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# Asymptotic Behaviour of Solutions to a Nonlinear Third Order P.D.E. Modeling Physical Phenomena

Salvatore Rionero

*To the memory of Enrico Magenes*

**Abstract.** – *The longtime behaviour of the solutions to the initial boundary value problem* (1.1)-(1.3) *modeling various physical phenomena, either in the autonomous case or in the nonautonomous case, is studied. Conditions guaranteeing ultimately boundedness and conditions guaranteeing nonlinear asymptotic global stability of the null solution are obtained. Boundary conditions, different from* $(1.2)_1$-$(1.2)_2$, *are also considered (Section* 9*).*

## 1. – Introduction

The present paper is concerned with the initial boundary problem (I.B.V.P.)

$$u_{tt} + au_t = C(t)u_{xx} + \varepsilon(t)u_{xxt} + F(u), \tag{1.1}$$

$$\begin{cases} u(0,t) = u(1,t) = 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = v_1(x), \end{cases} \tag{1.2}$$

with $u_0$ and $v_1$ assigned regular functions such that

$$u_0 = v_1 = 0, \qquad x = 0, 1, \tag{1.3}$$

with

$$\begin{cases} 0 \leq \varepsilon \in C^1(\mathbb{R}^+),\ \mathbb{R}^+ = [0, \infty[,\ \bar{C} \leq C \in C^1(\mathbb{R}^+), \\ \bar{C} = \text{ positive constant}, \ \ 0 \leq a = a(t) \in C^1(\mathbb{R}^+), \\ F(0) = 0,\ F \text{ generally nonlinear function of } u. \end{cases} \tag{1.4}$$

The I.B.V.P. (1.1)-(1.3), either in the autonomous case or in the nonautonomous case, has attracted the attention of several authors {cfr. [1], [16], [20] and the references therein}.

This is because (1.1)-(1.3) arises in modeling various physical phenomena. We confine ourselves to recall that (1.1)-(1.3): i) for

$$F(u) = b \sin u, \quad b = \text{const}, \tag{1.5}$$

reduces to a perturbed Sine-Gordon equation used for describing the classical Josephson effect in the superconductors theory [3]-[6]; ii) arises in modeling heat conduction at low temperature [7],[20]; sound propagation in viscous gases [8]; propagation of plane waves in M.H.D.[9]; motions of viscoelastic fluids [10]-[12].

Various qualitative analysis of (1.1)-(1.3), either in the autonomous case or in the nonautonomous case, have been done in [13]-[16]. In particular conditions sufficient for the stability have been found assuming that at least two of the inequalities

$$uF(u) \le \bar{k}u^2, \;\; uF(u) \le 0, \;\; \varphi(u) = \int_0^u F(z)\,dz < 0, \tag{1.6}$$

with $\bar{k}$ positive constant, hold (1).

Our aim here is to study the longtime behaviour of the solutions either in the autonomous case or in the nonautonomous case. Precisely our aim is to obtain conditions:

1) guaranteeing the ultimately boundedness of solutions;
2) necessary and sufficient for the global asymptotic nonlinear stability of the null solution.

Apart from Section 2 devoted to some preliminaries, the paper is divided in two parts. Sections 3-5 are devoted to the autonomous case, while the nonautonomous case is considered in the Sections 6-9. The paper ends with some final remarks (Section 10) and an Appendix (Section 11) where some proofs are sketched.

## 2. – Preliminaries

In view of the boundary conditions, we embed the problem in the space $L_2^*(0,1)$ such that $\Phi \in L_2^*(0,1)$ requires

i) $\Phi = 0$ for $x = 0, 1$;
ii) $\Phi \in W^{2,2}(0,1)$,
iii) $\Phi, \Phi_t, \Phi_x, \Phi_{xx}, \Phi_{xxt}$ can be expanded in Fourier series absolutely uniformly convergent in $[0,1]$, $\forall t \in \mathbb{R}^+$.

(1) $(1.6)_3$ is implied by $(1.6)_2$ { cfr. [20] exercise 2.12 }. In [16] can be found various examples of forcing terms $F(u)$ fulfilling two of inequalities (1.6). We confine ourselves to mentioning the cases $F(u) = -b|u|^q u$, with $b$ and $q$ positive constants. Then $uF(u) = -b|u|^q u^2 < 0$. Obviously $(1.6)_1$ is also implied by $(1.6)_2$.

