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Fractional Interior Differentiability of the Stress Velocities to Elastic Plastic Problems with Hardening

JENS FREHSE - MARIA SPECOVIVS-NEUGEBAUER

In memoriam Enrico Magenes

Abstract. – *We consider classical variational inequalities modeling elastic plastic problems with kinematic and isotropic hardening. It is shown that the stress velocities have fractional derivatives of order $1/2 - \delta$ in L^2 in time direction on the whole existence interval. In space direction an analogous result holds in the interior of the domain. In the case of kinematic hardening, these results are also true for the strain velocity.*

1. – Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial\Omega$. The domain Ω represents a solid body which undergoes an elastic plastic deformation, hence the case $n = 3$ is the natural application, however, the study of arbitrary dimensions $n \geq 2$ gives additional mathematical insight. Our aim is to prove a new regularity result for a classical variational inequality which models elastic plastic deformation with isotropic and kinematic hardening.

Let us first fix some general notation. For $n \times n$ -matrices τ, σ , the scalar product $\sigma : \tau$, Euclidean norm $|\sigma|$, the trace $\text{tr } \sigma$ and the deviator σ_D are given by:

$$\sigma : \tau = \sum_{i,k=1}^n \sigma_{ik} \tau_{ik}, \quad |\sigma| = (\sigma : \sigma)^{1/2}, \quad \text{tr } \sigma = \sum_i \sigma_{ii}, \quad \sigma_D = \sigma - \frac{\text{tr } \sigma}{n} \mathbb{I},$$

where \mathbb{I} is the unit matrix. By $\mathbb{R}_{\text{sym}}^{n \times n}$, we denote the set of all symmetric $n \times n$ matrices $\tau \in \mathbb{R}^{n \times n}$.

Under the influence of a volume force with density $f(t, x)$ and an external loading $p = p(t, x)$ there appear stresses $\sigma(t, x) \in \mathbb{R}_{\text{sym}}^{n \times n}$, where $x \in \Omega$ or $x \in \partial\Omega$. The parameter $t \in [0, T]$ is a so called loading parameter, but with some abuse of notation is often referred to as ‘time’-variable. Assuming that the body is clamped in a region $\Gamma \subset \partial\Omega$, the balance of forces implies

$$(1.1) \quad -\text{div } \sigma = f \quad \text{in } \Omega, \quad v \cdot \sigma = p \quad \text{on } \partial\Omega \setminus \Gamma,$$

where $\nu = \nu(x)$ is the outer unit normal vector at $x \in \partial\Omega$. For the formulation of the classical hardening problem we need a set of hardening variables $\xi = \xi(t, x)$, which is but a scalar function for isotropic hardening, or a symmetric tensor function -the so-called back stress- in the case of kinematic hardening. In addition we have a yield condition specified to

$$(1.2) \quad |\sigma_D| - \xi \leq \kappa \quad (\text{'isotropic hardening'}) \text{ or}$$

$$(1.3) \quad |\sigma_D - \xi_D| \leq \kappa \quad (\text{'kinematic hardening'}),$$

respectively, where $\kappa > 0$ is a given constant.

In general the yield condition is formulated as

$$F(\sigma; \xi) \leq 0$$

with a convex function F , but we confine us to the so called von-Mises-yield-condition (1.2) or (1.3). The problem is completed with constitutive laws, namely a stress-strain relation and a hardening law which describes the evolution of ξ , they are indicated in the formulae (1.17) and (1.18) below.

The proper mathematical formulation as a variational inequality involves some well known function spaces: The symbol $L^q(\Omega)$, with $1 \leq q \leq \infty$, denotes the usual Lebesgue-space, where we do not distinguish between scalar-, vector-, or tensor-valued functions as long as no confusion arises. In all cases we indicate the $L^2(\Omega)$ -scalar product with brackets $(\cdot, \cdot)_\Omega$. For $T > 0$ and a given Banach space X (which is always a function space in the sequel), the symbol $L^q(0, T; X)$ stands for the Bochner space of measurable and q -summable functions defined on the interval $[0, T]$ with values in X . For $X = L^q(\Omega)$, we frequently shorten the notation to $L^p(L^q)$, if no confusion arises. The space of functions in $L^2(\Omega)$ with derivatives up to order $m \in \mathbb{N}$ is denoted by $H^m(\Omega)$, furthermore

$$H^1_\Gamma(\Omega) = \{\varphi \in H^1(\Omega, \mathbb{R}^n) \mid \varphi|_\Gamma = 0\}$$

the boundary condition has to be understood in the sense of traces, of course. The part $\Gamma \subset \partial\Omega$ is either void or a relatively open subset.

For convenience, we assume that the volume force density f and the external loading p fulfil the following regularity assumptions:

$$(1.4) \quad f, \dot{f} \in L^\infty(0, T; L^\infty(\Omega)), \ddot{f} \in L^1(0, T; L^2),$$

$$(1.5) \quad p, \dot{p} \in L^\infty(0, T; L^\infty(\partial\Omega)), \ddot{p} \in L^1(0, T; L^2(\partial\Omega)),$$

here the dot indicates the derivative with respect to the 'time' variable t .

DEFINITION 1.1 (Admissible stresses and hardening variables). – $\mathbb{K}(t)$ is the set of all pairs (τ, η) with the following properties:

$$(1.6) \quad \tau \in L^2(\Omega; \mathbb{R}_{\text{sym}}^{n \times n}), \eta \in L^2(\Omega, \mathbb{R}^m)$$

where τ fulfills the balance of forces in the weak form:

$$(1.7) \quad (\tau, \nabla \varphi)_\Omega = (f, \varphi)_\Omega + \int_{\partial\Omega} p\varphi \, d\sigma \text{ for all } \varphi \in H^1_\Gamma(\Omega).$$

In the case of isotropic hardening we have $m = 1$ and:

$$(1.8) \quad \eta \in L^2(\Omega; \mathbb{R}), \quad |\tau_D| - \eta \leq \kappa,$$

in the case of kinematic hardening we have $m = n(n + 1)/2$ ⁽¹⁾ and:

$$(1.9) \quad \eta \in L^2(\Omega; \mathbb{R}^{n \times n}_{\text{sym}}), \quad |\tau_D - \eta_D| \leq \kappa.$$

We assume that the hardening variables start at zero, that is $\zeta(0) = 0$, while for the initial value σ_0 of the stresses and the pair $(\sigma_0, 0)$ we require

$$(1.10) \quad \sigma_0 \in H^2(\Omega), \quad (\sigma_0, 0) \in \mathbb{K}(0).$$

Finally, we need the so called compliance tensor or inverse elasticity tensor $A = (a_{ik}^{vu})$, a given symmetric tensor of rank four, and the hardening tensor $H \in \mathbb{R}^{m \times m}$. We assume that A and H satisfy the usual ellipticity condition

$$(1.11) \quad \tau : A\tau \geq c_0|\tau|^2, \quad H\eta \cdot \eta \geq c_1|\eta|^2$$

for all $\tau \in \mathbb{R}^{n \times n}_{\text{sym}}$, $\eta \in \mathbb{R}^m$, respectively, with constants $c_0, c_1 > 0$. In order to limit the technical details we formally consider only the case of constant A and H here, however, all the results remain true if the entries of A and H are Lipschitz continuous functions on Ω , since this generalization will cause only pollution terms in the proofs. The tensors A and H have to fulfill the condition (1.11) uniformly in x in this case. Note that in the case of isotropic hardening the term H is just a scalar function with $H(x) \geq c_1 > 0$ uniformly on Ω .

With this notation the classical variational inequality for isotropic or, respectively, kinematic hardening is the following

HARDENING PROBLEM. – Let σ_0 be a given initial stress, such that $(\sigma_0, 0) \in \mathbb{K}(0)$. Find $\sigma \in L^\infty(L^2)$, $\zeta \in L^\infty(L^2)$ such that

$$(1.12) \quad \dot{\sigma} \in L^2(L^2), \quad \dot{\zeta} \in L^2(L^2)$$

$$(1.13) \quad (\sigma(t), \zeta(t)) := (\sigma(t, \cdot), \zeta(t, \cdot)) \in \mathbb{K}(t), \quad t \in [0, T]$$

$$(1.14) \quad \sigma(0) = \sigma_0, \quad \zeta(0) = 0$$

$$(1.15) \quad (A\dot{\sigma}, \sigma - \tau) + (H\dot{\zeta}, \zeta - \eta) \leq 0 \quad \text{a.e. in } [0, T] \text{ for all } (\tau, \eta) \in \mathbb{K}(t).$$

⁽¹⁾ In order to have a unified notation also in the calculations needed lateron we identify ζ with a vector of $\mathbb{R}^{n(n+1)/2}$.

