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Some Remarks on Vector Potentials for Maxwell's Equations in Space-Time Carnot Groups

ANNALISA BALDI - BRUNO FRANCHI [1]

This article is dedicated to the memory of Enrico Magenes

Abstract. – *In this paper we prove a Γ-convergence result for time-depending variational functionals in a space-time Carnot group $\mathbb{R}\times\mathbb{G}$ arising in the study of Maxwell's equations in the group. Indeed, a Carnot groups $\mathbb{G}$ (a connected simply connected nilpotent stratified Lie group) can be endowed with a complex of "intrinsic" differential forms that provide the natural setting for a class of "intrinsic" Maxwell's equations. Our main results states precisely that a the vector potentials of a solution of Maxwell's equation in $\mathbb{R}\times\mathbb{G}$ is a critical point of a suitable functional that is in turn a Γ-limit of a sequence of analogous Riemannian functionals.*

1. – Introduction

A connected and simply connected Lie group $(\mathbb{G},\cdot)$ (in general non-commutative) is said a *Carnot group of step κ* if its Lie algebra $\mathfrak{g}$ admits a *step κ stratification*, i.e. there exist linear subspaces $V_1,\dots,V_\kappa$ such that

$$\mathfrak{g}=V_1\oplus\ldots\oplus V_\kappa,\quad [V_1,V_i]=V_{i+1},\quad V_\kappa\neq\{0\},\quad V_i=\{0\}\text{ if } i>\kappa,$$

where $[V_1,V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X,Y]$ with $X\in V_1$ and $Y\in V_i$. The first layer V_1, the so-called *horizontal layer*, plays a key role in the theory, since it generates $\mathfrak{g}$ by commutation.

The Carnot group $\mathbb{G}$ is said to be *free* if its Lie algebra is free, i.e, if the commutators satisfy no linear relations other than antisymmetry and Jacobi identity.

A Carnot group $\mathbb{G}$ can be always identified, through exponential coordinates, with the Euclidean space $(\mathbb{R}^n,\cdot)$, where n is the dimension of $\mathfrak{g}$, endowed with a suitable group operation.

One of the main properties of Carnot groups is that they are endowed with two family of important transformations: the *(left) translation* $\tau_x:\mathbb{G}\to\mathbb{G}$ de-

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fined as $z \mapsto \tau_x z := x \cdot z$, and the (non-isotropic) *group dilations* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$, that are associated with the stratification of $\mathfrak{g}$ and are automorphisms of the group (see Section 2 for details. In general, we refer to [9] or [5] for an exhaustive introduction to Carnot groups).

The Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ can be identified with the tangent space at the origin e of $\mathbb{G}$, and hence the horizontal layer of $\mathfrak{g}$ can be identified with a subspace $H\mathbb{G}_e$ of $T\mathbb{G}_e$. By left translation, $H\mathbb{G}_e$ generates a subbundle $H\mathbb{G}$ of the tangent bundle $T\mathbb{G}$, called the horizontal bundle. A section of $H\mathbb{G}$ is called a horizontal vector field.

Obviously, Euclidean spaces $(\mathbb{R}^n, +)$ are commutative Carnot groups, and, more precisely, the only commutative Carnot groups. Indeed, in this case the stratification of the algebra consists of only one layer, i.e. the Lie algebra reduces to the horizontal layer. The simplest example of non-Abelian Carnot group of step 2 is given by the first Heisenberg group $\mathbb{H}^1 \equiv \mathbb{R}^3$, with variables (x, y, z) and product

$$(x, y, z) \cdot (\xi, \eta, \zeta) = \Big(x + \xi,\, y + \eta,\, z + \zeta - \frac{1}{2}(y\xi - x\eta)\Big).$$

Indeed, let us set $X := \partial_x - \frac{1}{2} y\partial_z$, $Y := \partial_y + \frac{1}{2} x\partial_z$, $Z := \partial_z$. Since $Z = [X, Y]$, the stratification of the algebra $\mathfrak{g}$ is given by $\mathfrak{g} = V_1 \oplus V_2$, where $V_1 = \text{span}\,\{X, Y\}$ and $V_2 = \text{span}\,\{Z\}$.

It is well known that Carnot groups are endowed with an intrinsic geometry, the so-called Carnot-Carathéodory geometry (see for instance, choosing in a wide literature, [9], [5], [15] [10]). From now on, we use the word “intrinsic” when we want to stress a privileged role played by the horizontal layer and by group translations and dilations in $(\mathbb{R}^n, \cdot)$. On the contrary, the word “Euclidean” is used when dealing with the special commutative group $(\mathbb{R}^n, +)$.

It is worth stressing that Carnot-Carathéodory geometry is not Riemannian at any scale (see [20]). In addition, Carnot groups have a privileged role, akin to that of Euclidean spaces versus Riemannian manifolds, acting as rigid “tangent” spaces to more general metric structures (the so-called sub-Riemannian spaces). Here, the word “rigid” is meant to stress their invariance under left translations and group dilations. Thus, Carnot spaces provide a natural setting for Maxwell’s equations, similar to that of Euclidean spaces in special relativity, that is, roughly speaking, a “tangent theory” for general relativity.

The aim of this paper is to study, from a variational point of view, a new class of vector-valued equations in Carnot groups that, though not hyperbolic, we can still call “wave equations” because of their origin, as “equations for a vector potential”, from a class of intrinsic Maxwell’s equations, precisely as it holds in the Euclidean setting, where the potential vector associated with classical Maxwell’s equations satisfies a d’Alembert wave equation. Let us remind this procedure in the Euclidean setting.

Consider the space-time $\mathbb{R} \times \mathbb{R}^3$ of special relativity, where we denote by $s \in \mathbb{R}$ the time variable and by $x \in \mathbb{R}^3$ the space variable. If (Ω^*, d) is the de Rham complex of differential forms in $\mathbb{R} \times \mathbb{R}^3$, classical Maxwell's equations can be formulated in their simplest form as follows: we fix the standard volume form dV in $\mathbb{R}^3$, and we consider a 2-form $F \in \Omega^2$ (Faraday's form), that can be always written as $F = ds \wedge E + B$, where E is the electric field 1-form and B is the magnetic induction 2-form. Then, if we assume for sake of simplicity all "physical" constants (i.e. magnetic permeability and electric permittivity) to be 1, classical Maxwell's equations become

$$dF = 0 \quad \text{and} \quad d(*_M F) = \mathcal{J}. \tag{1}$$

Here $*_M$ is the Hodge-star operator associated with the space-time Minkowskian metric and the volume form $ds \wedge dV$ in $\mathbb{R} \times \mathbb{R}^3$, and $\mathcal{J} = ds \wedge *J - \rho$ is a closed 3-form in $\mathbb{R} \times \mathbb{R}^3$, where $*J$ and $\rho = \rho_0 dV$ are respectively the current density 2-form and the charge density 3-form (here $*$ is the standard Hodge-star operator in $\mathbb{R}^3$ associated with the Euclidean metric and the volume form dV). Since $dF = 0$, we can always assume that $F = dA$, where A (the electromagnetic potential 1-form) can be written as $A = A_\Sigma + \varphi ds$. If, in addition, A_Σ and φ satisfy suitable gauge conditions, then they satisfy the wave equations

$$\frac{\partial^2 A_\Sigma}{\partial s^2} = -\Delta A_\Sigma - J \tag{2}$$

$$\frac{\partial^2 \varphi}{\partial s^2} = -\Delta\varphi + \rho_0, \tag{3}$$

where ΔA_Σ is the positive Hodge Laplacian on 1-forms

$$\Delta A_\Sigma = (d^*d + dd^*)A_\Sigma.$$

We remind that, in the Euclidean space, the Hodge Laplace operator Δ acts diagonally on 1-forms, i.e.

$$\Delta A_\Sigma := \Delta\left(\sum_i A_{\Sigma,i}\, dx^i\right) = \sum_i (\Delta A_{\Sigma,i})\, dx^i,$$

so that equation (2) reduces to a system of uncoupled wave equations.