In the sequel we denote by
$\|\cdot\|$ the norm in $L_2^*(0,1)$;
$\langle\cdot,\cdot\rangle$ the scalar product in $L_2^*(0,1)$, and further assume assume that $\varepsilon(t), C(t)$ and $a(t)$ are bounded functions.

Since $\{\sin n\pi x\}$, $(n = 1, 2, \ldots)$, is a complete orthogonal system in $L_2^*(0,1)$, according to *iii*), it follows that

$$u = \sum_{n=1}^{\infty} u_n, \tag{2.1}$$

with

$$u_n = X_n(t)\sin n\pi x, \tag{2.2}$$

implies

$$u_t = \sum_{n=1}^{\infty} v_n(t), \tag{2.3}$$

with

$$\begin{cases} v_n(t) = \displaystyle\sum_{n=1}^{\infty} Y_n(t)\sin n\pi x, \\ Y_n(t) = \dfrac{dX_n}{dt}. \end{cases} \tag{2.4}$$

Setting

$$u_t = v, \qquad \gamma_{12} = C(t), \qquad \gamma_{22} = \varepsilon, \tag{2.5}$$

(1.1)-(1.3) is reduced to the binary reaction-diffusion of P.D.E. with self and cross diffusion given by [18]

$$\begin{cases} u_t = v, \\ v_t = -av + \gamma_{22}v_{xx} + \gamma_{12}u_{xx} + F(u) \end{cases}, \quad x \in \,]0,1[, \tag{2.6}$$

under the I.B.C.

$$\begin{cases} u(x,0) = u_0(x), \;\; v(x,0) = v_1(x), \\ u = v = 0, \qquad x = 0, 1. \end{cases} \tag{2.7}$$

Setting

$$v = \mu v^*, \tag{2.8}$$

with $\mu$ positive constant scaling to be chosen suitably later, and omitting the star, it follows that

$$\text{(2.9)} \qquad \begin{cases} u_t = \mu v, \\ v_t = -av + \gamma_{22} v_{xx} + \gamma_{12}\mu^{-1} u_{xx} + \mu^{-1} F(u), \end{cases}$$

under the I.B.C. (1.3).

We end this section by recalling the basic conditions guaranteeing stability-instability for nonautonomous systems [19].

i) *Stability*. As concerns the stability, the *main stability theorems* of the Direct Method for *nonautonomous* systems guarantee that: *the existence of a positive definite function W i.e.*

$$\text{(2.10)} \qquad W \geq m(\|u\|^2 + \|v\|^2), \quad m = \text{positive constant},$$

*implies*

- *stability if the temporal derivative along the solutions is semidefinite negative* {*i.e.* $\dot{W} \leq 0$ };
- *asymptotic stability if admits an upper bound which is infinitely small at the origin* {*i.e.* $W \leq m_1(\|u\|^2 + \|v\|^2)$, $m_1$ *positive constant*}, *and its temporal derivative along the solutions is negative definite* {*i.e.* $\dot{W} < 0$ *for* $\|u\|^2 + \|v\|^2 \neq 0$ }.

ii) *Instability*. As concerns the instability, the Cetaev instability theorem guarantees that: *If exists a function W*

- *taking positive values in any disk centered at* $(u = v = 0)$,
- *(for all* $t \geq t_0 > 0$ *in which W is bounded),* $\dot{W}$ *is positive definite then the null solution is unstable.*

## 3. – Ultimately boundedness in the autonomous case

Setting

$$\text{(3.1)} \qquad G_1 = \gamma_{22}(v_{xx} + \pi^2 v), \qquad G_2 = \gamma_{12}\mu(u_{xx} + \pi^2 u^2),$$

(2.9) becomes

$$\text{(3.2)} \qquad \begin{cases} u_t = \mu v, \\ v_t = \mu^{-1} A u + I v + G_1 + G_2 + \mu^{-1} F(u), \end{cases}$$

with

$$\text{(3.3)} \qquad A = \gamma_{12}\pi^2 = \pi^2 C, \qquad I = -(a + \gamma_{22}\pi^2) = -(a + \varepsilon\pi^2).$$