This problem has a unique solution [9]. The inequality (1.15) contains the constitutive law, under additional regularity conditions it is equivalent to an a.e. point-wise equation provided that the so called safe load condition holds:

DEFINITION 1.2 (Safe load condition). – *There exist $\hat{\sigma} \in L^\infty(L^2)$, $\hat{\xi} \in L^\infty(L^2)$ with*

$$(1.16) \quad \begin{aligned} \dot{\hat{\sigma}} &\in L^\infty(L^2), \quad \ddot{\hat{\sigma}} \in L^1(L^2), \quad \dot{\hat{\xi}} \in L^\infty(L^2) \\ (\hat{\sigma}(0), 0) &\in \mathbb{K}(0), \quad \hat{\xi}|_{t=0} = 0 \\ (\hat{\sigma}(t, \cdot), \hat{\xi}(t, \cdot)) &\in \mathbb{K}(t), \end{aligned}$$

and there exists a $\delta > 0$ such that

$$|\hat{\sigma}_D| - \xi \leq \kappa - \delta \text{ or } |\hat{\sigma}_D - \hat{\xi}_D| \leq \kappa - \delta, \text{ respectively.}$$

By a theorem of Johnson [9] it is known that there exists a displacement

$$u \in L^\infty(0, T; H_F^{1,2}(\Omega, \mathbb{R}^n)) \text{ with } \dot{u} \in L^\infty(0, T; H_F^{1,2}(\Omega, \mathbb{R}^n)),$$

furthermore, there exists a multiplier [3]

$$\dot{\lambda} \in L^\infty(0, T; L^2(\Omega, \mathbb{R}))$$

such that in the case of isotropic hardening, a.e. in $[0, T] \times \Omega$

$$(1.17) \quad \begin{cases} \frac{1}{2}(\nabla \dot{u} + \nabla \dot{u}^T) &= A\dot{\sigma} + \dot{\lambda}\sigma_D|\sigma_D|^{-1} \\ 0 &= H\dot{\xi} - \dot{\lambda} \end{cases}$$

and in the case of kinematic hardening

$$(1.18) \quad \begin{cases} \frac{1}{2}(\nabla \dot{u} + \nabla \dot{u}^T) &= A\dot{\sigma} + \dot{\lambda}(\sigma_D - \xi_D)|\sigma_D - \xi_D|^{-1} \\ 0 &= H\dot{\xi} - \dot{\lambda}(\sigma_D - \xi_D)|\sigma_D - \xi_D|^{-1}. \end{cases}$$

The multiplier function $\dot{\lambda}$ satisfies $\dot{\lambda} \geq 0$ a.e. and

$$\dot{\lambda}(|\sigma_D| - \kappa - \xi) = 0$$

in the case of isotropic hardening or

$$\dot{\lambda}(|\sigma_D - \xi_D| - \kappa) = 0,$$

respectively, in the case of kinematic hardening. This implies that $\dot{\lambda} = 0$ if $|\sigma_D| = 0$ in the case of isotropic hardening, and $\dot{\lambda} = 0$ if $|\sigma_D - \xi_D| = 0$ in the case of kinematic hardening, so (1.17) und (1.18) can be defined. Vice versa, from (1.17) and (1.18) one recovers the variational inequality (cf. [3] for a simple proof concerning the construction of λ). Equation (1.17) and (1.18) have the advantage,

that they are defined point-wise and that the deformation velocity \dot{u} appears explicitly.

Concerning interior regularity of the solution (σ, ξ) and the displacements the following results are known: If f is sufficiently regular, that is

$$(1.19) \quad f \in L^\infty(H^1_{loc}),$$

then in the case of kinematic hardening [13]:

$$\sigma \in L^\infty(0, T; H^1_{loc}), \quad \xi \in L^\infty(0, T; H^1_{loc}), \quad u \in L^\infty(0, T; H^2_{loc})$$

In the case of isotropic hardening the $L^\infty(H^1_{loc})$ -property is known only for σ and ξ cf. [13], while the $L^\infty(H^1_{loc})$ -property for ∇u is an interesting open problem. In [4] it was shown that $\nabla u \in L^\infty(L^6_{loc})$, if $n = 3$. Up to now there are no regularity results for $\dot{\sigma}, \dot{\xi}$, besides a result of [5], where the inclusion $\dot{\sigma}, \dot{\xi} \in L^\infty(L^{2+2\delta})$ for small $\delta > 0$ is proved. For more general models describing elastic-plastic deformation with hardening see also [1, 2, 7]. These papers use the so called primal formulation, i.e the principal unknowns are the displacements rather than the stresses. Our main result states that in $[0, T] \times \Omega_0, \Omega_0 \subset\subset \Omega$, the functions $\dot{\sigma}, \dot{\xi}$ have fractional derivatives in time *and* space direction of order $\frac{1}{2} - \delta, \delta > 0$. In the kinematic case we obtain this regularity property also for $\nabla \dot{u}$, in the isotropic case we reach only $\nabla \dot{u} \in L^{\frac{8}{3}-\delta_1}(L^{\frac{8}{3}-\delta_1}), n = 3$, for all $\delta_1 > 0$ (cf. Remark 4.4, [6]).

For the result in time direction, the method of the proof is related to a recent paper [6] of the authors about the Prandtl-Reuss problem, where fractional differentiability of order $\frac{1}{2} - \delta_1$ was achieved for the stress velocities. The ideas of this paper can be adapted to gain a similar result in the setting with hardening, as it is considered here, however, the method to achieve also the fractional differentiability of order $\frac{1}{2} - \delta_1$ with respect to space direction needs an additional consideration. This is the purpose of our paper.

A counterexample of D. Knees [10] indicates that our regularity result is optimal.

2. – The main results

We formulate the main results, starting with the regularity in time: The stress velocities $\dot{\sigma}$ and the time derivatives $\dot{\xi}$ of the hardening variables ξ have fractional derivatives of order $\frac{1}{2}$ in time direction, in a weak sense. This result holds up to the boundary $\partial\Omega$, for arbitrary dimension n , and both for kinematic and isotropic hardening. We recall the notation of difference operators: Let e_i

denote the i -th unit vector, for any $w = w(t, x)$, and $s > 0$ we put

$$(2.1) \quad \mathcal{A}_t^s w(t, x) = w(t + h, x) - w(t, x), \quad \mathcal{A}_i^s w(t, x) = w(t, x + se_i) - w(t, x).$$

THEOREM 2.1 (Regularity in time). – *Let the data f , p and σ_0 fulfill the regularity assumptions (1.4), (1.5) and (1.10), assume that the ellipticity condition (1.11) for the tensors A and H and further the safe load condition (cf. Def. 1.2) are satisfied. Then for the solution σ , ξ of the hardening problem introduced in Section 1, there holds the estimate*

$$(2.2) \quad h^{-2} \int_0^h \int_0^{T-h} \int_{\Omega} [|\mathcal{A}_t^s \dot{\sigma}|^2 + |\mathcal{A}_t^s \dot{\xi}|^2] dx dt ds \leq C$$

uniformly for $0 < h < h_0$.

REMARK 2.2. – The inequality (2.2) is a weak version of the Nikolskii - space property

$$\sup_{0 < h < h_0} \frac{1}{h} \int_0^{T-h} \int_{\Omega} [|\mathcal{A}_t^s \dot{\sigma}|^2 + |\mathcal{A}_t^s \dot{\xi}|^2] dx dt \leq C,$$

(which we do not prove), see the discussion in [6]. Theorem 2.1 implies that the Fourier- coefficients $c_m = c_m(x)$ of $\dot{\xi}$, $\dot{\sigma}$ in time-direction gain the following summability property (see [6, Lemma A.1] for the details)

$$\sum_{m=-\infty}^{\infty} m^{1-\delta} \int_{\Omega} |c_m(x)|^2 dy \leq C_{\delta} \quad \text{for all } \delta > 0.$$

In the case of kinematic hardening, Theorem 2.1 implies the fractional differentiability in time for the strain velocities:

THEOREM 2.3. – *Assume the requirements of Theorem 2.1 are met and the pair (σ, ξ) is the solution to the problem with kinematic hardening. Then the corresponding displacement field u satisfies the estimate*

$$(2.3) \quad h^{-2} \int_0^{T-h} \int_0^h \int_{\Omega} |\mathcal{A}_t^s \nabla u|^2 dx ds dt \leq C$$

with a constant independent of $0 < h \leq h_0$.

Using the result (2.2) for the time we can also prove that the velocities $\dot{\sigma}$, $\dot{\xi}$ of the stresses and the hardening parameters have fractional derivatives of order $\frac{1}{2}$ in space, here again for isotropic and kinematic hardening.

THEOREM 2.4 (Local regularity in space). – *Assuming the requirements of Theorem 2.1 (σ, ξ) be again the solution pair of the hardening problems formulated in Section 1. Then the velocities $\dot{\sigma}, \dot{\xi}$ have local fractional derivatives of order $1/2$ in space direction, in the following sense*

$$(2.4) \quad \sup_{0 < h \leq h_0} h^{-1} \int_0^{T-h} \int_{\Omega_0} |A_i^h \dot{\sigma}|^2 + |A_i^h \dot{\xi}|^2 dx dt \leq C, \quad i = 1, \dots, n$$

for any domain Ω_0 such that $\overline{\Omega_0} \subset \Omega$ and $h_0 \leq \text{dist}(\partial\Omega, \partial\Omega_0)$.

REMARK 2.5 (Possible generalizations). – Apart from passing to Lipschitz continuous entries in the tensors A and H - as already mentioned in Section 1 it is also possible to consider more general flow rules for the hardening variables ξ . For example, one may add in (1.17), (1.18) globally Lipschitz continuous functions $g(t, x, \sigma, \xi)$. Terms of this type create pollution terms which can be treated via Gronwall's inequality. However, we are mostly interested in the classical case as in Johnson [8], mainly because they present the essential mathematical difficulties.

