}}} \le Ce^{-\beta\ell} \quad \text{(i.e. (3.7))}$$

which implies changing $\frac{\ell}{2}$ into ℓ

$$|u_\infty|_{V_\ell} \le Ce^{-2\beta\ell} + |u_{2\ell}|_{V_\ell} \le Ce^{2\delta\ell} \quad \text{by (3.15)}. \tag{3.17}$$

This is not yet the last row of (3.6) and we need to improve the estimate. For that we take in (3.6) $v = \varphi^+$ and ρ the function of X_1 defined by

$$\rho(X_1) = \min\big(1, \operatorname{dist}(X_1, \mathbb{R}^p \backslash B_p(0, \ell+1))\big).$$

Note that one has

$$0 \le \rho \le 1,\ \rho = 1 \text{ on } B_p(0,\ell),\ \rho = 0 \text{ outside } B_p(0,\ell+1), \tag{3.18}$$

$$|\nabla_{X_1}\rho(X_1)| \le 1. \tag{3.19}$$

We obtain

$$\int_{\Omega_{\ell+1}} A(x)\nabla u_\infty \cdot \nabla((\varphi^+ - u_\infty)\rho)dx \geq \langle f, (\varphi^+ - u_\infty)\rho\rangle.$$

This can be written as

$$\int_{\Omega_{\ell+1}} \big(A(x)\nabla u_\infty \cdot \nabla u_\infty\big)\rho dx \leq -\int_{\Omega_{\ell+1}} \big(A(x)\nabla u_\infty \cdot \nabla\rho\big)u_\infty dx + \int_{\Omega_{\ell+1}} A(x)\nabla u_\infty \cdot \nabla(\rho\varphi^+)dx - \langle f, (\varphi^+ - u_\infty)\rho\rangle.$$

We denote by D_ℓ the set defined as $D_\ell = \Omega_{\ell+1}\backslash\Omega_\ell$. We then derive from above (note (3.1), (3.2), (3.18), (3.19))

$$\lambda\int_{\Omega_{\ell+1}} |\nabla u_\infty|^2\rho dx \leq \Lambda\int_{D_\ell} |\nabla u_\infty||u_\infty|dx + \Lambda\int_{D_\ell} |\nabla u_\infty||\varphi^+|dx + \Lambda\int_{\Omega_{\ell+1}} |\nabla u_\infty||\nabla\varphi^+|\rho dx + |\langle f, (\varphi^+ - u_\infty)\rho\rangle|. \tag{3.20}$$

We then estimate each term on the right hand side of this inequality.

By the Young and Poincaré inequalities we get

$$\Lambda\int_{D_\ell} |\nabla u_\infty||u_\infty|dx \leq \frac{\Lambda}{2}\int_{D_\ell} |\nabla u_\infty|^2dx + \frac{\Lambda}{2}\int_{D_\ell} |u_\infty|^2dx \leq \frac{\Lambda}{2}(1 + c_p^2(\omega))\int_{D_\ell} |\nabla u_\infty|^2dx. \tag{3.21}$$

Again by the Young and Poincaré inequalities

$$\begin{aligned}\Lambda\int_{D_\ell} |\nabla u_\infty||\varphi^+|dx &\leq \frac{\Lambda}{2}\int_{D_\ell} |\nabla u_\infty|^2dx + \frac{\Lambda}{2}\int_{D_\ell} (\varphi^+)^2dx\\ &\leq \frac{\Lambda}{2}\int_{D_\ell} |\nabla u_\infty|^2dx + \frac{\Lambda}{2}c_p^2(\omega)\int_{D_\ell} |\nabla\varphi^+|^2dx\\ &\leq \frac{\Lambda}{2}\int_{D_\ell} |\nabla u_\infty|^2dx + \frac{\Lambda}{2}c_p^2(\omega)\int_{\Omega_{\ell+1}} |\nabla\varphi^+|^2dx.\end{aligned}$$

For the third term by the Young inequality it comes

$$\Lambda\int_{\Omega_{\ell+1}} |\nabla u_\infty||\nabla\varphi^+|\rho dx \leq \varepsilon\int_{\Omega_{\ell+1}} |\nabla u_\infty|^2\rho dx + \frac{\Lambda^2}{4\varepsilon}\int_{\Omega_{\ell+1}} |\nabla\varphi^+|^2dx.$$