Recently, in a series of papers ([4], [12], [11], [2]), the authors introduced and studied the notion of "intrinsic" Maxwell's equations in free Carnot groups (see (16)). The setting for these equations is provided by a subcomplex of "intrinsic" differential forms (E_0^*, d_c) – homotopic to de Rham's complex (Ω^*, d) – introduced by Rumin in [19], [18], (see also [3]). The main features of this theory are sketched in Section 2. Here is important to stress that, when acting on intrinsic 1-forms, the "exterior differential" d_c is an operator of order κ (the step of the group) in the horizontal derivatives. Therefore, the associated Laplacian

$\Delta_{\mathbb{G},1} := d_c^* d_c + (d_c d_c^*)^\kappa$ is a (maximal hypoelliptic) operator of order 2κ. Assume now (as we can always do) the higher order gauge condition

$$(4) \qquad (-\Delta_{\mathbb{G}})^{\kappa-1}\delta_c A_\Sigma + \frac{\partial \varphi}{\partial s} = 0.$$

where $\Delta_{\mathbb{G}} := \sum_{j=1}^m X_j^2$ is the usual subelliptic Laplacian in $\mathbb{G}$, $\{X_1, \cdots, X_m\}$ being a suitable basis of V_1. Then the corresponding "wave equations for a vector potential" $A_\Sigma + \varphi\, ds$ takes the form:

$$(5) \qquad \frac{\partial^2 A_\Sigma}{\partial s^2} = -\Delta_{\mathbb{G},1} A_\Sigma$$

$$(6) \qquad \frac{\partial^2 \varphi}{\partial s^2} = -(-\Delta_{\mathbb{G}})^\kappa \varphi\ ,$$

provided (4) holds.

It is important to notice that the equation for A_Σ cannot be diagonalized, unlike in the Euclidean case. But the main new phenomenon is that the "wave equations" we obtain utterly differ even in the scalar case from what one could imagine as "wave equations in the group", i.e.

$$(7) \qquad \frac{\partial^2 \varphi}{\partial s^2} - \Delta_{\mathbb{G}} \varphi = 0.$$

Indeed, the equations we obtain are by no means hyperbolic equations, by [16], Theorem 5.5.2, since they contain second order derivatives in s and 2κ-th order derivatives in x, so that their principal parts are (degenerate) elliptic. Thus, we should not expect any hyperbolic behavior, as, for instance, finite speed of propagation like in (3) (see, e.g., [17], [14]).

Another interesting feature of "wave equations" (5) has been already pointed out in [12]. In case of cylindrical symmetry in $\mathbb{H}^1$ (i.e. when dealing with functions depending only on the horizontal variables), the components of A_Σ, as well as φ, solve the equation

$$\frac{\partial^2 u}{\partial s^2} = -\Delta^2 u \quad \text{in } \mathbb{R}^2.$$

In this way, we recover a classical equation of elasticity, the so-called Germain-Lagrange equation for the vibration of plates (see e.g. [21], Section 9).

In [2], we proved that intrinsic time-harmonic Maxwell's equations in a bounded domain of $\mathbb{G}$ are variational limits of classical time-harmonic Maxwell's equations in the matter in presence of strongly anisotropic electric permittivity and magnetic permeability. In this paper we prove a corresponding result in terms of Γ-convergence for time-depending variational functionals associated with the wave equations (5) in the space-time Carnot group $\mathbb{R} \times \mathbb{G}$ (that actually is not free).

More precisely, we prove that the vector potential $A_\Sigma + \varphi\, ds$ is a critical point of a variational "energy functional" $\mathcal{L}$, and that $\mathcal{L}$ is the Γ-limit of a sequence of functionals $\mathcal{L}_\varepsilon$ associated with approximated Riemannian energies. It is worth to stress here that this result is not meant in the perspective of obtaining existence of critical points for $\mathcal{L}$, but only to show in what sense vector potentials in the group can be seen as limits of "usual" vector potentials in the de Rham complex.

The paper is organized as follows: in Section 2 we collect some known results on Carnot groups and we present the main results of Rumin's theory of differential forms in Carnot groups; in Section 3 we remind the notion of Maxwell's equations in space-time Carnot groups, and finally in Section 4 we prove our main convergence result.

2. – Differential Forms in Carnot groups

To keep this paper self-contained we remind some defintions and properties concerning the"intrinsic" complex of differential forms in Carnot groups, as defined by Rumin in [19], [18], (see also [3]). Let $(\mathbb{G}, \cdot)$ be a Carnot group of step κ and let $\mathfrak{g}$ be its Lie algebra with dimension n.

DEFINITION 2.1. – *Let $e_1, \dots, e_n$ be a basis of $\mathfrak{g}$ adapted to the stratification, and let $X = \{X_1, \dots, X_n\}$ be the family of left invariant vector fields such that $X_i(0) = e_i$, $i = 1, \dots, n$. The Lie algebra $\mathfrak{g}$ can be endowed with a scalar product $\langle \cdot, \cdot \rangle$, making $\{X_1, \dots, X_n\}$ an orthonormal basis.*

DEFINITION 2.2. – *Let $m \geq 2$ and $\kappa \geq 1$ be fixed integers. We say that $\mathfrak{f}_{m,\kappa}$ is the free Lie algebra with m generators $x_1, \dots, x_m$ and nilpotent of step κ if:*

i) *$\mathfrak{f}_{m,\kappa}$ is a Lie algebra generated by its elements $x_1, \dots, x_m$, i.e. $\mathfrak{f}_{m,\kappa} = \mathrm{Lie}(x_1, \dots, x_m)$;*

ii) *$\mathfrak{f}_{m,\kappa}$ is nilpotent of step κ;*

iii) *for every Lie algebra $\mathfrak{n}$ nilpotent of step κ and for every map φ from the set $\{x_1, \dots, x_m\}$ to $\mathfrak{n}$, there exists a (unique) homomorphism of Lie algebras φ from $\mathfrak{f}_{m,\kappa}$ to $\mathfrak{n}$ which extends φ.*

The Carnot group $\mathbb{G}$ is said free if its Lie algebra $\mathfrak{g}$ is isomorphic to a free Lie algebra.

When $\mathbb{G}$ is a free group, we can assume $\{X_1, \dots, X_n\}$ a Grayson-Grossman-Hall basis of $\mathfrak{g}$ (see [13], [5], *Theorem* 14.1.10). *This makes several computations much simpler. In particular, $\{[X_i, X_j], X_i, X_j \in V_1, i<j\}$ provides an orthonormal basis of V_2.*

Since $\mathbb{G}$ is written in exponential coordinates, a point $p \in \mathbb{G}$ is identified with the n-tuple $(p_1, \dots, p_n) \in \mathbb{R}^n$ and we can identify $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$, where the explicit expression of the group operation $\cdot$ is determined by the Campbell-Hausdorff formula.

For any $x \in \mathbb{G}$, the *(left) translation* $\tau_x : \mathbb{G} \to \mathbb{G}$ is defined as

$$z \mapsto \tau_x z := x \cdot z.$$

For any $\lambda > 0$, the *dilation* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$, is defined as

$$\delta_\lambda(x_1, \dots, x_n) = (\lambda^{d_1} x_1, \dots, \lambda^{d_n} x_n),$$

where $d_i \in \mathbb{N}$ is called *homogeneity of the variable* x_i in $\mathbb{G}$ (see [9], Chapter 1). The dilations δ_λ are group automorphisms, since $\delta_\lambda x \cdot \delta_\lambda y = \delta_\lambda(x \cdot y)$.