Introducing the positive definite functional

$$V = \frac{1}{2}\int_0^1 \Big[A(u^2+v^2)+\mu^{-2}A^2v^2+(\mu v - Iu)^2\Big]\,dx, \tag{3.4}$$

the temporal derivative of $V$ along the solutions is easily find to be [17], [21], [22]

$$\dot V = AI\int_0^1 (u^2+v^2)\,dx + \Psi_1 + \Psi, \tag{3.5}$$

with

$$\begin{cases} \Psi_1 = \langle A_2 v - A_3 u, G_1 + G_2\rangle,\ \ \Psi = \dfrac{1}{\mu}\langle A_2 v - A_3 u, F(u)\rangle, \\ A_2 = A+\mu^2,\ \ A_3 = \mu I. \end{cases} \tag{3.6}$$

LEMMA 3.1. – *Let*

$$|I|\gamma_{22} > \gamma_{12}, \Leftrightarrow C < \varepsilon(a+\varepsilon\pi^2). \tag{3.7}$$

*Then*

$$\mu = \mu_* = \frac{A\gamma_{12}}{|I|\gamma_{22}-\gamma_{12}} = \frac{\pi^2 C}{\varepsilon(a+\varepsilon\pi^2)-C}, \tag{3.8}$$

*implies*

$$\Psi_1 \le 0, \qquad \forall t \ge 0. \tag{3.9}$$

PROOF. – In view of the Poincaré inequality holding in $L_2^*(0,1)$,

$$\|\nabla\varphi\|^2 \ge \pi^2\|\varphi\|^2, \quad \forall\varphi\in L_2^*(0.1), \tag{3.10}$$

one obtains

$$\begin{cases} \langle A_2 v - A_3 u, G_1\rangle = \langle A_2 v - A_3 u, \gamma_{22}(v_{xx}+\pi^2 v)\rangle = \\ (A+\mu^2)\gamma_{22}(-\|v_x\|^2+\pi^2\|v\|^2) + \mu|I|\gamma_{22}(-\langle u_x, v_x\rangle + \pi^2\langle u, v\rangle), \\ \langle A_2 v - A_3 u, G_2\rangle = \langle A_2 v + \mu|I|u, \gamma_{12}\mu^{-1}(u_{xx}+\pi^2 u)\rangle = \\ (A+\mu^2)\gamma_{12}\mu^{-1}(-\langle u_x, v_x\rangle + \langle u, v\rangle) + \gamma_{12}|I|(-\|u_x\|^2+\pi^2\|u\|^2), \end{cases} \tag{3.11}$$

and hence

$$\Psi_1 = \begin{cases} -\Big\{\big[A+\mu^2\big]\gamma_{22}\|v_x\|^2 + \big[\mu|I|\gamma_{22}+(A+\mu^2)\gamma_{12}\mu^{-1}\big]\langle u_x, v_x\rangle + \gamma_{12}|I|\|u_x\|^2\Big\} \\ + \\ \pi^2\Big\{\big[A+\mu^2\big]\gamma_{22}\|v\|^2 + \big[\mu|I|\gamma_{22}+(A+\mu^2)\gamma_{12}\mu^{-1}\big]\langle u, v\rangle + \gamma_{12}|I|\|u\|^2\Big\} \end{cases} \tag{3.12}$$

Requiring to $\mu$ to verify the equation

$$\mu|I|\gamma_{22} + (A+\mu^2)\gamma_{12}\mu^{-1} = 2\sqrt{(A+\mu^2)\gamma_{22}\gamma_{12}|I|}, \tag{3.13}$$

i.e.

$$\left(\sqrt{\mu|I|\gamma_{22}} - \sqrt{(A+\mu^2)\gamma_{12}\mu^{-1}}\right)^2 = 0, \tag{3.14}$$

one obtains that one has to require

$$(|I|\gamma_{22} - \gamma_{12})\mu^2 = A\gamma_{12}. \tag{3.15}$$

Therefore, when (3.7) holds, for $\mu = \mu_*$, one obtains

$$\Psi_1 = -\|\nabla\varphi_1\|^2 + \pi^2\|\varphi_1\|^2, \tag{3.16}$$

with

$$\varphi_1 = \sqrt{\gamma_{12}|I|}u + \sqrt{(A+\mu^2)\gamma_{22}}v, \tag{3.17}$$

and (3.9) is immediately implied by (3.10). We can now show that (3.7) and

$$I<0, \quad |F(u)|<\tilde{m} = \text{positive constant}, \tag{3.18}$$

guarantee the ultimately boundedness in the autonomous case. In fact, in view of (3.5) and (3.9), it follows that

$$\dot{V} \le -A|I|\int_0^1 (u^2+v^2)\,dx + \frac{A+\mu_*^2}{\mu_*^2}|\langle v, F(u)\rangle| + |I|\cdot|\langle u, F(u)\rangle|. \tag{3.19}$$