Finally for the last term one has

$$(3.22)\quad |\langle f,(\varphi^+-u_\infty)\rho\rangle| \le |f|_{V^*_{\ell+1}} ||\nabla((\varphi^+-u_\infty)\rho)||_{2,\Omega_{\ell+1}} \le \frac{1}{2\varepsilon}|f|^2_{V^*_{\ell+1}} + \frac{\varepsilon}{2}||\nabla((\varphi^+-u_\infty)\rho)||^2_{2,\Omega_{\ell+1}}$$

and

$$(3.23)\quad \int_{\Omega_{\ell+1}} |\nabla((\varphi^+-u_\infty)\rho)|^2 dx$$
$$\le 4\int_{\Omega_{\ell+1}} \{|\rho\nabla\varphi^+|^2 + |\varphi^+\nabla\rho|^2 + |\rho\nabla u_\infty|^2 + |u_\infty\nabla\rho|^2\}dx$$
$$= 4\int_{\Omega_{\ell+1}} \{|\rho\nabla\varphi^+|^2 + |\rho\nabla u_\infty|^2\}dx + 4\int_{D_\ell}\{|\varphi^+\nabla\rho|^2 + |u_\infty\nabla\rho|^2\}dx$$
$$\le 4\int_{\Omega_{\ell+1}} \{|\nabla\varphi^+|^2 + |\nabla u_\infty|^2\rho\}dx + 4\int_{D_\ell}\{|\varphi^+|^2 + |u_\infty|^2\}dx$$
$$\le 4\int_{\Omega_{\ell+1}} \{|\nabla\varphi^+|^2 + |\nabla u_\infty|^2\rho\}dx + 4c_p^2(\omega)\int_{D_\ell}\{|\nabla\varphi^+|^2 + |\nabla u_\infty|^2\}dx$$
$$\le 4\int_{\Omega_{\ell+1}} |\nabla u_\infty|^2\rho dx + 4c_p^2(\omega)\int_{D_\ell} |\nabla u_\infty|^2 dx + 4(1+c_p^2(\omega))\int_{\Omega_{\ell+1}} |\nabla\varphi^+|^2 dx$$

now by (3.22) and (3.23) we have

$$(3.24)\quad |\langle f,(\varphi^+-u_\infty)\rho\rangle| \le \frac{1}{2\varepsilon}|f|^2_{V^*_{\ell+1}} + 2\varepsilon\int_{\Omega_{\ell+1}} |\nabla u_\infty|^2\rho dx + 2\varepsilon c_p^2(\omega)\int_{D_\ell} |\nabla u_\infty|^2 dx + 2\varepsilon(1+c_p^2(\omega))\int_{\Omega_{\ell+1}} |\nabla\varphi^+|^2 dx.$$

Collecting the estimates (3.20)-(3.24) it comes

$$(3.25)\quad \lambda\int_{\Omega_{\ell+1}} |\nabla u_\infty|^2\rho dx \le \left\{\frac{\Lambda}{2}(1+c_p^2(\omega)) + \frac{\Lambda}{2} + 2\varepsilon c_p^2(\omega)\right\}\int_{D_\ell} |\nabla u_\infty|^2 dx$$
$$+3\varepsilon\int_{\Omega_{\ell+1}} |\nabla u_\infty|^2\rho dx + \left\{\frac{\Lambda}{2}c_p^2(\omega) + \frac{\Lambda^2}{4\varepsilon} + 2\varepsilon(1+c_p^2(\omega))\right\}\int_{\Omega_{\ell+1}} |\nabla\varphi^+|^2 dx + \frac{1}{2\varepsilon}|f|^2_{V^*_{\ell+1}}$$

hence by (3.5)

$$\lambda\int_{\Omega_{\ell+1}} |\nabla u_\infty|^2\rho dx \le 3\varepsilon\int_{\Omega_{\ell+1}} |\nabla u_\infty|^2\rho dx + C(\varepsilon)\int_{D_\ell} |\nabla u_\infty|^2 dx + C(\varepsilon)e^{2\delta(\ell+1)}.$$

Choosing $3\varepsilon = \dfrac{\lambda}{2}$ we arrive to

$$\int_{\Omega_{\ell+1}} |\nabla u_\infty|^2 \rho dx \le C_1 e^{2\delta(\ell+1)} + C_2 \int_{D_\ell} |\nabla u_\infty|^2 dx$$

for some constants C_1 and C_2. From this we derive

$$\int_{\Omega_\ell} |\nabla u_\infty|^2 dx \le C_1 e^{2\delta(\ell+1)} + C_2 \int_{\Omega_{\ell+1}} |\nabla u_\infty|^2 dx - C_2 \int_{\Omega_\ell} |\nabla u_\infty|^2 dx \tag{3.26}$$

$$\iff \int_{\Omega_\ell} |\nabla u_\infty|^2 dx \le C_0 e^{2\delta(\ell+1)} + \gamma \int_{\Omega_{\ell+1}} |\nabla u_\infty|^2 dx$$

where $C_0 = \dfrac{C_1}{1+C_2}$ and $\gamma = \dfrac{C_2}{1+C_2} < 1$. Note that C_0 and γ are independent of ℓ and δ.