The Haar measure of $\mathbb{G} = (\mathbb{R}^n, \cdot)$ is the Lebesgue measure $\mathcal{L}^n$ in $\mathbb{R}^n$. If $A \subset \mathbb{G}$ is $\mathcal{L}$-measurable, we write also $|A| := \mathcal{L}^n(A)$.

Following [9], we also adopt the following multi-index notation for higher-order derivatives. If $I = (i_1, \dots, i_n)$ is a multi-index, we set $X^I = X_1^{i_1} \cdots X_n^{i_n}$. By the Poincaré-Birkhoff-Witt theorem (see, e.g. [6], I.2.7), the differential operators X^I form a basis for the algebra of left invariant differential operators in $\mathbb{G}$. Furthermore, we set $|I| := i_1 + \cdots + i_n$ the order of the differential operator X^I, and $d(I) := d_1 i_1 + \cdots + d_n i_n$ its degree of homogeneity with respect to group dilations.

Let k be a positive integer, $1 \le p < \infty$, and let Ω be an open set in $\mathbb{G}$. The Folland-Stein Sobolev space $W^{k,p}_{\mathbb{G}}(\Omega)$ associated with the vector fields $X_1, \dots, X_m$ is defined to consist of all functions $f \in L^p(\Omega)$ with distributional derivatives $X^I f \in L^p(\Omega)$ for any X^I as above with $d(I) \le k$, endowed with the natural norm. We keep the subscript $\mathbb{G}$ to avoid misunderstanding with the usual Sobolev spaces $W^{k,p}(\Omega)$.

The dual space of $\mathfrak{g}$ is denoted by $\bigwedge^1 \mathfrak{g}$. The basis of $\bigwedge^1 \mathfrak{g}$, dual of the basis $X_1, \cdots, X_n$, is the family of covectors $\{\theta_1, \cdots, \theta_n\}$. We indicate by $\langle \cdot, \cdot \rangle$ also the inner product in $\bigwedge^1 \mathfrak{g}$ that makes $\theta_1, \cdots, \theta_n$ an orthonormal basis. We point out that, except for the trivial case of the commutative group $\mathbb{R}^n$, the forms $\theta_1, \cdots, \theta_n$ may have polynomial (hence variable) coefficients. Following Federer (see [8] 1.3), the exterior algebras of $\mathfrak{g}$ and of $\bigwedge^1 \mathfrak{g}$ are the graded algebras indicated as $\bigwedge_* \mathfrak{g} = \bigoplus_{h=0}^{n} \bigwedge_h \mathfrak{g}$ and $\bigwedge^* \mathfrak{g} = \bigoplus_{h=0}^{n} \bigwedge^h \mathfrak{g}$ where $\bigwedge_0 \mathfrak{g} = \bigwedge^0 \mathfrak{g} = \mathbb{R}$ and, for $1 \le h \le n$,

$$\bigwedge_h \mathfrak{g} := \mathrm{span}\{X_{i_1} \wedge \cdots \wedge X_{i_h} : 1 \le i_1 < \cdots < i_h \le n\},$$

$$\bigwedge^h \mathfrak{g} := \mathrm{span}\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_h} : 1 \le i_1 < \cdots < i_h \le n\}.$$

The elements of $\bigwedge_h \mathfrak{g}$ and $\bigwedge^h \mathfrak{g}$ are called *h-vectors* and *h-covectors*, respectively. We denote by Ω_h and Ω^h the spaces of all sections of $\bigwedge_h \mathfrak{g}$ and $\bigwedge^h \mathfrak{g}$, respectively for $h = 0, 1, \dots, n$. We refer to elements of Ω_h as to fields of h-vectors and to elements of Ω^h as to h-forms and to (Ω^*, d) as to the de Rham complex.

We denote by Θ^h the basis $\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_h} : 1 \leq i_1 < \cdots < i_h \leq n\}$ of $\bigwedge^h \mathfrak{g}$.

The dual space $\bigwedge^1 (\bigwedge_h \mathfrak{g})$ of $\bigwedge_h \mathfrak{g}$ can be naturally identified with $\bigwedge^h \mathfrak{g}$.

The inner product $\langle \cdot, \cdot \rangle$ extends canonically to $\bigwedge_h \mathfrak{g}$ and to $\bigwedge^h \mathfrak{g}$ making the bases $X_{i_1} \wedge \cdots \wedge X_{i_h}$ and $\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}$ orthonormal.

DEFINITION 2.3. – *We define linear isomorphisms (Hodge duality: see* [8] 1.7.8)

$$* : \bigwedge_h \mathfrak{g} \longleftrightarrow \bigwedge_{n-h} \mathfrak{g} \quad \text{and} \quad * : \bigwedge^h \mathfrak{g} \longleftrightarrow \bigwedge^{n-h} \mathfrak{g},$$

for $1 \leq h \leq n$, *putting, for* $v, w \in \bigwedge_h \mathfrak{g}$ *and* $\varphi, \psi \in \bigwedge^h \mathfrak{g}$

$$v \wedge *w = \langle v, w \rangle X_1 \wedge \cdots \wedge X_n, \qquad \varphi \wedge *\psi = \langle \varphi, \psi \rangle \theta_1 \wedge \cdots \wedge \theta_n.$$

From now on, we refer to the n-form

$$dV := \theta_1 \wedge \cdots \wedge \theta_n$$

as to the canonical volume form in $\mathbb{G}$.

If d is the usual de Rham exterior differential, we denote by $\delta = d^*$ its formal adjoint in $L^2(\mathbb{G}, \Omega^*)$.

DEFINITION 2.4. – *If* $\alpha \in \bigwedge^1 \mathfrak{g}$, $\alpha \neq 0$, *we say that* α *has pure weight* k, *and we write* $w(\alpha) = k$, *if its dual vector* $\alpha^\natural$ *is in* V_k. *More generally, if* $\alpha \in \bigwedge^h \mathfrak{g}$, *we say that* α *has pure weight* k *if* α *is a linear combination of covectors* $\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}$ *with* $w(\theta_{i_1}) + \cdots + w(\theta_{i_h}) = k$.

REMARK 2.5. – As shown in [3], if $\alpha, \beta \in \bigwedge^h \mathfrak{g}$ and $w(\alpha) \neq w(\beta)$, then $\langle \alpha, \beta \rangle = 0$, and we have ([3], formula (16))

$$\bigwedge^h \mathfrak{g} = \bigoplus_{p=M_h^{\min}}^{M_h^{\max}} \bigwedge^{h,p} \mathfrak{g}, \tag{8}$$

where $\bigwedge^{h,p} \mathfrak{g}$ is the linear span of the h-covectors of weight p and $M_h^{\min}$, $M_h^{\max}$ are respectively the smallest and the largest weight of left-invariant h-covectors.

Keeping in mind the decomposition (8), we can define in the same way several left invariant fiber bundles over $\mathbb{G}$, that we still denote with the same symbol $\bigwedge^{h,p} \mathfrak{g}$. Notice also that the fiber $\bigwedge_x^h \mathfrak{g}$ (and hence the fiber $\bigwedge_x^{h,p} \mathfrak{g}$) can be endowed with a natural scalar product $\langle \cdot, \cdot \rangle_x$.

We denote by $\Omega^{h,p}$ the vector space of all smooth h-forms in $\mathbb{G}$ of pure weight p, i.e. the space of all smooth sections of $\bigwedge^{h,p} \mathfrak{g}$. We have

$$\Omega^h = \bigoplus_{p=M_h^{\min}}^{M_h^{\max}} \Omega^{h,p}. \tag{9}$$

The following crucial property of the weight follows from Cartan identintity: see [19], Section 2.1:

LEMMA 2.6. – *We have* $d(\bigwedge^{h,p}\mathfrak{g}) \subset \bigwedge^{h+1,p}\mathfrak{g}$, *i.e., if* $\alpha \in \bigwedge^{h,p}\mathfrak{g}$ *is a left invariant h-form of weight p with* $d\alpha \neq 0$, *then* $w(d\alpha) = w(\alpha)$.