On the other hand

$$\begin{cases} |\langle v, F(u)\rangle| \le \langle |v|, m\rangle \le \dfrac{\eta}{2}\|v\|^2 + \dfrac{\tilde{m}^2}{2\eta}, \\ |\langle u, F(u)\rangle| \le \dfrac{\eta}{2}\|u\|^2 + \dfrac{\tilde{m}^2}{2\eta}, \end{cases} \tag{3.20}$$

with $0<\eta<1$, hence (3.19) reduces to

$$\dot{V} \le -A|I|\int_0^1 (u^2+v^2)\,dx + \eta\alpha(\|u\|^2 + \|v\|^2) + \frac{\alpha}{\eta}\tilde{m}^2, \tag{3.21}$$

with

$$\alpha = \frac{1}{2}\left(\frac{A+\mu_*^2}{\mu_*^2}|I|\right). \tag{3.22}$$

Choosing

$$\eta = \bar{\eta} = \frac{1}{2}\frac{A|I|}{\alpha}, \tag{3.23}$$

it follows that

$$\dot{V} \le -\frac{A|I|}{2}\int_0^1 (u^2 + v^2)\,dx + \gamma, \tag{3.24}$$

with

$$\gamma = \frac{2\alpha^2}{A|I|}\tilde{m}^2, \tag{3.25}$$

and in view of

$$V \ge \frac{A}{2}\int_0^2 (u^2 + v^2)\,dx, \tag{3.26}$$

one obtains

$$\dot{V} \le -|I|V + \gamma. \tag{3.27}$$

THEOREM 3.1. – *Let* (3.7) *and* (3.8) *hold. Then (in the autonomous case), the set* $S_\sigma$ *of the phase space* $(u, v)$ *such that*

$$S_\sigma : \left\{ u, v \in S_\sigma \Rightarrow V < (1+\sigma)\frac{\gamma}{|I|} \right\}, \tag{3.28}$$

*with* $\sigma > 0$, *is an absorbing set.*

PROOF. – The proof is easily reached by following the standard procedure associate to (3.27) { cfr. [23], p. 259 }. For the sake of completeness a sketch of the proof is given in the Appendix.

REMARK 3.1. – The assumption $|F(u)| \in L(\infty)$ reflects essentially the case (1.6). Obviously the ultimately boundedness can be obtained under weaker assumptions on $F(u)$. In fact also

$$\frac{A + \mu_*^2}{\mu_*^2}|\langle v, F(u)\rangle| + |I| \cdot |\langle \boldsymbol{u}, \boldsymbol{F}(\boldsymbol{u})\rangle| \le h(\|u\|^2 + \|v\|^2) + h_1, \tag{3.29}$$

with $0 < h < |I|A$, $h_1 = \text{const.} > 0$, implies the ultimately boundedness.

This is the case, for instance, of $F(u)$ sublinear. In fact, let $F(u) < r|u|^s$, with $r$ and $s$ positive constants with $s<1$. Then

$$\langle v, F(u)\rangle \le \frac{\eta}{2}\|v\|^2 + \frac{r^2}{2\eta}\|u^s\|, \quad \eta = \text{const.} > 0. \tag{3.30}$$

On the other hand – via the Young inequality – one obtains

$$|u|^{2s} \le \frac{\eta_1^p |u|^{2sp}}{p} + \frac{1}{\eta_1^q q} \qquad 0<\eta_1 = \text{const}, \tag{3.31}$$

with $p, q \ge 1$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Therefore, choosing $p = \dfrac{1}{s}(>1)$ and $q = \dfrac{1}{1-s}(>1)$, one obtains

$$\begin{cases} \langle v, F(u)\rangle \le \dfrac{\eta}{2}\|v\|^2 + \eta^{\frac{1}{1-s}}\dfrac{r^2 s}{2\eta}\|u\|^2 + \dfrac{r^2(1-s)}{2\eta_1^{\frac{1}{1-s}}\eta}, \\ \langle u, F(u)\rangle \le \dfrac{\eta}{2}\|u\|^2 + \eta_1^{\frac{1}{s}}\dfrac{r^2 s}{2\eta}\|u\|^2 + \dfrac{r^2(1-s)}{2\eta_1^{\frac{1}{1-s}}\eta}. \end{cases} \tag{3.32}$$