We iterate the inequality (3.26) $[\ell]$ times to get

$$\int_{\Omega_\ell} |\nabla u_\infty|^2 dx \le C_0 e^{2\delta(\ell+1)} + \gamma \Big\{ C_0 e^{2\delta(\ell+2)} + \gamma \int_{\Omega_{\ell+2}} |\nabla u_\infty|^2 dx \Big\}$$

$$\le C_0 e^{2\delta(\ell+1)} \Big\{ 1 + \gamma e^{2\delta} + (\gamma e^{2\delta})^2 + \cdots + (\gamma e^{2\delta})^{[\ell]-1} \Big\} + \gamma^{[\ell]} \int_{\Omega_{\ell+[\ell]}} |\nabla u_\infty|^2 dx$$

and if we choose

$$\gamma e^{2\delta} < 1 \iff \delta < \frac{1}{2} \ln\left(\frac{1}{\gamma}\right)$$

then

$$1 + \gamma e^{2\delta} + (\gamma e^{2\delta})^2 + \cdots + (\gamma e^{2\delta})^{[\ell]-1} \le 1 + \gamma e^{2\delta} + (\gamma e^{2\delta})^2 + \cdots = \frac{1}{1-\gamma e^{2\delta}}$$

so since $\ell - 1 < [\ell] \le \ell$ we have

$$\int_{\Omega_\ell} |\nabla u_\infty|^2 dx \le C_0 e^{2\delta(\ell+1)} \frac{1}{1-\gamma e^{2\delta}} + \frac{1}{\gamma} \gamma^\ell \int_{\Omega_{2\ell}} |\nabla u_\infty|^2 dx.$$

This implies by (3.17)

$$\int_{\Omega_\ell} |\nabla u_\infty|^2 dx \le \frac{C_0 e^{2\delta}}{1-\gamma e^{2\delta}} e^{2\delta\ell} + C e^{-\ell \ln(\frac{1}{\gamma})} e^{8\delta\ell}.$$

Now if we also choose

$$8\delta - \ln\left(\frac{1}{\gamma}\right) < 2\delta \iff \delta < \frac{1}{6} \ln\left(\frac{1}{\gamma}\right)$$

we obtain

$$\int_{\Omega_\ell} |\nabla u_\infty|^2 dx \leq Ce^{2\delta\ell}$$

i.e. the last line of (3.6).

STEP 6. Uniqueness of the solution to (3.6).

Let ρ be the function defined by (3.8), u_∞, u'_∞ two solutions to (3.6). Taking $v = u'_\infty$ in (3.6) and $v = u_\infty$ in (3.6) written for u'_∞ one gets

$$\int_{\Omega_\ell} A(x)\nabla u_\infty \cdot \nabla((u'_\infty - u_\infty)\rho)dx \geq \big\langle f, (u'_\infty - u_\infty)\rho\big\rangle$$

and

$$\int_{\Omega_\ell} A(x)\nabla u'_\infty \cdot \nabla((u_\infty - u'_\infty)\rho)dx \geq \big\langle f, (u_\infty - u'_\infty)\rho\big\rangle.$$

Adding these two inequalities it comes

$$\int_{\Omega_\ell} A(x)\nabla(u_\infty - u'_\infty) \cdot \nabla\big((u_\infty - u'_\infty)\rho\big)dx \leq 0$$

and then arguing as after (3.11) with u_ℓ, $u_{\ell+r}$ replaced respectively by u_∞, u'_∞ we arrive to (see (3.14))

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\infty - u'_\infty)|^2 dx \leq Ce^{-\frac{\alpha\ell}{2}} \int_{\Omega_\ell} |\nabla(u_\infty - u'_\infty)|^2 dx$$

for α satisfying (3.13). It follows that

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\infty - u'_\infty)|^2 dx \leq 2Ce^{-\frac{\alpha\ell}{2}} \left\{ \int_{\Omega_\ell} |\nabla u_\infty|^2 dx + \int_{\Omega_\ell} |\nabla u_\infty|^2 dx \right\} \leq 4Ce^{-\frac{\alpha\ell}{2}} e^{2\delta\ell}$$

and the uniqueness follows by choosing $2\delta < \frac{\alpha}{2}$ since the right hand side above converges towards 0.