DEFINITION 2.7. – *Let now* $\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i \, \theta_i^h \in \Omega^{h,p}$ *be a (say) smooth form of pure weight p. Then we can write*

$$d\alpha = d_0\alpha + d_1\alpha + \ldots + d_\kappa\alpha,$$

where

$$d_0\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i d\theta_i^h$$

does not increase the weight, and

$$d_i\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \sum_{X_j \in V_i} (X_j\alpha_i)\theta_j \wedge \theta_i^h,$$

increases the weight by i for $i = 1, \ldots, \kappa$. In particular, d_0 is an algebraic operator.

LEMMA 2.8. – $d_0^2 = 0$, *i.e.* (Ω^*, d_0) *is a complex.*
Moreover, let $\alpha \in \Omega^h$ *be left-invariant. We have:*

i) $d\alpha = d_0\alpha$;
ii) $d_0\alpha$ *is left-invariant.*

The following definition of intrinsic covectors (and therefore of intrinsic forms) is due to M. Rumin ([19], [18]).

DEFINITION 2.9. – *If* $0 \leq h \leq n$ *we set*

$$E_0^h := \ker d_0 \cap (\mathrm{Im}\, d_0)^\perp \subset \Omega^h$$

The elements of E_0^h are *intrinsic h-forms on* $\mathbb{G}$. Since the construction of E_0^h is left invariant, this space of forms can be seen as the space of sections of a fiber subbundle of $\bigwedge^h \mathfrak{g}$, generated by left translation and still denoted by E_0^h. In particular E_0^h inherits from $\bigwedge^h \mathfrak{g}$ the scalar product on the fibers.

Moreover, there exists a left invariant orthonormal basis $\Xi_0^h = \{\xi_j\}$ of E_0^h that is adapted to the filtration (8).

Since it is easy to see that $E_0^1 = \mathrm{span}\,\{\theta_1, \ldots, \theta_m\}$, without loss of generality, we can take $\xi_j = \theta_j$ for $j = 1, \ldots, m$.

If we set $E_0^{h,p} := E_0^h \cap \Omega^{h,p}$, then

$$E_0^h = \bigoplus_p E_0^{h,p}.$$

We define now a (pseudo) inverse of d_0 as follows (see [3], Lemma 2.11):

LEMMA 2.10. – *If $\beta \in \bigwedge^{h+1} \mathfrak{g}$, then there exists a unique $\alpha \in \bigwedge^h \mathfrak{g} \cap (\ker d_0)^\perp$ such that $d_0\alpha - \beta \in \mathcal{R}(d_0)^\perp$. We set $\alpha := d_0^{-1}\beta$. Notice that d_0^{-1} preserves the weights.*

The following theorem summarizes the construction of the intrinsic differential d_c (for details, see [19] and [3], Section 2) .

THEOREM 2.11. – *The de Rham complex (Ω^*, d) splits in the direct sum of two sub-complexes (E^*, d) and (F^*, d), with*

$$E := \ker d_0^{-1} \cap \ker (d_0^{-1} d) \quad \text{and} \quad F := \mathcal{R}(d_0^{-1}) + \mathcal{R}(d d_0^{-1}).$$

We have

i) *Let Π_E be the projection on E along F (that is not an orthogonal projection). Then for any $\alpha \in E_0^{h,p}$, if we denote by $(\Pi_E\alpha)_j$ the component of $\Pi_E\alpha$ of weight j, then*

$$\begin{aligned} &(\Pi_E\alpha)_p = \alpha \\ &(\Pi_E\alpha)_{p+k+1} = -d_0^{-1}\Big(\sum_{1\le \ell\le k+1} d_\ell(\Pi_E\alpha)_{p+k+1-\ell}\Big). \end{aligned} \tag{10}$$

ii) *Π_E is a chain map, i.e.*

$$d\Pi_E = \Pi_E d.$$

iii) *Let Π_{E_0} be the orthogonal projection from Ω^* on E_0^*, then*

$$\Pi_{E_0} = Id - d_0^{-1}d_0 - d_0 d_0^{-1}, \quad \Pi_{E_0}^\perp = d_0^{-1}d_0 + d_0 d_0^{-1}. \tag{11}$$

Set now

$$d_c = \Pi_{E_0}\, d\, \Pi_E : E_0^h \to E_0^{h+1}, \quad h = 0, \dots, n-1.$$

We have:

iv) $d_c^2 = 0$;

v) *the complex $E_0 := (E_0^*, d_c)$ is exact;*

vi) *with respect to the bases Ξ^*, the intrinsic differential d_c can be seen as a matrix-valued operator such that, if α has weight p, then the component of weight q of $d_c\alpha$ is given by an homogeneous differential operator in the horizontal derivatives of order $q - p \ge 1$, acting on the components of α.*

From now on, we restrict ourselves to assume $\mathbb{G}$ is a free group of step κ. The technical reason for this choice relies in the following property.

THEOREM 2.12 ([11], Theorem 5.9). – *Let $\mathbb{G}$ be a free group of step κ. Then all forms in E_0^1 have weight 1 and all forms in E_0^2 have weight $\kappa+1$.*

Moreover, if $\xi \in \bigwedge^{2,p}\mathfrak{g}$ with $p \neq \kappa+1$, then $\Pi_{E_0}\xi = 0$. Indeed, $\Pi_{E_0}\xi$ has weight p, and therefore has to be zero, since $\Pi_{E_0}\xi \in \bigwedge^{2,\kappa+1}\mathfrak{g}$.

We denote by $\delta_c = \delta_{c,\mathbb{G}} = d_c^* = d_{c,\mathbb{G}}^*$ the formal adjoint of d_c in $L^2(\mathbb{G}, E_0^*)$. The following assertion holds.

DEFINITION 2.13. – *If $\Omega \subset \mathbb{G}$ is an open set, $0 \le h \le n$, $k \in \mathbb{N}$ and $1 \le p \le \infty$, then we denote by $W^{k,p}_{\mathbb{G}}(\Omega, E_0^h)$ the space of all forms in E_0^h with coefficients in $W^{k,p}_{\mathbb{G}}(\Omega)$, endowed with its natural norm. It is easy to see that this definition is independent of the basis of $\bigwedge^h\mathfrak{g}$ we have chosen. The spaces $L^p(\Omega, E_0^h)$ and $\mathcal{D}(\Omega, E_0^h)$ are defined analogously starting from $L^p(\Omega)$ and from the space of test functions $\mathcal{D}(\Omega)$, respectively.*

3. – Space-time Carnot groups and Maxwell's equations

From now on, we denote by x a "space" point in the Carnot group $\mathbb{G}$, and by $s \in \mathbb{R}$ the "time", and we choose in $\mathbb{R}\times\mathbb{G}$ the canonical volume form $ds \wedge dV$, where, as above, $dV = \theta_1 \wedge \cdots \wedge \theta_n$ is the canonical volume form in $\mathbb{G}$. Moreover, we denote by $(\Omega^*_{\mathbb{G}}, d_{\mathbb{G}})$ and $(\Omega^*_{\mathbb{R}\times\mathbb{G}}, d_{\mathbb{R}\times\mathbb{G}})$ the de Rham complex of forms on $\mathbb{G}$ and on $\mathbb{R}\times\mathbb{G}$, respectively. Notice that, in general, even if $\mathbb{G}$ is a free group, then $\mathbb{R}\times\mathbb{G}$ may fail to be free. For sake of brevity, we write

$$\begin{aligned} \Omega^* := \Omega^*_{\mathbb{G}} \quad &\text{and} \quad \hat{\Omega}^* := \Omega^*_{\mathbb{R}\times\mathbb{G}}, \\ d := d_{\mathbb{G}} \quad &\text{and} \quad \hat{d} := d_{\mathbb{R}\times\mathbb{G}}, \\ \delta := d^*_{\mathbb{G}} \quad &\text{and} \quad \hat{\delta} := d^*_{\mathbb{R}\times\mathbb{G}}. \end{aligned}$$