3. – Penalty approximation

The hardening problem (1.12) can be approximated in several ways via penalty approximations. In this paper we follow the approach in [3, 4, 5]. We introduce the penalty potentials

$$G_\mu^{\text{iso}}(\sigma, \xi) = \frac{1}{2} \mu^{-1} [|\sigma_D| - (\kappa + \xi)]_+^2$$

in the case of isotropic hardening, and

$$G_\mu^{\text{kin}}(\sigma, \xi) = \frac{1}{2} \mu^{-1} [|\sigma_D - \xi_D| - \kappa]_+^2$$

in the case of kinematic hardening, where for any real valued function ϕ the expression $[\phi]_+ = \max(\phi, 0)$ is the positive part, and $\mu > 0$ a small parameter. Then we obtain

$$G_{1\mu}^{\text{iso}}(\sigma, \xi) := \frac{\partial}{\partial \sigma} G_\mu^{\text{iso}}(\sigma, \xi) = \mu^{-1} [|\sigma_D| - (\kappa + \xi)]_+ \sigma_D |\sigma_D|^{-1}$$

$$G_{2\mu}^{\text{iso}}(\sigma, \xi) := \frac{\partial}{\partial \xi} G_\mu^{\text{iso}}(\sigma, \xi) = -\mu^{-1} [|\sigma_D| - (\kappa + \xi)]_+,$$

and in particular the (point-wise) relation

$$(3.1) \quad |G_{1\mu}^{\text{iso}}| = |G_{2\mu}^{\text{iso}}|.$$

For kinematic hardening we have

$$G_{1\mu}^{\text{kin}}(\sigma, \xi) := \frac{\partial}{\partial \sigma} G_\mu(\sigma, \xi) = \mu^{-1}[|\sigma_D - \xi_D| - \kappa]_+(\sigma_D - \xi_D)|\sigma_D - \xi_D|^{-1}$$

$$G_{2\mu}^{\text{kin}}(\sigma, \xi) := \frac{\partial}{\partial \xi} G_\mu(\sigma, \xi) = -\mu^{-1}[|\sigma_D - \xi_D| - \kappa]_+(\sigma_D - \xi_D)|\sigma_D - \xi_D|^{-1},$$

that is

$$(3.2) \quad G_{1\mu}^{\text{kin}} = -G_{2\mu}^{\text{kin}}$$

Note that for $|\sigma_D| = 0$ or $|\sigma_D - \xi_D| = 0$, the terms $G_{j\mu}^{\text{kin}}$ can be continuously extended by 0; for $G_{2\mu}^{\text{iso}}$ this follows from $\xi \geq 0$.

Due to our conventions in notation, the mathematical formulations of the penalty problem in the kinematic and the isotropic look the same, thus in the following we simply write $G_{j\mu}$ instead of $G_{1\mu}^{\text{kin}}$ or $G_{1\mu}^{\text{iso}}$, if the arguments run parallel. We formulate the penalty approximation of the hardening problem:

DEFINITION 3.1 (Penalty problem). – Find $\sigma_\mu, \xi_\mu \in L^\infty(0, T; L^2(\Omega))$ such that $\dot{\sigma}_\mu, \dot{\xi}_\mu \in L^\infty(0, T; L^2(\Omega))$, the pairs (σ_μ, ξ_μ) fulfill the initial condition (1.14) and the balance of forces (1.7) in the weak form for almost every t , further

$$(3.3) \quad (A\dot{\sigma}_\mu + G_{1\mu}(\sigma_\mu, \xi_\mu), \tau)_\Omega = 0$$

for all $\tau \in L^2(\Omega, \mathbb{R}_{\text{sym}}^{n \times n})$ which satisfy $(\tau, \nabla \varphi)_\Omega = 0$ for all $\varphi \in H^1_\Gamma(\Omega; \mathbb{R}^n)$,

$$(3.4) \quad H\dot{\xi}_\mu + G_{2\mu}(\sigma_\mu, \xi_\mu) = 0$$

By the L^2 -Helmholtz decomposition theorem for symmetric tensors we may replace (3.3) by the point-wise equation

$$(3.5) \quad \frac{1}{2}(\nabla v + \nabla v^T) = A\dot{\sigma} + G_{1\mu}(\sigma, \xi)$$

with the so called deformation velocity $v = \dot{u}$, $v \in H^1_\Gamma(\Omega, \mathbb{R}^n)$.

It is well known (see the discussion and references in [3]) that the penalty problem has a unique solution (σ_μ, ξ_μ) . Moreover, we have the following estimates independent of $\mu \in (0, \mu_0]$, provided that the safe load condition holds:

$$(3.6) \quad \|\sigma_\mu\|_{L^\infty(L^2)} + \|\dot{\sigma}_\mu\|_{L^\infty(L^2)} + \|u_\mu\|_{L^\infty(L^2)} + \|\dot{u}_\mu\|_{L^\infty(L^2)} \leq C,$$

$$(3.7) \quad \|\xi_\mu\|_{L^\infty(L^2)} + \|\dot{\xi}_\mu\|_{L^\infty(L^2)} \leq C,$$

$$(3.8) \quad \|\nabla \dot{u}_\mu\|_{L^\infty(L^2)} \leq C$$

The estimates (3.6), (3.7) have been worked out with a related penalty term in [9]. It is a routine matter to adapt this proof to our case. An alternative reference is [12]. By an argument of Johnson [9] (‘Johnson’s trick’), involving the relations (3.1) and (3.2), one has (3.8). Compared to the Prandtl-Reuss law, which corre-

sponds to $\zeta = 0$, where only an $L^\infty(L^1)$ -estimate is available, the estimate (3.8) makes the analysis much easier and gives better regularity results.

By monotonicity methods we have the convergence (see, e.g. [12]) $\sigma_\mu \rightarrow \sigma$, $\xi_\mu \rightarrow \xi$ strongly in $L^2(L^2)$ while $\nabla \dot{u}_\mu \rightarrow \nabla \dot{u}$, as $\mu \rightarrow 0$. Here (σ, ξ) is the solution of the original hardening problem. In [5, Sec. 3] it was shown that even

$$(3.9) \quad \dot{\sigma}_\mu \rightarrow \dot{\sigma}, \quad \dot{\xi}_\mu \rightarrow \dot{\xi} \text{ strongly in } L^2(L^2),$$

which implies also $\nabla \dot{u}_\mu \rightarrow \nabla \dot{u}$ in the kinematic case. For the proof it is essential to know that (in the case of isotropic hardening)

$$G_{1\mu}^{\text{iso}} \rightharpoonup \dot{\lambda} \frac{\sigma_D}{|\sigma_D|}, \quad G_{2\mu}^{\text{iso}} \rightharpoonup -\dot{\lambda}$$

weakly in L^2 , with a function $\dot{\lambda} \in L^\infty(L^2)$ enjoying the properties

$$\dot{\lambda} \geq 0, \quad \dot{\lambda} = 0 \text{ if } |\sigma_D| - \zeta < \kappa.$$

This result was proved in [3], the case of kinematic hardening runs in a completely analogous way. Using the multiplier $\dot{\lambda}$, the constitutive equation can be written in the form (1.17).

Furthermore, there are local uniform estimates

$$(3.10) \quad \|\nabla \sigma_\mu\|_{L^\infty(L^2(\Omega_0))} + \|\nabla \xi_\mu\|_{L^\infty(L^2(\Omega_0))} \leq C_{\Omega_0}, \quad \Omega_0 \subset\subset \Omega,$$

cf. [13, 12]. In addition, in the case of kinematic hardening, one has $u_\mu \in L^\infty(H_{\text{loc}}^2)$ and

$$(3.11) \quad \|u_\mu\|_{L^\infty(H^2(\Omega_0))} \leq C_{\Omega_0}, \quad \Omega_0 \subset\subset \Omega, \quad \mu \rightarrow 0.$$

It is an interesting open problem to obtain (3.11) also in the case of isotropic hardening. In [4], the authors were only able to prove $\nabla u \in L^\infty(L_{\text{loc}}^6)$, $n = 3$, in the latter case. Finally, it is known [5] that

$$(3.12) \quad \dot{\sigma} \in L^\infty(L^{2+2\delta}), \quad \dot{\xi} \in L^\infty(L^{2+2\delta}), \quad \nabla \dot{u} \in L^\infty(L^{2+2\delta}),$$

for some small $\delta > 0$.

4. – The regularity in time

4.1 – Auxiliary estimates

The proofs of the regularity results involve various auxiliary results for the penalty terms. We start with the proof that for almost every t , the penalty potential tends to 0 in $L^1(\Omega)$, as $\mu \rightarrow 0$.