Choosing

$$\eta_1 = \bar{\eta}_1 = \frac{1}{(r^2 s)^s}, \tag{3.33}$$

one obtains (3.20) with

$$\tilde{m}^2 = \frac{r^2(1-s)}{\eta_1^{\frac{1}{1-s}}}. \tag{3.34}$$

## 4. – Linear instability in the autonomous case

In order to know which are the *best stability conditions in the autonomous case*, it is useful to obtain the conditions necessary and sufficient for the linear instability.
In view of (2.1)-(2.4) and

$$\Delta u_n = -n^2\pi^2 u_n, \quad \Delta v_n = -n^2\pi^2 v_n,$$

disregarding $F(u)$, (3.2) imply

$$\begin{cases} \dfrac{dX_n}{dt} = 0 \cdot X_n + \mu Y_n, \\ \dfrac{dY_n}{dt} = -\mu^{-1}\pi^2 n^2 \gamma_{12} X_n - (a + \pi^2 n^2 \gamma_{22}) Y_n. \end{cases} \tag{4.1}$$

The eigenvalues of the matrix

$$\begin{pmatrix} 0 & \mu \\ -\mu^{-1}\pi^2 n^2 \gamma_{12} & -(a+\pi^2 n^2 \gamma_{22}) \end{pmatrix} \tag{4.2}$$

have negative real parts if and only if the (Routh-Hurwitz) conditions

$$A_n = \pi^2 n^2 \gamma_{12} > 0, \qquad I_n = -(a+\pi^2 n^2 \gamma_{22}) < 0, \quad \forall n \in \mathbb{N},$$

hold. In view of ($\varepsilon \geq 0, C > 0$), the following theorem holds.

THEOREM 4.1. – *The null solution of* (1.1)-(1.3) *is linearly asymptotically stable if and only if*

$$a + \varepsilon\pi^2 > 0. \tag{4.3}$$

## 5. – Nonlinear stability in the autonomous case

Returning to the nonlinear equations (3.2), and introducing the functional

$$W = V + k\int_0^1 e^{-\varphi(u)}\, dx, \tag{5.1}$$

with $V$ given by (3.4), $k$ being a positive constant (to be chosen later) and $\varphi$ given by (1.6)$_3$ i.e. $\varphi = \int_0^u F(z)\, dz$, the temporal derivative of $W$ along the solutions of (3.2), in view of

$$\frac{d}{dt}\int_0^1 e^{-\varphi(u)}\, dx = -\int_0^1 e^{-\varphi(u)}\varphi' u_t\, dx = -\int_0^1 e^{-\varphi(u)} vF(u)\, dx, \tag{5.2}$$

is given by

$$\dot{W} = AI\int_0^1 (u^2+v^2)\, dx + \Psi_1 + \Psi - k\int_0^1 e^{-\varphi(u)} vF(u)\, dx, \tag{5.3}$$

with $\Psi_1$ and $\Psi$ given by (3.6).

THEOREM 5.1. – *Let*

$$a + \varepsilon\pi^2 > 0, \;\; C < \varepsilon(a+\varepsilon\pi^2), \tag{5.4}$$

*together with either*

$$uF(u) \leq 0, \tag{5.5}$$

*or*

$$\varphi(u) \leq 0, \ uF(u) < \bar{A}u^2, \bar{A} < A, \tag{5.6}$$

*hold. Then (in the autonomous case) the null solution is globally asymptotically stable.*

PROOF. – By virtue of Lemma 3.1, choosing $\mu = \mu_*$ (5.3) reduces to

$$\dot{W} = -AI\int_0^1 (u^2 + v^2)\, dx + \Psi - k\int_0^1 e^{-\varphi(u)} vF(u)\, dx, \tag{5.7}$$

with

$$\Psi = \frac{A + \mu_*^2}{\mu_*}\langle v, F(u)\rangle + |I|\langle u, F(u)\rangle. \tag{5.8}$$