When dealing with intrinsic forms, we denote by $(E^*_{0,\mathbb{G}}, d_{c,\mathbb{G}})$ and $(E^*_{0,\mathbb{R}\times\mathbb{G}}, d_{c,\mathbb{R}\times\mathbb{G}})$ the complex of intrinsic forms on $\mathbb{G}$ and on $\mathbb{R}\times\mathbb{G}$, respectively. Again, we write

$$E_0^* := E^*_{0,\mathbb{G}} \quad \text{and} \quad \hat{E}_0^* := E^*_{0,\mathbb{R}\times\mathbb{G}},$$

as well as

$$d_c := d_{c,\mathbb{G}} \quad \text{and} \quad \hat{d}_c := d_{c,\mathbb{R}\times\mathbb{G}},$$

$$\delta_c := d^*_{c,\mathbb{G}} \quad \text{and} \quad \hat{\delta}_c := d^*_{c,\mathbb{R}\times\mathbb{G}}.$$

Denote by S the vector field $\dfrac{\partial}{\partial s}$. The Lie group $\mathbb{R} \times \mathbb{G}$ is a Carnot group; its Lie algebra $\hat{\mathfrak{g}}$ admits the stratification

$$\hat{\mathfrak{g}} = \hat{V}_1 \oplus V_2 \oplus \cdots \oplus V_\kappa, \tag{12}$$

where $\hat{V}_1 = \text{span}\,\{S, V_1\}$. Since the adapted basis $\{X_1, \dots, X_n\}$ has been already fixed once and for all, the associated orthonormal fixed basis for $\hat{\mathfrak{g}}$ will be $\{S, X_1, \dots, X_n\}$. Consider the Lie derivative $\mathcal{L}_S$ along S. When acting on h-forms α in $\mathbb{G}$, without risk of misunderstandings, we write $S\alpha$ for $\mathcal{L}_S\alpha$.

We point out that S commutes with d. Thus, if $\alpha \in \Omega^h$ and its coefficients depend on s and x (and is identified with a h-form in $\hat{\Omega}^h$), then

$$\hat{d}\alpha = d\alpha + ds \wedge (S\alpha). \tag{13}$$

Let us state preliminarily a structure lemma for intrinsic forms in $\mathbb{R} \times \mathbb{G}$. The proof can be found in [4] and also in [11].

LEMMA 3.1. – *If $1 \leq h \leq n$, then a h-form α belongs to $\hat{E}_0^h$ if and only if it can be written as*

$$\alpha = ds \wedge \beta + \gamma, \tag{14}$$

where $\beta \in E_0^{h-1}$ and $\gamma \in E_0^h$ are respectively intrinsic $(h-1)$-forms and h-forms in $\mathbb{G}$ with coefficients depending on x and s.

As in special relativity, the space-time $\mathbb{R} \times \mathbb{G}$ can be endowed with a Minkowskian scalar product $\langle \cdot, \cdot \rangle_M$ in $\bigwedge_* \hat{\mathfrak{g}}$ and $\bigwedge^* \hat{\mathfrak{g}}$. For a precise definition see [11], Definition 4.1.

DEFINITION 3.2. – *If $1 \leq h \leq n$, we set*

$$\langle ds \wedge \beta + \gamma, ds \wedge \beta' + \gamma' \rangle_M := \langle \gamma, \gamma' \rangle - \langle \beta, \beta' \rangle.$$

for $\beta, \beta' \in E_0^{h-1}$ and $\gamma,\ \gamma' \in E_0^h$. In addition, we denote by $*_M$ the Hodge operator $*_M : \bigwedge^h \hat{\mathfrak{g}} \to \bigwedge^{n-h} \hat{\mathfrak{g}}$ associated with the Minkowskian scalar product in $\bigwedge^* \hat{\mathfrak{g}}$, with respect to the volume form $ds \wedge dV$, by

$$\alpha \wedge *_M\beta = \langle \alpha, \beta \rangle_M ds \wedge dV.$$

REMARK 3.3. – If $\alpha = ds \wedge \beta + \gamma \in \hat{E}_0^h$, then

$$*_M\alpha = (-1)^h ds \wedge *\gamma - *\beta.$$

PROPOSITION 3.4 ([11], Proposition 4.7). – *If $1 \leq h \leq n$, and $\alpha = ds \wedge \beta + \gamma \in \hat{E}_0^h$, then*

$$\hat{d}_c\alpha = ds \wedge (S\gamma - d_c\beta) + d_c\gamma. \tag{15}$$

Let $\mathcal{J}$ be a fixed closed intrinsic n-form in $\mathbb{R}\times\mathbb{G}$ (a source form). We can write $\mathcal{J}=ds\wedge *J-\rho$, where $J=J(s,\cdot)$ is an intrinsic 1-form on $\mathbb{G}$ and $\rho(s,\cdot)=\rho_0(s,\cdot)\,dV$ is a volume form on $\mathbb{G}$ for any fixed $s\in\mathbb{R}$.

If $F\in\hat{E}_0^2$, we call *Maxwell's equations in* $\mathbb{G}$ the system

$$\hat{d}_c F=0 \quad\text{and}\quad \hat{d}_c(\,*_M F)=\mathcal{J} \tag{16}$$

(for sake of simplicity, we assume all "physical" constants to be 1). This system corresponds to a particular choice of the so-called constitutive relations. We refer to [4], [11] for further comments (in particular for invariance under suitable contact Lorentz transformation).

If F is a solution of (16), then it is in particular a closed form. Therefore it admits a vector potential

$$A:=A_\Sigma+\varphi\,ds\in\hat{E}_0^1 \quad\text{such that}\quad \hat{d}_c A=F. \tag{17}$$

If F satisfies (16) and A is a vector potential associated with F as in (17), then A is a stationary point of the functional

$$\int_{\hat{\mathbb{G}}}\big(\hat{d}_c A\wedge *_M\hat{d}_c A-A\wedge\mathcal{J}\big). \tag{18}$$

4. – The main Γ-convergence results

We recall briefly the definition of sequential Γ-convergence. For an accurate and exhaustive study of Γ-convergence, we refer to the monograph [7].

DEFINITION 4.1. – *Let X be a separated topological space, and let*

$$F_\varepsilon,\ F:\ X\longrightarrow[-\infty,+\infty]$$

with $\varepsilon>0$ be functionals on X. We say that $\{F_\varepsilon\}_{\varepsilon>0}$ sequentially Γ-converges to F on X as ε goes to zero if the following two conditions hold:

1) *for every $u\in X$ and for every sequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ with $\varepsilon_k\to 0$ as $k\to\infty$, which converges to u in X, there holds*

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_{\varepsilon_k})\ge F(u); \tag{19}$$

2) *for every $u\in X$ and for every sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ with $\varepsilon_k\to 0$ as $k\to\infty$ there exists a subsequence (still denoted by $\{\varepsilon_k\}_{k\in\mathbb{N}}$) such that $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ converges to u in X and*

$$\limsup_{k\to\infty} F_{\varepsilon_k}(u_{\varepsilon_k})\le F(u) \tag{20}$$

To avoid cumbersome notations, from now on we write systematically $\lim_{\varepsilon\to 0}$ to mean a limit with $\varepsilon=\varepsilon_k$, where $\{\varepsilon_k\}_{k\in\mathbb{N}}$ is any sequence with $\varepsilon_k\to 0$ as $k\to\infty$.