LEMMA 4.1. – Let σ_μ, ξ_μ be the solution of the penalty problem 3.1, where the data f, p and σ_0 fulfill the regularity assumptions (1.4), (1.5) and (1.10). Then

$$\int_0^T \int_{\Omega} G_\mu(\sigma_\mu(t, x), \xi_\mu(t, x)) dx dt \rightarrow 0 \quad \text{as } \mu \rightarrow 0,$$

in particular there exists a sequence $\mu_n \rightarrow 0$ such that ⁽²⁾

$$\int_{\Omega} G_{\mu_n}(\sigma_{\mu_n}(t, x), \xi_{\mu_n}(t, x)) dx \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{for a.e. } t \in [0, T].$$

PROOF. – We only give the details in the case of kinematic hardening, the isotropic case can be done in an analogous way. Using the pair $(\sigma_\mu - \sigma, \xi_\mu - \xi)$ as test functions in the relations (3.3) and (3.4) of the penalty problem and observing (3.2), we obtain

$$\begin{aligned} 0 &= \int_0^T (A\dot{\sigma}_\mu, \sigma_\mu - \sigma)_\Omega + (H\dot{\xi}_\mu, \xi_\mu - \xi)_\Omega dt \\ &+ \int_0^T \int_{\Omega} \mu^{-1} [|\sigma_{\mu D} - \xi_{\mu D}| - \kappa]_+ \frac{\sigma_{\mu D} - \xi_{\mu D}}{|\sigma_{\mu D} - \xi_{\mu D}|} (\sigma_{\mu D} - \sigma_D - \xi_{\mu D} + \xi_D) dx dt =: \mathcal{I}_1 + \mathcal{I}_2. \end{aligned}$$

We have

$$\begin{aligned} \mathcal{I}_1 &= \frac{1}{2} \int_0^T \frac{\partial}{\partial t} (A(\sigma_\mu - \sigma), \sigma_\mu - \sigma)_\Omega + \int_0^T (A\dot{\sigma}, \sigma_\mu - \sigma)_\Omega dt + \\ &\frac{1}{2} \int_0^T \frac{\partial}{\partial t} (H(\xi_\mu - \xi), \xi_\mu - \xi)_\Omega + \int_0^T (H\dot{\xi}, \xi_\mu - \xi)_\Omega dt \\ &= \frac{1}{2} \left((A(\sigma_\mu - \sigma), \sigma_\mu - \sigma)_\Omega + (H(\xi_\mu - \xi), \xi_\mu - \xi)_\Omega \right) \Big|_{t=T} + o(1), \end{aligned}$$

because the remaining integrals tend to 0 as $\mu \rightarrow \infty$ due to the weak convergence of σ_μ, ξ_μ . Note the first term on the right-hand side is nonnegative thanks to the positivity condition (1.11). The integrand of \mathcal{I}_2 can be treated as follows:

⁽²⁾ In the following we omit the explicit mentioning of a subsequence in order to keep the proofs as simple as possible.

$$\begin{aligned} & \mu^{-1}[\sigma_{\mu D} - \xi_{\mu D} - \kappa]_+ \left(|\sigma_{\mu D} - \xi_{\mu D}| - \frac{\sigma_{\mu D} - \xi_{\mu D}}{|\sigma_{\mu D} - \xi_{\mu D}|} (\sigma_D - \xi_D) \right) \\ & \geq \mu^{-1}[\sigma_{\mu D} - \xi_{\mu D} - \kappa]_+ (|\sigma_{\mu D} - \xi_{\mu D}| - |\sigma_D - \xi_D|) \\ & \geq \mu^{-1}[\sigma_{\mu D} - \xi_{\mu D} - \kappa]_+ (|\sigma_{\mu D} - \xi_{\mu D}| - \kappa) = 2G_\mu^{\text{kin}} \geq 0, \end{aligned}$$

the last inequality holds, since $|\sigma_D - \xi_D| \leq \kappa$. Hence we have

$$0 \leq \int_0^T \int_\Omega G_\mu^{\text{kin}} dx dt \leq -K + o(1),$$

from which the assertion follows. □

Now we establish estimates involving difference quotients. In addition to the notation (2.1) we use the expressions

$$\begin{aligned} Iw(t, x) &= w(t, x), \quad E_t^h w(t, x) = w(t + h, x), \quad D_t^h w(t, x) = \frac{1}{h} \Delta_t^h w(t, x), \quad \text{hence} \\ \Delta_t^h w(t, x) &= (E_t^h - I)w(t, x) = hD_t^h w(t, x). \end{aligned}$$

The uniform $L^\infty(H^1)$ - and $H^1(L^2)$ -estimates (3.6)-(3.8) imply certain estimates of the penalty term, which we want to fix. Recall that $G_{j\mu}$ is either $G_{j\mu}^{\text{kin}}$ or $G_{j\mu}^{\text{iso}}$.

LEMMA 4.2. – *Assume the hypotheses of Lemma 4.1, then*

$$\begin{aligned} \mathcal{P}_0 &:= \int_0^{T-h} (D_t^h G_{1\mu}(\sigma_\mu, \xi_\mu), D_t^h \sigma_\mu)_\Omega + (D_t^h G_{2\mu}(\sigma_\mu, \xi_\mu), D_t^h \xi_\mu)_\Omega dt \leq C \\ &\text{uniformly for } 0 < h < h_0, \quad 0 < \mu < \mu_0. \end{aligned}$$

PROOF. – During this proof, we drop the index μ and simply write σ, ξ and u . We apply the operation D_t^h to (3.5) and (3.4) and use $D_t^h \sigma$ and $D_t^h \xi$, respectively, as a test function. Then we obtain

$$\begin{aligned} & \frac{1}{2} \int_0^{T-h} \frac{\partial}{\partial t} (D_t^h \sigma, AD_t^h \sigma)_\Omega + \frac{\partial}{\partial t} (D_t^h \xi, HD_t^h \xi)_\Omega dt + \mathcal{P}_0 = \int_0^{T-h} (D_t^h \nabla \dot{u}, D_t^h \sigma)_\Omega dt \\ (4.1) \quad & = \int_0^{T-h} (D_t^h \nabla \dot{u}, D_t^h (\sigma - \hat{\sigma}))_\Omega dt + \int_0^{T-h} (D_t^h \nabla \dot{u}, D_t^h \hat{\sigma})_\Omega dt =: \mathcal{I}_3 + \mathcal{I}_4, \end{aligned}$$

where $\hat{\sigma}$ is defined in the definition 1.2 of the safe load condition. Since $\dot{u} \in L^\infty(H_T^1(\Omega))$ we have $D_t^h \dot{u}(t) \in H_T^1(\Omega)$ for almost every t , hence the first integrand vanishes for almost every $t \in [0, T - h]$, since both σ and $\hat{\sigma}$ fulfill the balance of forces (1.7). We can transform the term \mathcal{I}_4 with the help of (1.7), too,

then we integrate by parts with respect to t and obtain

$$\begin{aligned} \mathcal{I}_4 &= \int_0^{T-h} (D_t^h \dot{u}, D_t^h f)_\Omega dt + \int_0^{T-h} \int_{\partial\Omega} D_t^h \dot{u} \cdot D_t^h p \, do dt \\ &= - \int_0^{T-h} (D_t^h u, D_t^h \dot{f})_\Omega dt + (D_t^h u, D_t^h f)_\Omega \Big|_0^{T-h} \\ &\quad - \int_0^{T-h} \int_{\partial\Omega} D_t^h u \cdot D_t^h \dot{p}_0 \, do dt + \int_{\partial\Omega} D_t^h u \cdot D_t^h p_0 \, do \Big|_0^{T-h}. \end{aligned}$$

After a possible redefinition on a set of measure zero (in time) we have for all $t \in [0, T - h]$.

$$\|D_t^h u(t)\|_{L^2(\Omega)} \leq \|\dot{u}\|_{L^\infty(0, T; L^2(\Omega))} \leq C$$

due to (3.6). To estimate the boundary integrals we need the trace theorem [11] in addition:

$$\|D_t^h u(t)\|_{L^2(\partial\Omega)} \leq \|\dot{u}\|_{L^\infty(0, T; L^2(\partial\Omega))} \leq \|\nabla \dot{u}\|_{L^\infty(0, T; L^2(\Omega))} \leq C$$

due to (3.8). Using the assumptions on the data we see that all terms in \mathcal{I}_4 are bounded, thus (4.1) leads to

$$\begin{aligned} &\mathcal{P}_0 + \frac{1}{2} \left((D_t^h \sigma, AD_t^h \sigma)_\Omega + \frac{1}{2} (D_t^h \zeta, HD_t^h \zeta)_\Omega \right) \Big|_{t=T-h} \\ &\leq C + \frac{1}{2} \left((D_t^h \sigma, AD_t^h \sigma)_\Omega + \frac{1}{2} (D_t^h \zeta, HD_t^h \zeta)_\Omega \right) \Big|_{t=0} \\ &\leq C \left(1 + h^{-1} \int_0^h \int_\Omega |\dot{\sigma}|^2 + |\dot{\zeta}|^2 \, dx \, dt \right) \leq C \end{aligned}$$

again since $\dot{\sigma}$ and $\dot{\zeta}$ are bounded in $L^\infty(L^2)$. This finishes the proof of Lemma 4.2. □

The following lemma serves as an auxiliary tool to control the quantity

$$\lim_{\mu \rightarrow 0} h^{-2} \int_0^h \int_0^{T-h} (A_t^s \dot{\sigma}_\mu, A \dot{\sigma}_\mu)_\Omega + (A_t^s \dot{\zeta}_\mu, H \dot{\zeta}_\mu)_\Omega \, dt \, ds,$$

which is needed in the proof of Theorem 2.1 to estimate the fractional derivative of $\dot{\sigma}$ and $\dot{\zeta}$ in time direction. The proof as well as the arguments for lemma 4.3 below run analogously to the proof of a corresponding result concerning the Prandtl-Reuss law [6].