This completes the proof of the theorem. □

REMARK 3. – One can consider general force terms

$$f \in \mathcal{D}'(\Omega) \ \text{ such that } \ \forall \ell > 0, f \in H^{-1}(\Omega_\ell)$$

and define the localized forces $f_\ell \in V_\ell^*$ by

$$\langle f_\ell, \cdot \rangle = \langle f, \zeta_\ell \cdot \rangle$$

where

$$\zeta_\ell(X_1) = \min\big(1, \operatorname{dist}(X_1, \mathbb{R}^p \backslash B_p(0,\ell))\big)$$

then one may show that $|f_\ell|_{V_\ell^*} \le C|f|_{H^{-1}(\Omega_\ell)}$.

Now if we consider u_ℓ solution to

$$\begin{cases} u_\ell \in \mathcal{K}_\ell = \Big\{ v \in V_\ell \;\Big|\; v(x) \ge \varphi(x) \text{ a.e. in } \Omega_\ell \Big\}, \\ \displaystyle\int_{\Omega_\ell} A(x)\nabla u_\ell \cdot \nabla(v - u_\ell)dx \ge \langle f_\ell, v - u_\ell \rangle, \; \forall v \in \mathcal{K}_\ell \end{cases}$$

the theorem 2 will hold assuming the growth condition

$$|f|_{H^{-1}(\Omega_\ell)} = O(e^{\delta\ell}).$$

4. – Concluding remarks

1. First it is clear that the existence and uniqueness result can be carried out for nonlinear monotone operators of the type

$$-\operatorname{div}(A(x, \nabla u))$$

under suitable assumptions.

2. It is particularly clear in section 2 one does not need Ω to be bounded in one direction but to be able to apply the Poincaré inequality in one direction, i.e. the result applies for domains (in $\mathbb{R}^2$) of the type depicted in figure 3.

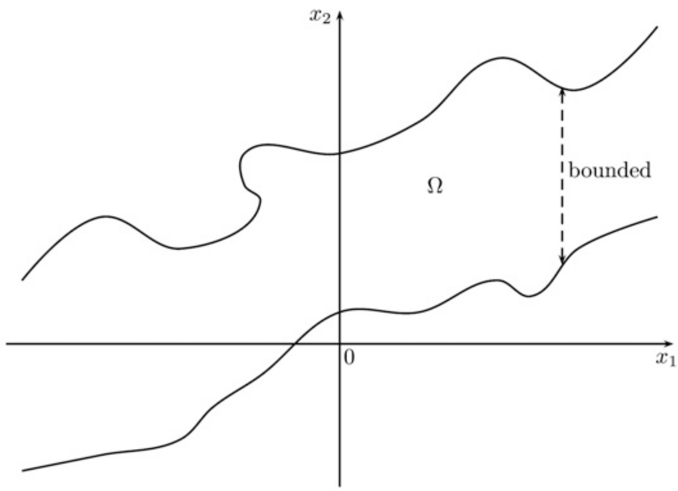

Fig. 3. – General domain.

3. One can consider more general constraint convex sets as for instance the ones of the double obstacle problem.

Acknowledgments. The first author has been supported by the Swiss National Science Foundation under the contract #21-129807/1.

REFERENCES

[C0] M. Chipot, *ℓ goes to plus infinity*, Birkäuser, 2002.

[C1] M. Chipot, *Variational inequalities and flows in porous media*, Springer, Berlin, 1984.

[C2] M. Chipot, *Elements of nonlinear analysis*, Birkäuser, 2000.

[CM] M. Chipot - S. Mardare, To appear.

[CY] M. Chipot - K. Yeressian, *Exponential rates of convergence by an iteration technique*, C. R. Acad. Sci. Paris Ser. I, **346** (2008), 21-26.

[DL] R. Dautray - J. L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Springer-Verlag, Berlin, 1990.

[KS] D. Kinderlehrer - G. Stampacchia, *An introduction to variational inequalities and their applications*, Academic Press, New York, 1980.

[LS] J. L. Lions - G. Stampacchia, *Variational inequalities*, Comm. Pure Applied Math, **20** (1967), 493-519.

[R] J. F. Rodrigues, *Obstacle problems in mathematical physics*, Mathematical Studies 134, North-Holland, 1987.

[Y] K. Yeressian, *Spatial Asymptotic Behavior of Elliptic Equations and Variational Inequalities*, Thesis, University of Zurich, 2010.

Michel Chipot, Institute for Mathematics, University of Zurich
Winterthurerstrasse 190, CH-8057 Zurich
E-mail: m.m.chipot@math.uzh.ch

Karen Yeressian, Institute for Mathematics, RWTH Aachen University
Templergraben 55, 52062 Aachen, Germany
E-mail: karen@instmath.rwth-aachen.de

Received December 29, 2011 and in revised form January 26, 2012