DEFINITION 4.2. – *We set*

$$\mathcal{W}_{\mathbb{G}}\Big(\bigwedge^1 \hat{\mathfrak{g}}\Big) := \Big\{A = A_\Sigma + \varphi\, ds,$$
$$\text{with } A_\Sigma \in L^2\Big(\mathbb{R}, W^{\kappa,2}_{\mathbb{G}}(\mathbb{G}, \bigwedge^1 \mathfrak{g})\Big) \cap W^{1,2}\Big(\mathbb{R}, L^2(\mathbb{G}, \bigwedge^1 \mathfrak{g})\Big)$$
$$\text{and } \varphi \in L^2\Big(\mathbb{R}, W^{\kappa,2}_{\mathbb{G}}(\mathbb{G}, \bigwedge^0 \mathfrak{g})\Big)\Big\},$$

endowed with the norm

$$\Big(\int_{\mathbb{R}} \|A_\Sigma(s,\cdot)\|^2_{W^{\kappa,2}_G(\mathbb{G},\bigwedge^1 \mathfrak{g})}\, ds + \int_{\mathbb{R}} \Big\|\frac{\partial A_\Sigma}{\partial s}(s,\cdot)\Big\|^2_{L^2\left(\mathbb{G},\bigwedge^1 \mathfrak{g}\right)}\, ds\Big)^{1/2}$$
$$+\Big(\int_{\mathbb{R}} \|\varphi(s,\cdot)\|^2_{W^{\kappa,2}_{\mathbb{G}}\left(\mathbb{G},\bigwedge^0 \mathfrak{g}\right)}\, ds\Big)^{1/2}.$$

We set also

$$\mathcal{W}_{\mathbb{G}}(\hat{E}^1_0) := \Big\{A = A_\Sigma + \varphi\, ds \in \mathcal{W}_{\mathbb{G}}\big(\bigwedge^1 \hat{\mathfrak{g}}\big),\ \text{with } A_\Sigma \in E^1_0\Big\}.$$

Suppose now $\mathcal{J} = ds \wedge *J - \rho_0\, dV$, where $J \in L^2(\hat{\mathbb{G}}, E^1_0)$ and $\rho_0 \in L^2(\hat{\mathbb{G}}, E^0_0)$, so that $\mathcal{J} \in L^2(\hat{\mathbb{G}}, \hat{E}^n_0)$. Consider the functional

$$\int_{\hat{\mathbb{G}}} \big(\hat{d}_c A \wedge *_M \hat{d}_c A - A \wedge \mathcal{J}\big), \tag{21}$$

introduced in (18), that is finite for $A = A_\Sigma + \varphi\, ds \in \mathcal{W}_{\mathbb{G}}(\hat{E}^1_0)$.

It is easy to see that, up a factor $(-1)^n$, (21) can be written also as

$$\int_{\hat{\mathbb{G}}} \big(|d_c A_\Sigma|^2 - |SA_\Sigma - d_c\varphi|^2 + \langle A_\Sigma, J\rangle + \varphi\rho_0\big) ds \wedge dV. \tag{22}$$

For sake of simplicity, from now on we denote by $\mathcal{L}_1(A)$ the source term

$$\mathcal{L}_1(A) := \int_{\mathbb{R}} ds \int_{\mathbb{G}} (\langle A_\Sigma, J\rangle + \varphi\rho_0)\, dV.$$

Eventually, if $A = A_\Sigma + \varphi\, ds \in \mathcal{W}_{\mathbb{G}}(\bigwedge^1 \hat{\mathfrak{g}})$, we can define a functional in $\mathcal{W}_{\mathbb{G}}(\bigwedge^1 \hat{\mathfrak{g}})$ as

$$\mathcal{L}(A) = \begin{cases} \displaystyle\int_{\hat{\mathbb{G}}} \big(|d_c A_\Sigma|^2 - |SA_\Sigma - d_c\varphi|^2\big) ds \wedge dV + \mathcal{L}_1(A) & \text{if } A \in \mathcal{W}_{\mathbb{G}}(\hat{E}^1_0) \\ +\infty \quad \text{otherwise.} & \end{cases}$$

Let now $\varepsilon > 0$ be given. If $A \in \mathcal{W}_{\mathbb{G}}(\bigwedge^1 \hat{\mathfrak{g}})$ we set,

$$\mathcal{L}_\varepsilon(A) = \frac{1}{\varepsilon^{2\kappa}} \int_{\hat{\mathbb{G}}} |d_\varepsilon A_\Sigma|^2 \, ds \wedge dV - \frac{1}{\varepsilon^2} \int_{\hat{\mathbb{G}}} |\varepsilon S A_\Sigma - d_\varepsilon \varphi|^2 \, ds \wedge dV + \mathcal{L}_1(A),$$

where

$$d_\varepsilon = d_0 + \varepsilon d_1 + \cdots + \varepsilon^\kappa d_\kappa$$

(notice $\mathcal{L}_1(A)$ is independent of ε).

We stress that $\mathcal{L}_\varepsilon(A)$ is always finite, since $W^{\kappa,2}_{\mathbb{G}}(\mathbb{G}, \bigwedge^1 \mathfrak{g}) \subset W^{1,2}(\mathbb{G}, \bigwedge^1 \mathfrak{g})$ and $L^2(\mathbb{R}, W^{\kappa,2}_{\mathbb{G}}(\mathbb{G}, \bigwedge^0 \mathfrak{g})) \subset L^2(\mathbb{R}, W^{1,2}(\mathbb{G}, \bigwedge^0 \mathfrak{g}))$.

DEFINITION 4.3. – *We say that a sequence* $(A^n)_{n\in\mathbb{N}} := (A^n_\Sigma + \varphi^n ds)_{n\in\mathbb{N}}$ *in* $\mathcal{W}_{\mathbb{G}}(\bigwedge^1 \hat{\mathfrak{g}})$ *$\mathcal{M}$-converges to* $A := A_\Sigma + \varphi\, ds \in \mathcal{W}_{\mathbb{G}}(\bigwedge^1 \hat{\mathfrak{g}})$ *(briefly* $A^n \xrightarrow{\mathcal{M}} A$*) if*

- $A^n_\Sigma \to \alpha$ *weakly in* $L^2(\mathbb{R}, W^{\kappa,2}_{\mathbb{G}}(\mathbb{G}, \bigwedge^1 \mathfrak{g}))$;
- $A^n_\Sigma \to \alpha$ *strongly in* $W^{1,2}(\mathbb{R}, L^2(\mathbb{G}, \bigwedge^1 \mathfrak{g}))$;
- $\varphi^n \to \varphi$ *weakly in* $L^2(\mathbb{R}, W^{\kappa,2}_{\mathbb{G}}(\mathbb{G}, \bigwedge^0 \mathfrak{g}))$;
- $\varphi^n \to \varphi$ *strongly in* $L^2(\mathbb{R}, W^{1,2}_{\mathbb{G}}(\mathbb{G}, \bigwedge^0 \mathfrak{g}))$.