LEMMA 4.3. – Let $\zeta = \zeta(x)$ be a Lipschitz continuous function depending only on the space variables, denote $G_{j\mu}(t, x) = G_{j\mu}(\sigma_\mu(t, x), \xi_\mu(t, x))$, then

$$\mathcal{T}_0(t_1, t_2) =: \limsup_{\mu \rightarrow 0} \int_0^h \int_{t_1}^{t_2-h} (\zeta^2 G_{1\mu}, \Delta_t^s \dot{\sigma}_\mu)_\Omega + (\zeta^2 G_{2\mu}, \Delta_t^s \dot{\xi}_\mu)_\Omega dt ds \leq 0$$

a.e. with respect to $t_1, t_2 \in [0, T]$, such that $0 \leq t_1 \leq t_2 - h \leq T - h$. In particular this holds true for $\zeta \equiv 1$.

PROOF. – We split

$$\begin{aligned} (4.2) \quad \mathcal{T}_{0\mu} &:= \int_0^h \int_{t_1}^{t_2-h} (\zeta^2 G_{1\mu}, \Delta_t^s \dot{\sigma}_\mu)_\Omega + (\zeta^2 G_{2\mu}, \Delta_t^s \dot{\xi}_\mu)_\Omega dt ds \\ &= \int_0^h \int_{t_1}^{t_2-h} (\zeta^2 G_{1\mu}, E_t^s \dot{\sigma}_\mu)_\Omega + (\zeta^2 G_{2\mu}, E_t^s \dot{\xi}_\mu)_\Omega dt ds - \int_0^h \int_{t_1}^{t_2-h} \int_\Omega \zeta^2 \frac{\partial}{\partial t} G_\mu dx dt ds \\ &=: \mathcal{I}_5 + \int_0^h \int_\Omega \zeta^2 G_\mu dx \Big|_{t_1}^{t_2-h} ds = \mathcal{I}_5 + h \int_\Omega \zeta^2 G_\mu dx \Big|_{t_1}^{t_2-h} \end{aligned}$$

The last term tends to zero as $\mu \rightarrow 0$ due to Lemma 4.1, a.e. for $t_1, t_2 \in [0, T]$, $t_2 - h \geq t_1$. For the first term we use the identities $\int_0^h E_t^s \dot{\sigma}_\mu ds = \Delta_t^h \sigma_\mu$, $\int_0^h E_t^s \dot{\xi}_\mu ds = \Delta_t^h \xi_\mu$, hence

$$\mathcal{I}_5 = \int_{t_1}^{t_2-h} (\zeta^2 G_{1\mu}, \Delta_t^h \sigma_\mu)_\Omega + (\zeta^2 G_{2\mu}, \Delta_t^h \xi_\mu)_\Omega dt.$$

Due to the convexity of G_μ we have

$$(4.3) \quad G_{1\mu} : \Delta_t^h \sigma_\mu + G_{2\mu} \cdot \Delta_t^h \xi_\mu \leq \Delta_t^h G_\mu$$

and hence

$$\mathcal{I}_5 \leq \int_{t_1}^{t_2-h} \int_\Omega \zeta^2 \Delta_t^h G_\mu dx dt.$$

For fixed $h > 0$, the latter term tends to 0 a.e. with respect to $t_1, t_2 \in [0, T]$, as $\mu \rightarrow 0$, here we used Lemma 4.1 again. This proves Lemma 4.3. \square

4.2 – Proof of Theorem 2.1

Our approach here is very similar to the arguments used in [6]. Due to the strong convergence (3.9) of $\dot{\sigma}_\mu$ and $\dot{\zeta}_\mu$, we have for any fixed h :

$$(4.4) \quad \int_0^h \int_0^{T-h} \int_\Omega |A_t^s \dot{\sigma}|^2 + |A_t^s \dot{\zeta}|^2 dx dt ds = \lim_{\mu \rightarrow 0} \int_0^h \int_0^{T-h} \int_\Omega |A_t^s \dot{\sigma}_\mu|^2 + |A_t^s \dot{\zeta}_\mu|^2 dx dt ds.$$

In order to control the term on the right-hand side we choose $\int_0^h A_t^s \dot{\sigma}_\mu ds$ and $\int_0^h A_t^s \dot{\zeta}_\mu ds$ as test-functions in (3.5) and (3.4), then using the notation $\mathcal{T}_{0\mu} = \mathcal{T}_{0\mu}(t_1, t_2)$ from (4.2) (here with $\zeta \equiv 1$), we obtain

$$(4.5) \quad \begin{aligned} & \int_{t_1}^{t_2-h} \int_0^h (A \dot{\sigma}_\mu, A_t^s \dot{\sigma}_\mu)_\Omega ds dt + \int_{t_1}^{t_2-h} \int_0^h (H \dot{\zeta}_\mu, A_t^s \dot{\zeta}_\mu)_\Omega ds dt + \mathcal{T}_{0\mu} \\ &= \int_{t_1}^{t_2-h} \int_0^h (\nabla \dot{u}_\mu, A_t^s \dot{\sigma}_\mu)_\Omega ds dt =: \mathcal{I}. \end{aligned}$$

Next we recall an elementary identity, which holds for $\tau(t) \in \mathbb{R}_{\text{sym}}^{n \times n}$, and any symmetric tensor A ,

$$(4.6) \quad A\tau : A_t^s \tau = -\frac{1}{2} A A_t^s \tau : A_t^s \tau + \frac{1}{2} A_t^s (A\tau : \tau),$$

if τ and A are scalar functions, this is even simpler. Relation (4.6) turns (4.5) into

$$\begin{aligned} \mathcal{L}(\dot{\sigma}_\mu, \dot{\zeta}_\mu) &= \int_{t_1}^{t_2-h} \int_0^h (A A_t^s \dot{\sigma}_\mu, A_t^s \dot{\sigma}_\mu)_\Omega + (H A_t^s \dot{\zeta}_\mu, A_t^s \dot{\zeta}_\mu)_\Omega ds dt \\ &= \int_{t_1}^{t_2-h} \int_0^h \int_\Omega A_t^s (A \dot{\sigma}_\mu : \dot{\sigma}_\mu) + A_t^s (H \dot{\zeta}_\mu : \dot{\zeta}_\mu) dx ds dt + 2\mathcal{T}_{0\mu} - 2\mathcal{I} := \mathcal{R}_\mu \end{aligned}$$

Since \mathcal{L} defines a lower semi-continuous functional with respect to the $L^2(L^2)$ -norm, the weak convergence of $(\dot{\sigma}_\mu, \dot{\zeta}_\mu)$ implies

$$\mathcal{L}(\dot{\sigma}, \dot{\zeta}) \leq \liminf_{\mu \rightarrow 0} \mathcal{L}(\dot{\sigma}_\mu, \dot{\zeta}_\mu),$$

note, that here even the limit exists since we have the strong convergence (3.9).

Due to the positivity conditions for A and H (and Fubini's theorem) we get

$$(4.7) \quad \lim_{\mu \rightarrow 0} \int_0^h \int_{t_1}^{t_2-h} \int_{\Omega} |\mathcal{A}_t^s \dot{\sigma}_\mu|^2 + |\mathcal{A}_t^s \dot{\xi}_\mu|^2 dx dt ds \leq C_0 \lim_{\mu \rightarrow 0} \mathcal{L}(\dot{\sigma}_\mu, \dot{\xi}_\mu),$$

hence the assertion (2.2) is true, if we show

$$(4.8) \quad \limsup_{\mu \rightarrow 0} \mathcal{R}_\mu \leq Ch^2.$$

To estimate the first summand of \mathcal{R}_μ , we put

$$\varphi(t) = \int_{\Omega} A \dot{\sigma}_\mu : \dot{\sigma}_\mu + H \dot{\xi}_\mu : \dot{\xi}_\mu dx$$

and use the following argument

$$(4.9) \quad \left| \int_{t_1}^{t_2-h} \int_0^h \mathcal{A}_t^s \varphi(t) ds dt \right| = \left| \int_0^h \int_{t_2-h}^{t_2-h+s} \varphi(t) dt ds - \int_0^h \int_{t_1}^{t_1+s} \varphi(t) dt ds \right| \leq 2 \|\varphi\|_{L^\infty} h^2,$$

due to the bounds (3.6) for $\|\dot{\sigma}_\mu\|_{L^\infty(L^2)}$ and (3.7) for $\|\dot{\xi}_\mu\|_{L^\infty(L^2)}$. Note that the constant here is also independent from t_1 and t_2 . Using the safe load from Definition 1.2 we rewrite the term \mathcal{I} (cf. (4.5)):

$$\begin{aligned} \mathcal{I} &= \int_{t_1}^{t_2-h} \int_0^h (\nabla \dot{u}_\mu, \mathcal{A}_t^s (\dot{\sigma}_\mu - \dot{\sigma}))_{\Omega} ds dt + \int_{t_1}^{t_2-h} \int_0^h (\nabla \dot{u}_\mu, \mathcal{A}_t^s \dot{\sigma})_{\Omega} ds dt \\ &= \int_0^h \int_{t_1}^{t_2-h} (\nabla \dot{u}_\mu, \mathcal{A}_t^s \dot{\sigma})_{\Omega} dt ds, \end{aligned}$$

since $\text{div}(\dot{\sigma}_\mu - \dot{\sigma}) = 0$ (and by using Fubini's theorem). Unlike in the proof for the Prandtl Reuss case we may already use the bound $\|\nabla \dot{u}_\mu\|_{L^\infty(L^2)} \leq C_T$, together with the assumption (1.16) for $\dot{\sigma}$ this implies

$$(4.10) \quad \begin{aligned} |\mathcal{I}| &\leq \int_0^h \int_0^{T-h} \|\nabla \dot{u}_\mu\|_{L^2(\Omega)} \|\mathcal{D}_t^s \dot{\sigma}\|_{L^2(\Omega)} dt ds \\ &\leq \|\nabla \dot{u}_\mu\|_{L^\infty(L^2)} h \int_0^h \int_0^T \|\ddot{\sigma}\|_{L^2(\Omega)} dt ds \leq C(T)h^2 \end{aligned}$$

where $C(T)$ is independent of $\mu \leq \mu_0$ and $h < T$, and of t_1 and t_2 , of course. From Lemma 4.3 we get then (4.8) for almost all t_1 and t_2 , where C is independent of h and t_1, t_2 . Since the integrals in (4.7) depend continuously on t_1 and t_2 , we can pass to the limit $t_1 \rightarrow 0$, and $t_2 \rightarrow T$, which finishes the proof. \square