Both in the cases (5.5)-(5.6), choosing $k = \dfrac{A + \mu_*^2}{\mu_*}$, it follows that (5.7) reduces to

$$\dot{W} = -A|I|\int_0^1 (u^2 + v^2)\, dx + |I|\langle u, F(u)\rangle. \tag{5.9}$$

and hence

$$\langle u, F(u)\rangle \leq 0 \Rightarrow \dot{W} < -A|I|\int_0^1 (u^2 + v^2)\, dx, \tag{5.10}$$

while

$$\langle u, F(u)\rangle \leq \bar{A}u^2 \Rightarrow \dot{W} < -(A - \bar{A})|I|\int_0^1 (u^2 + v^2)\, dx\langle 0, \tag{5.11}$$

which proves the theorem in both the cases (5.5)-(5.6).

## 6. – Ultimately boundedness in the nonautonomous case

In the nonautonomous case $A$ and $I$ depend on $t$, hence the temporal derivative of $V$ along the solutions, in view of (3.4)-(3.5), is given by

$$\dot{V} = AI\int_0^1 (u^2 + v^2)\, dx + \Psi_1 + \Psi + \frac{\partial V}{\partial t}, \tag{6.1}$$

with

$$\frac{\partial V}{\partial t} = \frac{1}{2}\int_0^1 \left[\dot{A}(u^2+v^2) + \mu^{-2}\frac{dA^2}{dt}v^2 - 2(\mu v - Iu)\frac{dI}{dt}u\right]dx, \tag{6.2}$$

and hence

$$2\dot{V} = \int_0^1 \big(P(t)u^2 + Q(t)v^2 - 2R(t)uv\big)\,dx + 2(\Psi_1 + \Psi), \tag{6.3}$$

with

$$P(t) = \dot{A} + 2AI + \frac{dI^2}{dt},\ Q(t) = \dot{A} + 2AI + \mu^{-2}\frac{dA^2}{dt},\ R(t) = \mu\frac{dI}{dt}. \tag{6.4}$$

LEMMA 6.1. – *Let*

$$\varepsilon = \frac{(\pi^2 C + \delta^2)C}{(a + \varepsilon\pi^2)\delta^2} \Leftrightarrow \gamma_{22} = \frac{(A+\delta^2)}{|I|\delta^2}\gamma_{12}, \quad \forall t \geq 0, \tag{6.5}$$

*with $\delta$ positive constant. Then, in the nonautonomous case, choosing $\mu = \delta$ it follows that*

$$\Psi_1 \leq 0, \quad \forall t \geq 0. \tag{6.6}$$

PROOF. – In fact (3.12) continues to hold also in the nonautonomous case. Then it is enough to verify that (6.5) and (3.15) coincide, $\forall t \geq 0$, for $\mu = \delta$.

For any function $f : \mathbb{R}^+ \to \mathbb{R}$, we set

$$f_* = \inf_{\mathbb{R}^+} f, \quad f^* = \sup_{\mathbb{R}^+} f. \tag{6.7}$$

LEMMA 6.2. – *Let* (6.5) *hold. Then, in the nonautonomous case, the functional*

$$V = \frac{1}{2}\int_0^1 \left[A(u^2+v^2) + \frac{A^2}{\delta^2}v^2 + (\delta v - Iu)^2\right]dx, \tag{6.8}$$

*has the following properties*

i) *at any instant $\bar{t} \in \mathbb{R}^+$, in any disk centered at ($u = v = 0$) of the phase space $(u, v)$, exists a domain that verifies the inequality*

$$V(\bar{t}, u, v) > 0, \tag{6.9}$$

ii) *$V$ is positive definite and admits an upper bound which is infinitely small at the origin*

iii) *the temporal derivative of V along the solutions, is given by*

$$2\dot{V} = \int_0^1 \mathcal{P}\,dx + 2\Psi, \tag{6.10}$$

*with*

$$\mathcal{P} = P(t)u^2 + Q(t)v^2 - 2R(t)uv. \tag{6.11}$$

PROOF. – As concerns *i)-ii)* it is enough to remark that, since

$$A_* \geq \bar{C}\pi^2 > 0, \tag{6.12}$$

it immediately follows that

$$V \geq \frac{1}{2}\pi^2\bar{C}(u^2 + v^2). \tag{6.13}$$

Hence $V$ is positive definite. Moreover, since $\varepsilon, a, C$ are bounded, in view of (10.5), it follows that

$$V \leq M(u^2 + v^2), \tag{6.14}$$

with

$$M = \frac{1}{2}\left[A^* + \left(\frac{A^*}{\delta}\right)^2 + 2(\delta^2 + |I|^{2*})\right], \tag{6.15}$$

and hence $V$ admits an upper bound which is infinitely small at the origin. Finally iii) is immediately implied by (6.3) and Lemma 6.1.