LEMMA 4.4. – *Suppose* $J \in L^2(\hat{\mathbb{G}}, E^1_0)$ *and* $\rho_0 \in L^2(\hat{\mathbb{G}}, E^0_0)$*. Then* $\mathcal{L}_1$ *is continuous with respect to the $\mathcal{M}$-convergence.*

THEOREM 4.5. – *Let* $\mathbb{G}$ *be a free Carnot group of step* κ *and consider the Carnot group* $\hat{\mathbb{G}} = \mathbb{R} \times \mathbb{G}$*. Suppose* $J \in L^2(\hat{\mathbb{G}}, E^1_0)$ *and* $\rho_0 \in L^2(\hat{\mathbb{G}}, E^0_0)$*. Then*

$$\mathcal{L}_\varepsilon \textit{ sequentially } \Gamma\textit{-converges to } \mathcal{L} \textit{ in the } \mathcal{M}\textit{-topology}$$

as $\varepsilon \to 0$.

PROOF. – Without loss of generality, we can take $\mathcal{L}_1 \equiv 0$, by Lemma 4.4, arguing as in [7], Proposition 6.21.

Let $A^\varepsilon := A^\varepsilon_\Sigma + \varphi^\varepsilon ds \xrightarrow{\mathcal{M}} A := A_\Sigma + \varphi ds$ as $\varepsilon \to 0$. First, keeping in mind that $d_0 \varphi^\varepsilon = 0$, and that $d_c = d_1$ on functions, we notice that

$$\begin{aligned} &\frac{1}{\varepsilon^2} \int_{\hat{\mathbb{G}}} |\varepsilon S A^\varepsilon_\Sigma - d_\varepsilon \varphi^\varepsilon|^2 \, ds \wedge dV \\ &= \int_{\hat{\mathbb{G}}} |S A^\varepsilon_\Sigma - d_1 \varphi^\varepsilon - \varepsilon d_2 \varphi^\varepsilon - \cdots - \varepsilon^{\kappa-1} d_\kappa \varphi^\varepsilon|^2 \, ds \wedge dV \\ &\to \int_{\hat{\mathbb{G}}} |S A_\Sigma - d_c \varphi|^2 \, ds \wedge dV \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{23}$$

Thus, in order to show that

$$\mathcal{L}(A) \le \liminf_{\varepsilon\to 0} \mathcal{L}_\varepsilon(A^\varepsilon). \tag{24}$$

we are left to prove that

$$\int_{\hat{\mathbb{G}}} |d_c A_\Sigma|^2 ds\wedge dV \le \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^{2\kappa}} \int_{\hat{\mathbb{G}}} |d_\varepsilon A^\varepsilon_\Sigma|^2\, ds\wedge dV. \tag{25}$$

Without loss of generality, we can assume that

$$\liminf_{\varepsilon\to 0} \mathcal{L}_\varepsilon(A^\varepsilon) < \infty, \tag{26}$$

and therefore, also by (23), we can assume that the right hand side of formula (25) is finite. In particular, it follows that $A \in \mathcal{W}_{\mathbb{G}}(\hat{E}^1_0)$.

Keeping in mind (9), we write

$$A^\varepsilon_\Sigma = (A^\varepsilon_\Sigma)_1 + \cdots + (A^\varepsilon_\Sigma)_\kappa,$$

with $(A^\varepsilon_\Sigma)_i \in \hat{\Omega}^{1,i}$, $i = 1,\dots,\kappa$. Arguing as in [1], Theorem 5.1, we can write

$$\begin{aligned}\frac{1}{\varepsilon^{2\kappa}} \int_{\hat{\mathbb{G}}} |d_\varepsilon A^\varepsilon_\Sigma|^2\, ds\wedge dV = {}& \varepsilon^{-2\kappa} \sum_{2\le p\le\kappa} \int_{\hat{\mathbb{G}}} \Big|\sum_{i=0}^{p-1} \varepsilon^i d_i (A^\varepsilon_\Sigma)_{p-i}\Big|^2 ds\wedge dV\\ &+ \varepsilon^{2(1-\kappa)} \int_{\hat{\mathbb{G}}} \big|d_1(A^\varepsilon_\Sigma)_\kappa + \cdots + \varepsilon^{\kappa-1} d_\kappa (A^\varepsilon_\Sigma)_1\big|^2 ds\wedge dV\\ &+ \sum_{\kappa+2\le p\le 2\kappa} \varepsilon^{2(p-2\kappa)} \int_{\hat{\mathbb{G}}} \Big|\sum_{i=p-\kappa}^{\kappa} \varepsilon^{i-p+\kappa} d_i (A^\varepsilon_\Sigma)_{p-i}\Big|^2 ds\wedge dV.\end{aligned} \tag{27}$$

Assumption (26) implies that the three terms in the right hand side of (27) are uniformly bounded in $L^2(\hat{\mathbb{G}}, \bigwedge^2 \mathfrak{g})$. In particular, from the boundedness of the first term, it follows that

$$d_0(A^\varepsilon_\Sigma)_p + \varepsilon \sum_{i=1}^{p-1} \varepsilon^{i-1} d_i (A^\varepsilon_\Sigma)_{p-i} \longrightarrow 0 \tag{28}$$

as $\varepsilon \to 0$. On the other hand,

$$d_0(A^\varepsilon_\Sigma)_p \to d_0(A_\Sigma)_p \quad \text{weakly in } L^2(\hat{\mathbb{G}}, \textstyle\bigwedge^2 \mathfrak{g}), \tag{29}$$

since d_0 is algebraic and $(A^\varepsilon_\Sigma)_p \to (A_\Sigma)_p$ weakly in $L^2(\hat{\mathbb{G}}, \bigwedge^1 \mathfrak{g})$ for $p \ge 1$.

Combining (28) with the boundedness of $\{A^\varepsilon_\Sigma\}$ in $L^2(\mathbb{R}, W^{\kappa,2}_{\mathbb{G}}(\mathbb{G}, \bigwedge^1 \mathfrak{g}))$ and with (29), it follows that

$$d_0(A_\Sigma)_p = 0 \quad \text{for } p = 2,\dots,\kappa. \tag{30}$$

Hence $A_\Sigma \in \ker d_0 = E^1_0$.

We consider now

$$d_c A_\Sigma := \Pi_{E_0} d \Pi_E A_\Sigma .$$

We can replace $d\Pi_E(A_\Sigma)$ by $(d\Pi_E A_\Sigma)_{\kappa+1}$, since, by Theorem 5.9 in [11], Π_{E_0} vanishes on 2-forms of weight $p \neq \kappa + 1$. In other words,

$$d_c(A_\Sigma) = \Pi_{E_0}\Big(\sum_{\ell=1}^{\kappa} d_\ell(\Pi_E A_\Sigma)_{\kappa+1-\ell}\Big). \tag{31}$$

A slight modification of the arguments used in the proof of Theorem 5.1 in [1] yields

$$d_j(\Pi_E A_\Sigma)_{\kappa+1-\ell} = \lim_{\varepsilon\to 0} \varepsilon^{\ell-\kappa} d_j(A^\varepsilon_\Sigma)_{\kappa+1-\ell}, \tag{32}$$

in $\mathcal{D}'(\hat{\mathbb{G}}, \bigwedge^2 \mathfrak{g})$ for $\ell = 1, \dots, \kappa, j = 0, \dots, \ell$.

By (32), we get

$$\frac{1}{\varepsilon^{\kappa-1}}\Big(d_1(A^\varepsilon_\Sigma)_\kappa + \cdots + \varepsilon^{\kappa-1} d_\kappa(A^\varepsilon_\Sigma)_1\Big) \longrightarrow \sum_{\ell=1}^{\kappa} d_\ell(\Pi_E A_\Sigma)_{\kappa+1-\ell} \tag{33}$$

as $\varepsilon \to 0$ in the sense of distributions. We want to show that the limit in (33) is in fact a weak limit in $L^2(\hat{\mathbb{G}}, \bigwedge^2 \mathfrak{g})$. Indeed, again by (26),

$$\Big\{\frac{1}{\varepsilon^{\kappa-1}}\big(d_1(A^\varepsilon_\Sigma)_\kappa + \cdots + \varepsilon^{\kappa-1} d_\kappa(A^\varepsilon_\Sigma)_1\big)\Big\}_{\varepsilon>0}$$

is equibounded in $L^2(\hat{\mathbb{G}}, \bigwedge^2 \mathfrak{g})$ as $\varepsilon \to 0$. On the other hand, the limit $\sum_{\ell=1}^{\kappa} d_\ell(\Pi_E(A_\Sigma))_{\kappa+1-\ell}$ belongs to $L^2(\hat{\mathbb{G}}, \bigwedge^2 \mathfrak{g})$ (since $\Sigma \to d_\ell(\Pi_E A_\Sigma)_{\kappa+1-\ell}$ is an homogeneous differential operator in the horizontal derivatives of $\mathbb{G}$ of order κ).