4.3 – Proof of Theorem 2.3

Using the penalty equations (3.5) and (3.4) together with (3.2), we obtain in the case of kinematic hardening

$$\frac{1}{2}(\nabla u_\mu + (\nabla u_\mu)^\top) = A\dot{\sigma}_\mu + H\dot{\xi}_\mu.$$

Hence Theorem 2.3 follows from Theorem 2.1 and Korn’s inequality. □

5. – The regularity in space direction

5.1 – Auxiliary inequalities for the penalty terms

In the next lemma we derive a local bound for the spatial difference quotients of the penalty term similar to Lemma 4.2. Since we prove only local regularity in space direction we fix a localization function ζ with compact support in Ω such that $\nabla\zeta$ is Lipschitz continuous. We use difference and shift operators similar as in Section 4. Let e_j be the j -th unit-vector in \mathbb{R}^n , and $h > 0$. Apart from the notation Δ_j^h introduced in (2.1) we use also

$$E_j^h w(t, x) = w(t, x + he_j), \quad D_j^h = \frac{1}{h}\Delta_j^h.$$

LEMMA 5.1. – Let (σ_μ, ξ_μ) be the solution to the penalty problem introduced in Section 3, then the following estimate holds

$$\mathcal{P}_j := \int_0^T (D_j^h G_{1\mu}(\sigma_\mu, \xi_\mu), \zeta^2 D_j^h \sigma_\mu)_\Omega + (D_j^h G_{2\mu}(\sigma_\mu, \xi_\mu), \zeta^2 D_j^h \xi_\mu)_\Omega dt \leq C$$

uniformly for $0 < \mu < \mu_0$ and $0 < h < h_0$, where $h_0 < \text{dist}(\text{supp } \zeta, \partial\Omega)$.

PROOF. – This proof contains also the arguments for the H_{loc}^1 -regularity of σ . Just like in the proof of Lemma 4.2 we drop the index μ here. We apply the operation D_j^h to (3.5), (3.4), and then, similar as in the proof of Lemma 4.2, test the first relation with $\zeta^2 D_j^h \sigma$ and the second with $\zeta^2 D_j^h \xi$, then for any $T_1 \leq T$ we have

$$(5.1) \quad \frac{1}{2} \int_0^{T_1} \frac{\partial}{\partial t} (D_j^h \sigma, \zeta^2 A D_j^h \sigma)_\Omega + \frac{\partial}{\partial t} (D_j^h \xi, \zeta^2 H D_j^h \xi)_\Omega dt + \mathcal{P}_j = \int_0^{T_1} (D_j^h \nabla \dot{u}, \zeta^2 D_j^h \sigma)_\Omega dt.$$

We integrate by parts in the right-hand side and use $-\text{div } \sigma = f$, which

leads to

$$(5.2) \quad \int_0^{T_1} (D_j^h \nabla \dot{u}, \zeta^2 D_j^h \sigma)_\Omega dt = - \int_0^T (D_j^h \dot{u}, \zeta^2 f)_\Omega dt - 2 \int_0^T (D_j^h \dot{u}, \zeta \nabla \zeta D_j^h \sigma)_\Omega dt =: \mathcal{I}_1 + \mathcal{I}_2.$$

Since we have the uniform bound (3.8) for $\|\nabla \dot{u}\|_{L^\infty(L^2(\text{supp } \zeta))}$ as $\mu \rightarrow 0$, we obtain also $\|D_j^h \dot{u}\|_{L^\infty(L^2(\text{supp } \zeta))} \leq C$, uniformly in $0 < \mu < \mu_0$, $0 < h < h_0$. Therefore the term \mathcal{I}_1 turns out to be bounded due to the regularity assumptions for f . Using this argument once more, Hlder’s and Young’s inequalities imply

$$(5.3) \quad |\mathcal{I}_2| \leq C_\zeta \left(1 + \int_0^T \int_\Omega \zeta^2 |D_j^h \sigma|^2 dx dt \right).$$

Now we evaluate the left-hand side of (5.1). Since $\zeta(0) = 0$, we have

$$\begin{aligned} & \frac{1}{2} \int_0^T \frac{\partial}{\partial t} (D_j^h \sigma, \zeta^2 A D_j^h \sigma)_\Omega + \frac{\partial}{\partial t} (D_j^h \zeta, \zeta^2 H D_j^h \zeta)_\Omega dt \\ &= \frac{1}{2} \left((D_j^h \sigma A, \zeta^2 D_j^h \sigma)_\Omega + (D_j^h \zeta, \zeta^2 H D_j^h \zeta)_\Omega \right) \Big|_{t=T_1} - (D_j^h \sigma^0 A, \zeta^2 D_j^h \sigma^0)_\Omega \end{aligned}$$

The convexity of the penalty potential implies $\mathcal{P}_j \geq 0$, hence the bounds for \mathcal{I}_1 and (5.3) for \mathcal{I}_2 together with the positivity of A and H lead to the inequality

$$\begin{aligned} \int_\Omega \zeta^2 |D_j^h \sigma(T_1)|^2 dx &\leq (D_j^h \sigma, \zeta^2 A D_j^h \sigma)_\Omega \Big|_{t=T_1} + (D_j^h \zeta, \zeta^2 H D_j^h \zeta)_\Omega \Big|_{t=T_1} + 2\mathcal{P}_j \\ &\leq C \left(1 + \int_\Omega |D_j^h \sigma_0|^2 \zeta^2 dx + \int_0^{T_1} \int_\Omega \zeta^2 |D_j^h \sigma|^2 dx dt \right). \end{aligned}$$

Since $\sigma_0 \in H^1(\Omega)$, now Gronwall’s inequality implies that $|\mathcal{I}_2|$ is bounded independent of μ , but this implies also the desired bound for \mathcal{P}_j . □

We also need a result corresponding to Lemma 4.3 for difference quotients of $\dot{\sigma}_\mu$ in space direction. We recall that

$$(E_i^s E_i^h - I)w(t, x) = w(t + s, x + he_i) - w(t, x).$$

LEMMA 5.2. – Fix $h_0 > 0$ such that $\text{supp } E_i^h \zeta \subset \Omega$ for any h with $0 \leq h \leq h_0$. Then there exists a constant $\tilde{C} > 0$, independent of h, t_1 , and t_2 such that

$$\limsup_{\mu \rightarrow 0} \int_0^h \int_{t_1}^{t_2-h} \int_\Omega \left[G_{1\mu} : (E_i^s E_i^h - I)\dot{\sigma}_\mu + G_{2\mu} : (E_i^s E_i^h - I)\dot{\xi}_\mu \right] \zeta^2 dx dt ds \leq \tilde{C}h^2,$$

for almost every t_1, t_2 such that $0 \leq t_1 \leq t_2 - h \leq T - h$.

PROOF. – We set

$$\mathcal{T}_{i\mu} = \mathcal{T}_{i\mu}(t_1, t_2) = \int_0^h \int_{t_1}^{t_2-h} \int_{\Omega} \left[G_{1\mu} : (E_t^s E_i^h - I) \dot{\sigma}_\mu + G_{2\mu} : (E_t^s E_i^h - I) \dot{\zeta}_\mu \right] \zeta^2 dx dt ds$$

and decompose $\mathcal{T}_{i\mu} = \mathcal{T}_{i\mu}^1 - \mathcal{T}_{i\mu}^2$ where

$$\begin{aligned} \mathcal{T}_{i\mu}^1 &= \int_0^h \int_{t_1}^{t_2-h} \int_{\Omega} \left[G_{1\mu} : E_t^s E_i^h \dot{\sigma}_\mu + G_{2\mu} : E_t^s E_i^h \dot{\zeta}_\mu \right] \zeta^2 dx dt ds, \\ \mathcal{T}_{i\mu}^2 &= \int_0^h \int_{t_1}^{t_2-h} \int_{\Omega} \left(G_{1\mu} : \dot{\sigma}_\mu + G_{2\mu} : \dot{\zeta}_\mu \right) \zeta^2 dx dt ds \\ &= \int_0^h \int_{t_1}^{t_2-h} \int_{\Omega} \frac{d}{dt} G_\mu \zeta^2 dx dt ds = h \int_{\Omega} G_\mu \zeta^2 dx \Big|_{t_1}^{t_2-h} \end{aligned}$$