We can show now that

$$I^* < 0, \qquad |F(u)| < \tilde{m}, \tag{6.16}$$

together with

$$\begin{cases} P^* < -2h_1, \;\; Q^* \langle -2h_2, \;\; (|R|^*)^2 < 2dh_1h_2, \\ 0 < d < h_i, \quad h_i = \text{ const. } > 0, \;\; (i = 1, 2), \end{cases} \tag{6.17}$$

implies

$$2\dot{V} \leq \int_0^1 (-h_1u^2 - h_2v^2 + 2d\sqrt{h_1h_1}|uv|)\,dx + 2\Psi, \tag{6.18}$$

and hence

$$(6.19)\qquad \dot V \le -\frac{1}{2}\int_0^1 \left[(h_1-d)u^2+(h_2-d)v^2\right]dx + \frac{A^*+\delta^2}{\delta^2}|\langle v, F(u)\rangle| + |I^*|\cdot\langle u, F(u)\rangle.$$

Setting

$$(6.20)\qquad M = \frac{1}{2}\inf\,(h_1-d, h_2-d),$$

and following the procedure used for obtaining (3.20), one obtains

$$(6.21)\qquad \dot V \le -M\int_0^1 (u^2+v^2)dx + \eta\bar\alpha\|v\|^2 + \bar\alpha\frac{\tilde m^2}{\eta},$$

with $\bar\alpha$ given by (3.22) with $A^*$, $|I^*|$ and $\delta$ at the place of $A$, $|I|$ and $\mu_*$ respectively. Therefore one obtains

$$(6.22)\qquad \dot V \le -\frac{M}{2}\int_0^1 (u^2+v^2)\,dx + \bar\gamma,$$

with

$$(6.23)\qquad \bar\gamma = \frac{\bar\alpha}{\bar\eta}\tilde m^2, \qquad \bar\eta = \frac{1}{2}\frac{M}{\bar\alpha}.$$

In view of

$$(6.24)\qquad V \ge \frac{1}{2}\pi^2\bar C\int_0^1 (u^2+v^2)\,dx,$$

it turns out that

$$(6.25)\qquad \dot V \le -\bar M V + \bar\gamma,$$

with

$$(6.26)\qquad \bar M = \frac{M}{\pi^2\bar C},$$

and hence the ultimately boundedness theorem.

THEOREM 6.1. – *Let* (6.5) *and* (6.16)-(6.17) *hold. Then (in the nonautonomous case), the set*

$$(6.27)\qquad \bar S_\sigma : \left\{(u,v)\in\bar S_\sigma \Rightarrow V < (1+\sigma)\frac{\bar\gamma}{\bar M}\right\},$$

*with* $\sigma > 0$, *is an absorbing set.*

## 7. – Linear stability in the nonautonomous case

Disregarding $\Psi$ and letting (6.5) holds, (6.10) reduces to

$$\dot{V} = \frac{1}{2}\int_0^1 \mathcal{P}\, dx, \tag{7.1}$$

and the following theorem immediately follows.

THEOREM 7.1. – *Let* (6.5) *holds. Then the null solution in the nonautonomous case is linearly:*

1) *stable, if $\mathcal{P}$ is semidefinite negative $\forall t \geq 0$;*
2) *asymptotically stable, if $\mathcal{P}$ is definite negative $\forall t \geq 0$;*
3) *unstable, if $\mathcal{P}$ is definite positive $\forall t \geq 0$;*

*In particular, denoting by $h_1$ and $h_2$ two positive constants,* (6.17) *imply asymptotic stability, while*

$$P_* > h_1, \quad Q_* > h_2, \quad (|R|_*)^2 > 2d^* h_1 h_2, \tag{7.2}$$

*imply instability.*

## 8. – Global nonlinear stability in the nonautonomous case

THEOREM 8.1. – *Let the assumptions of linear stability of the nonautonomous case, together with* (5.5) *or* (5.6), *hold. Then the null solution – in the nonautonomous case – is globally asymptotically stable.*