Thus, by (31), (27) and taking into account that Π_{E_0} is an orthogonal projection, we obtain eventually

$$\begin{aligned}
\int_{\hat{\mathbb{G}}} |d_c A_\Sigma|^2 \, ds \wedge dV &= \int_{\hat{\mathbb{G}}} |\Pi_{E_0}\Big(\sum_{\ell=1}^{\kappa} d_\ell(\Pi_E A_\Sigma)_{\kappa+1-\ell}\Big)|^2 \, ds \wedge dV \\
&\le \int_{\hat{\mathbb{G}}} \Big|\sum_{\ell=1}^{\kappa} d_\ell(\Pi_E A_\Sigma)_{\kappa+1-\ell}\Big|^2 \, ds \wedge dV \\
&\le \liminf_{\varepsilon\to 0} \varepsilon^{2(1-\kappa)} \int_{\hat{\mathbb{G}}} |d_1(A^\varepsilon_\Sigma)_\kappa + \cdots + \varepsilon^{\kappa-1} d_\kappa(A^\varepsilon_\Sigma)_1|^2 \, ds \wedge dV \\
&\le \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^{2\kappa}} \int_{\hat{\mathbb{G}}} |d_\varepsilon A^\varepsilon_\Sigma|^2 \, ds \wedge dV.
\end{aligned}$$

This proves (25) and eventually (24).

We prove now that, if $A \in \mathcal{W}_{\mathbb{G}}(\hat{E}_0^1)$, then there exists a sequence $(A^\varepsilon)_{\varepsilon>0}$ in $\mathcal{W}_{\mathbb{G}}(\bigwedge^1 \hat{\mathfrak{g}})$ such that

i) $A^\varepsilon \stackrel{\mathcal{M}}{\longrightarrow} A$;
ii) $\mathcal{L}_\varepsilon(A^\varepsilon) \to \mathcal{L}(A)$ as $\varepsilon \to 0$.

By a pretty standard reduction argument, without loss of generality we may assume $A \in \mathcal{D}(\hat{\mathbb{G}}, E_0^1)$.

We choose $A^\varepsilon := A^\varepsilon_\Sigma + \varphi\, ds$, where

$$A^\varepsilon_\Sigma = A_\Sigma + \varepsilon(\Pi_E A_\Sigma)_2 + \cdots + \varepsilon^{\kappa-1}(\Pi_E A_\Sigma)_\kappa. \tag{34}$$

As above, it is easy to see that

$$\frac{1}{\varepsilon^2}\int_{\hat{\mathbb{G}}} |\varepsilon S A^\varepsilon_\Sigma - d_\varepsilon\varphi|^2\, ds\wedge dV \to \int_{\hat{\mathbb{G}}} |SA_\Sigma - d_c\varphi|^2\, ds\wedge dV$$

as $\varepsilon \to 0$, since

$$\varepsilon S A^\varepsilon_\Sigma - d_\varepsilon\varphi = \varepsilon\Big(SA_\Sigma - d_c\varphi + \sum_{j=2}^{\kappa} \varepsilon^{j-1}\big[(\Pi_E S A_\Sigma)_j - d_j\varphi\big]\Big).$$

On the other hand,

$$\begin{aligned}\int_{\hat{\mathbb{G}}} |d_\varepsilon A^\varepsilon_\Sigma|^2\, ds\wedge dV = \frac{1}{\varepsilon^{2\kappa}}\Bigg(&\int_{\hat G}\Big|\Pi_{E_0}\Big(d_\varepsilon\Big(\sum_{i=1}^{\kappa}\varepsilon^{i-1}(\Pi_E A_\Sigma)_i\Big)\Big)\Big|^2 ds\wedge dV \\ &+\int_{\hat G}\Big|\Pi_{E_0^\perp}\Big(d_\varepsilon\Big(\sum_{i=1}^{\kappa}\varepsilon^{i-1}(\Pi_E A_\Sigma)_i\Big)\Big)\Big|^2 ds\wedge dV\Bigg).\end{aligned} \tag{35}$$

Gathering the terms of weight $p = 2, \dots, 2\kappa$, we can write

$$\begin{aligned} d_\varepsilon\Big(\sum_{i=1}^{\kappa}\varepsilon^{i-1}(\Pi_E A_\Sigma)_i\Big) &= \sum_{2\le p\le\kappa}\varepsilon^{p-1}\sum_{i=0}^{p-1} d_i(\Pi_E A_\Sigma)_{p-i} \\ &+ \varepsilon^\kappa\big(d_1(\Pi_E A_\Sigma)_\kappa + \cdots + d_\kappa(\Pi_E A_\Sigma)_1\big) \\ &+ \sum_{\kappa+2\le p\le 2\kappa}\varepsilon^{p-1}\sum_{i=p-\kappa}^{\kappa} d_i(\Pi_E A_\Sigma)_{p-i} := I_1 + I_2 + I_3. \end{aligned}$$

First of all, notice that, by definition, $\varepsilon^{-\kappa}\Pi_{E_0} I_2 = \Pi_{E_0} d\Pi_E A_\Sigma = d_c A_\Sigma$.

Now, by Theorem 2.12,

$$\Pi_{E_0} I_1 = 0.$$

On the other hand, by the recursive formula (10) and Lemma 2.10, we can argue as in [1], Theorem 5.1, to prove that

$$\Pi^{\perp}_{E_0} I_1 = \Pi^{\perp}_{E_0} I_2 = 0. \tag{36}$$

Coming back to (35) we get,

$$\begin{aligned}
\int_{\hat{\mathbb{G}}} |d_\varepsilon A^\varepsilon_\Sigma|^2 \, ds\wedge dV &= \frac{1}{\varepsilon^{2\kappa}}\int_{\hat{G}} |\Pi_{E_0} I_2|^2 \, ds\wedge dV + \frac{1}{\varepsilon^{2\kappa}}\int_{\hat{G}} |I_3|^2 \, ds\wedge dV\\
&= \int_{\hat{G}} |\Pi_{E_0}\big(d_1(\Pi_E A_\Sigma)_\kappa + \cdots + d_\kappa(\Pi_E A_\Sigma)_1\big)|^2 \, ds\wedge dV\\
&+ \frac{1}{\varepsilon^{2\kappa}}\int_{\hat{G}} \Big| \sum_{\kappa+2\le p\le 2\kappa} \varepsilon^{p-1} \sum_{i=p-\kappa}^{\kappa} d_i(\Pi_E A_\Sigma)_{p-i}\Big|^2 ds\wedge dV;
\end{aligned}$$

observing that the second term in previous expression goes to zero as $\varepsilon \to 0$, we get

$$\lim_{\varepsilon\to 0} \int_{\hat{\mathbb{G}}} |d_\varepsilon A^\varepsilon_\Sigma|^2 \, ds\wedge dV = \int_{\hat{\mathbb{G}}} |d_c A_\Sigma|^2 \, ds\wedge dV.$$

This achieves the proof of the theorem. □

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