Lemma 4.1 implies

$$(5.4) \quad \lim_{\mu \rightarrow 0} \mathcal{T}_{i\mu}^2 = 0 \quad \text{a.e. with respect to } t_1, t_2,$$

such that $0 \leq t_1 \leq t_2 - h \leq T - h$. To analyze $\mathcal{T}_{i\mu}^1$ we perform the integration with respect to s , then we add and subtract the terms $G_{1\mu} : \sigma_\mu$ and $G_{2\mu} : \zeta_\mu$ in order to achieve an additional splitting:

$$\begin{aligned} \mathcal{T}_{i\mu}^1 &= \int_{t_1}^{t_2-h} \int_{\Omega} \left[G_{1\mu} : (E_t^h E_i^h \sigma_\mu - E_i^h \sigma_\mu) + G_{2\mu} : (E_t^h E_i^h \zeta_\mu - E_i^h \zeta_\mu) \right] \zeta^2 dx dt \\ &= \int_{t_1}^{t_2-h} \int_{\Omega} \left[G_{1\mu} : (E_t^h E_i^h \sigma_\mu - \sigma_\mu) + G_{2\mu} : (E_t^h E_i^h \zeta_\mu - \zeta_\mu) \right] \zeta^2 dx dt \\ &\quad - h \int_{t_1}^{t_2-h} \int_{\Omega} \left[G_{1\mu} : D_i^h \sigma_\mu + G_{2\mu} : D_i^h \zeta_\mu \right] \zeta^2 dx dt =: \mathcal{T}_{i\mu}^{1a} + \mathcal{T}_{i\mu}^{1b}. \end{aligned}$$

We first will get rid of $\mathcal{T}_{i\mu}^{1b}$ and rewrite

$$\begin{aligned} \mathcal{T}_{i\mu}^{1b} &= -h \int_{t_1}^{t_2-h} \int_{\Omega} \left[E_i^h G_{1\mu} : D_i^h \sigma_\mu + E_i^h G_{2\mu} D_i^h \zeta_\mu \right] \zeta^2 dx dt + \\ &\quad + h^2 \int_{t_1}^{t_2-h} \int_{\Omega} \left[D_i^h G_{1\mu} : D_i^h \sigma_\mu + D_i^h G_{2\mu} D_i^h \zeta_\mu \right] \zeta^2 dx dt \end{aligned}$$

Since the last integral is bounded according to Lemma 5.1 we conclude

$$\mathcal{T}_{i\mu}^{1b} \leq \tilde{C} h^2 - h \int_{t_1}^{t_2-h} \int_{\Omega} [E_i^h G_{1\mu} : D_i^h \sigma_{\mu} + E_i^h G_{2\mu} D_i^h \zeta_{\mu}] \zeta^2 dx dt$$

Now we exploit the convexity of G_{μ} again, which leads to

$$-(E_i^h G_{1\mu} : D_i^h \sigma_{\mu} + E_i^h G_{2\mu} D_i^h \zeta_{\mu}) \leq h^{-1} (G_{\mu} - E_i^h G_{\mu}) = -D_i^h G_{\mu}$$

This implies

$$\mathcal{T}_{i\mu}^{1b} \leq \tilde{C} h^2 - h \int_{t_1}^{t_2-h} \int_{\Omega} D_i^h G_{\mu} \zeta^2 dx,$$

here the last summand tends to zero as $\mu \rightarrow 0$ due to Lemma 4.1. Thus we arrive at

$$(5.5) \quad \limsup_{\mu \rightarrow \infty} \mathcal{T}_{i\mu}^{1b} \leq \tilde{C} h^2$$

A similar argument works for $\mathcal{T}_{i\mu}^{1a}$: From the convexity of G_{μ} we get

$$(5.6) \quad \mathcal{T}_{i\mu}^{1a} \leq \int_{t_1}^{t_2-h} \int_{\Omega} [E_i^h E_i^h G_{\mu} - G_{\mu}] \zeta^2 dx dt.$$

while the right-hand side term tends to zero as $\mu \rightarrow 0, h$ fixed, a.e. with respect to t_1, t_2 , again due to Lemma 4.1. Collecting (5.4), (5.5), (5.6), we arrive at $\lim_{\mu \rightarrow \infty} \mathcal{T}_{i\mu} \leq \tilde{C} h^2$ a.e. which is the statement of Lemma 5.2. \square

5.2 – Testing the strain velocity

We have to prepare one additional auxiliary estimate for the regularity result in space direction.

LEMMA 5.3. – *Let ζ be a localization function as in Lemma 5.1, and let $h_0 > 0$ be fixed such that $h_0 < \text{dist}(\text{supp } \zeta, \partial\Omega)$. Then*

$$(5.7) \quad \left| \int_0^h \int_{t_1}^{t_2-h} (\nabla \dot{u}, \zeta^2 (E_t^s E_i^h - I) \dot{\sigma})_{\Omega} dt \right| \leq C h^2$$

with h, t_1, t_2 as in Lemma 5.2 and C again independent of these parameters.

PROOF. – We denote

$$\mathcal{S} = \int_0^h \int_{t_1}^{t_2-h} (\nabla \dot{u}, \zeta^2(E_t^s E_i^h - I)\dot{\sigma})_{\Omega} dt ds = \mathcal{S}^1 + \mathcal{S}^2,$$

where

$$\begin{aligned} \mathcal{S}^1 &:= \int_0^h \int_{t_1}^{t_2-h} (\nabla \dot{u}, \zeta^2 E_t^s (E_i^h - I)\dot{\sigma})_{\Omega} dt ds = \int_0^h \int_{t_1}^{t_2-h} (\nabla \dot{u}, \zeta^2 E_t^s \mathcal{A}_i^h \dot{\sigma})_{\Omega} dt ds, \\ \mathcal{S}^2 &:= \int_0^h \int_{t_1}^{t_2-h} (\nabla \dot{u}, \zeta^2 (E_i^s - I)\dot{\sigma})_{\Omega} dt ds = \int_0^h \int_{t_1}^{t_2-h} (\nabla \dot{u}, \zeta^2 \mathcal{A}_i^s \dot{\sigma})_{\Omega} dt ds. \end{aligned} \tag{5.8}$$

STEP 1. – Estimates for $|\mathcal{S}^1|$.

To this end, we integrate by parts in the term \mathcal{S}^1 , then use the relation $-\operatorname{div} \sigma = f$, end up with

$$\mathcal{S}^1 = - \int_0^h \int_{t_1}^{t_2-h} (\dot{u} \zeta^2, E_t^s \mathcal{A}_i^h f)_{\Omega} dt ds - \int_0^h \int_{t_1}^{t_2-h} (\dot{u} \nabla \zeta^2, \mathcal{A}_i^s \dot{\sigma})_{\Omega} dt ds =: \mathcal{S}^{1a} + \mathcal{S}^{1b}.$$

Moving the operator \mathcal{A}_i^h from f to $\dot{u} \zeta^2$ yields

$$\mathcal{S}^{1a} = - \int_0^h \int_{t_1}^{t_2-h} (\mathcal{A}_i^{-h}(\dot{u} \zeta^2), E_i^s f)_{\Omega} dt ds.$$

Since

$$\|\mathcal{A}_i^{-h}(\dot{u} \zeta^2)\|_{L^\infty(L^2)} = h \|D_j^{-h}(\dot{u} \zeta^2)\|_{L^\infty(L^2)} \leq C(\|\dot{u}\|_{L^\infty(L^2)} + \|\nabla \dot{u}\|_{L^\infty(L^2)})h,$$

the uniform estimates (3.6), (3.8) together with the assumption $f \in L^\infty(L^2)$ (cf. (1.4)) lead to

$$|\mathcal{S}^{1a}| \leq h \int_0^h \|\dot{f}\|_{L^1(L^2)} \|D_i^{-h}(\dot{u} \zeta^2)\|_{L^\infty(L^2)} ds \leq C_T h^2,$$

where K_T is independent of $0 < \mu \leq \mu_0$, and $0 < h \leq h_0$. A similar argument works for the summand \mathcal{S}^{1b} , hence, again with (3.6) and (3.8),

$$\begin{aligned} |\mathcal{S}^{1b}| &= \left| \int_0^h \int_{t_1}^{t_2-h} (\mathcal{A}_i^{-h}(\dot{u} \nabla \zeta^2), E_t^s \dot{\sigma})_{\Omega} dt ds \right| \\ &\leq C h \int_0^h \|\dot{\sigma}\|_{L^1(L^2)} \|D_i^{-h}(\dot{u} \nabla \zeta^2)\|_{L^\infty(L^2)} ds \leq C_T h^2. \end{aligned}$$

STEP 2. – Estimates for $|S^2|$.