PROOF. – In fact the temporal derivative of

$$\begin{cases} W = V + k\displaystyle\int_0^1 e^{-\varphi(u)}\, dx, \\ \varphi(u) = \displaystyle\int_0^u F(z)\, dz, \end{cases} \tag{8.1}$$

is given by

$$\dot{W} = \frac{1}{2}\int_0^1 \mathcal{P}\, dx + \Psi, \tag{8.2}$$

with $\Psi$ given by (5.8) and hence $\Psi \leq 0$, by virtue of (5.5) or (5.6)

REMARK 8.1. – One easily verifies that, choosing:

i)

$$\delta = \pi, \ \ \varepsilon = \frac{C^2}{\pi^2}, \ \ a = \frac{\pi^2}{C}, \ \ C = C_0 + C_1 e^{-ht},$$

with

$$C_1, C_0, h \text{ suitable positive constants,}$$

all the assumptions of the stability-instability theorems can be verified.

ii)

$$\delta = 1 = a, \ \ \varepsilon = C = 1 + e^{-t},$$

it follows that all the assumptions of global stability are verified with

$$Pu^2 + Qv^2 - 2Ruv \leq -\pi^2(u^2 + v^2) - \pi^2(u + v)^2 < -\pi^2(u^2 + v^2).$$

## 9. – Final Remarks

i) *The paper is concerned with the longtime behaviour of the solutions to* (1.1)-(1.3), *either in the autonomous case or in the nonautonomous case.*

ii) *Conditions guaranteeing the ultimately boundedness of the solutions are found.*

iii) *The asymptotic nonlinear global stability of the null solution is studied.*

iv) *The procedures used continue to hold also when, instead of* (1.1), *one requires*

$$u(0, t) = 0, \qquad u_x(1, t) = 0, \qquad \forall t \geq 0, \tag{9.1}$$

*or*

$$u_x(0, t) = 0, \qquad u(1, t) = 0, \qquad \forall t \geq 0. \tag{9.2}$$

*In fact, in the case* (9.1), *for instance, at the place of* $\{\sin n\pi x\}$, *one has to substitute the sequence* $\left\{\sin\left(n - \frac{1}{2}\pi x\right)\right\}$ *and* $\frac{\pi^2}{4}$ *at the place of* $\pi^2$ *in* (3.10).

## 10. – Appendix

PROOF OF THEOREM 3.1. – i) $S_\sigma$ is an invariant set. In fact, let

$$V(t_0) < \left(1 + \frac{\sigma}{2^n}\right)\frac{\gamma}{|I|}, \qquad \text{for an } n \in \mathbb{N}. \tag{10.1}$$

A trajectory starting in $S_\sigma$ can leave $S_\sigma$ only if exists an instant $t_*$ such that

$$\begin{cases} V(t_*) = \left(1 + \dfrac{\sigma}{2^n}\right) \dfrac{\gamma}{|I|}, \\ \left(\dfrac{dV}{dt}\right)_{t=t_*} \geq 0. \end{cases} \tag{10.2}$$

But

$$\left(\frac{dV}{dt}\right)_{t=t_*} = -|I|V(t_*) + \gamma = -(1+\sigma)\gamma + \gamma = -\sigma\gamma, \tag{10.3}$$

i.e. $S_\sigma$ is invariant.

ii) $S_\sigma$ is an attractor. In fact let $\Sigma$ be a *bounded* set of the phase space and let

$$M = \sup_{\Sigma} V. \tag{10.4}$$

Since (3.25) implies

$$V(t) \leq V(0)e^{-|I|t} + \frac{\gamma}{|I|}, \tag{10.5}$$

$V(0) \in \Sigma$ implies

$$V(t) \leq Me^{-|I|t} + \frac{\gamma}{|I|}. \tag{10.6}$$

Requiring

$$Me^{-|I|t} + \frac{\gamma}{|I|} = (1+\sigma)\frac{\gamma}{|I|},$$

i.e.

$$\frac{M|I|}{\sigma\gamma} = e^{|I|t}, \tag{10.7}$$

it follows that for $t = \bar{t}$

$$\bar{t} = \frac{1}{|I|} \log \frac{|I|M}{\sigma m_1}, \tag{10.8}$$

any trajectory beginning in $\Sigma$ has reached $S_\sigma$ and hence will remains there $\forall t \geq \bar{t}$.

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