To show that this quantity is bounded by Ch^2 , it is not enough to use $\nabla \dot{u} \in L^\infty(L^2)$ together with (2.2), because then we only get the bound $Ch^{3/2}$. Instead we go back to the solutions of the penalized problem. Unfortunately the presence of the localization term ζ^2 prohibits to argue with the safe load as in (4.10), nevertheless this inequality gives already the desired estimate in the case $\zeta \equiv 1$. Recall that the system (3.4) and (3.5) leads to

$$(5.9) \quad \int_0^h \int_{t_1}^{t_2-h} (\nabla \dot{u}_\mu, \zeta^2 \mathcal{A}_t^s \dot{\sigma}_\mu)_\Omega dt ds = \int_0^h \int_{t_1}^{t_2-h} (A \dot{\sigma}_\mu, \zeta^2 \mathcal{A}_t^s \dot{\sigma}_\mu)_\Omega + (H \dot{\zeta}_\mu, \zeta^2 \mathcal{A}_t^s \dot{\zeta}_\mu)_\Omega dt ds + \int_0^h \int_{t_1}^{t_2-h} (G_{1\mu}, \zeta^2 \mathcal{A}_t^s \dot{\sigma}_\mu)_\Omega + (G_{2\mu}, \zeta^2 \mathcal{A}_t^s \dot{\zeta}_\mu)_\Omega dt ds =: S_\mu^{2a} + T_{0\mu},$$

where $T_{0\mu}$ was defined in (4.2). Using again “the product-rule” (4.6) we obtain

$$S_\mu^{2a} = -\frac{1}{2} \int_0^h \int_{t_1}^{t_2-h} (\zeta^2 \mathcal{A}_t^s \dot{\sigma}_\mu, \mathcal{A}_t^s \dot{\sigma}_\mu)_\Omega + (\zeta^2 H \mathcal{A}_t^s \dot{\zeta}_\mu, \mathcal{A}_t^s \dot{\zeta}_\mu)_\Omega dt ds + \frac{1}{2} \int_0^h \int_{t_1}^{t_2-h} \int_\Omega \zeta^2 \mathcal{A}_t^s (A \dot{\sigma}_\mu : \dot{\sigma}_\mu + H \dot{\zeta}_\mu : \dot{\zeta}_\mu) dx dt ds.$$

Note that $\lim_{\mu \rightarrow 0} S_\mu^{2a}$ as well as the limits for both summands on the right hand side exist due to (3.9). The limit of the first integral is bounded by Ch^2 due to Theorem 2.1 while for the second integral we get this bound following the same arguments as in (4.9), hence we have

$$(5.10) \quad \left| \int_0^h \int_{t_1}^{t_2-h} (A \dot{\sigma}, \zeta^2 \mathcal{A}_t^s \dot{\sigma})_\Omega + (H \dot{\zeta}, \zeta^2 \mathcal{A}_t^s \dot{\zeta})_\Omega dt ds \right| = \lim_{\mu \rightarrow 0} |S_\mu^{2a}| \leq Ch^2.$$

Since the limits of the other two terms in the equation (5.9) exist, we obtain that even $\lim_{\mu \rightarrow 0} T_{0\mu}$ exists. In particular, the representation (5.9) for $\zeta = 1$ (compare (4.5)) together with (4.10) then gives

$$(5.11) \quad \lim_{\mu \rightarrow 0} \left| \int_0^h \int_{t_1}^{t_2-h} (G_{1\mu}, \mathcal{A}_t^s \dot{\sigma}_\mu)_\Omega + (G_{2\mu}, \mathcal{A}_t^s \dot{\zeta}_\mu)_\Omega dt ds \right| \leq Ch^2$$

To extend this to the case where ζ is a proper localization function we use similar calculations as in Lemma 4.3, in particular the convexity of G_μ and Lemma 4.1. For fixed h , we get

$$\begin{aligned}
\lim_{\mu \rightarrow 0} |\mathcal{T}_{0\mu}| &= \lim_{\mu \rightarrow 0} \left| \int_{t_1}^{t_2-h} (\zeta^2 G_{1\mu}, \Delta_t^h \sigma_\mu)_\Omega + (\zeta^2 G_{2\mu}, \Delta_t^h \xi_\mu)_\Omega dt \right| \\
&= \lim_{\mu \rightarrow 0} \left| \int_{t_1}^{t_2-h} \int_\Omega \zeta^2 (A_t^h G_\mu - (G_{1\mu} : A_t^h \sigma_\mu + G_{2\mu} : A_t^h \xi_\mu)) dx dt \right| \\
&= \lim_{\mu \rightarrow 0} \int_{t_1}^{t_2-h} \int_\Omega \zeta^2 (A_t^h G_\mu - (G_{1\mu} : A_t^h \sigma_\mu + G_{2\mu} : A_t^h \xi_\mu)) dx dt \\
&\leq \max \zeta^2 \lim_{\mu \rightarrow 0} \int_{t_1}^{t_2-h} \int_\Omega A_t^h G_\mu - (G_{1\mu} : A_t^h \sigma_\mu + G_{2\mu} : A_t^h \xi_\mu) dx dt \\
&= C(\zeta) \lim_{\mu \rightarrow 0} \left| \int_{t_1}^{t_2-h} \int_\Omega (G_{1\mu} : A_t^h \sigma + G_{2\mu} : A_t^h \xi_\mu) dx dt \right| \\
&= C(\zeta) \lim_{\mu \rightarrow 0} \left| \int_0^h \int_{t_1}^{t_2-h} (G_{1\mu}, A_t^s \dot{\sigma}_\mu)_\Omega + (G_{2\mu}, A_t^s \dot{\xi}_\mu)_\Omega dt ds \right| \leq Ch^2,
\end{aligned}$$

observe, that the third equality and the following inequality are true because the integrand is non-negative almost everywhere (compare (4.3)), while the last inequality follows from (5.11). Together with (5.10) this gives the bound for $|\mathcal{S}^2|$. \square

5.3 – Proof of Theorem 2.4

We use the test function $\zeta^2(E_t^s E_i^h - I)\dot{\sigma}_\mu$ in (3.3) and test the relation (3.4) with $\zeta^2(E_t^s E_i^h - I)\dot{\xi}_\mu$. We sum the relations and integrate from $t = t_1$ to $t = t_2 - h$, this yields

$$\begin{aligned}
(5.12) \quad & \int_0^h \int_{t_1}^{t_2-h} (A\dot{\sigma}_\mu, \zeta^2(E_t^s E_i^h - I)\dot{\sigma}_\mu)_\Omega + (H\dot{\xi}_\mu, \zeta^2(E_t^s E_i^h - I)\dot{\xi}_\mu)_\Omega dt ds + \mathcal{T}_{i\mu}(t_1, t_2) = \\
& \int_0^h \int_{t_1}^{t_2-h} (\nabla \dot{u}_\mu, \zeta^2(E_t^s E_i^h - I)\dot{\sigma}_\mu)_\Omega dt ds =: \mathcal{S}_\mu(t_1, t_2),
\end{aligned}$$

where $\mathcal{T}_{i\mu}$ is the quantity coming from the penalty term and has the same meaning as in Lemma 5.2. Similar as in (4.6), for symmetric tensors A (with

constant entries) and $\tau = \tau(t, x)$ we have the relation

$$(5.13) \quad A\tau : (E_t^s E_i^h - I)\tau = \frac{1}{2}(E_t^s E_i^h - I)(A\tau : \tau) - \frac{1}{2}A(E_t^s E_i^h - I)\tau : (E_t^s E_i^h - I)\tau.$$

Put

$$\mathcal{R}_\mu(t_1, t_2) := \frac{1}{2} \int_0^h \int_{t_1}^{t_2-h} \int_\Omega \left[(E_t^s E_i^h - I)(A\dot{\sigma}_\mu : \sigma_\mu) + (E_t^s E_i^h - I)(H\dot{\zeta}_\mu : \dot{\zeta}_\mu) \right] \zeta^2 dx dt ds,$$

then (5.13) turns (5.12) into

$$(5.14) \quad \mathcal{L}_\mu(t_1, t_2) :=$$

$$\begin{aligned} & \int_0^h \int_{t_1}^{t_2-h} (A(E_t^s E_i^h - I)\dot{\sigma}_\mu, \zeta^2(E_t^s E_i^h - I)\dot{\sigma}_\mu)_\Omega + (H(E_t^s E_i^h - I)\dot{\zeta}_\mu, \zeta^2(E_t^s E_i^h - I)\dot{\zeta}_\mu)_\Omega dt ds \\ & = \mathcal{R}_\mu(t_1, t_2) + \mathcal{T}_{i\mu}(t_1, t_2) - \mathcal{S}_\mu(t_1, t_2). \end{aligned}$$

Since we have $(\overset{3}{\sigma})_\mu \rightarrow \dot{\sigma}$, $\dot{\zeta}_\mu \rightarrow \dot{\zeta}$ and $\nabla \dot{u}_\mu \rightarrow \nabla \dot{u}$ in $L^2(L^2)$ we obtain for any fixed h with $0 < h \leq h_0$:

$$(5.15) \quad \lim_{\mu \rightarrow 0} \mathcal{L}_\mu(t_1, t_2) =: \mathcal{L}(t_1, t_2) =$$

$$\begin{aligned} & \frac{1}{2} \int_0^h \int_{t_1}^{t_2-h} (A(E_t^s E_i^h - I)\dot{\sigma}, \zeta^2(E_t^s E_i^h - I)\dot{\sigma})_\Omega + (H(E_t^s E_i^h - I)\dot{\zeta}, \zeta^2(E_t^s E_i^h - I)\dot{\zeta})_\Omega dt ds, \\ & \lim_{\mu \rightarrow 0} \mathcal{S}_\mu(t_1, t_2) =: \mathcal{S}(t_1, t_2) = \int_0^h \int_{t_1}^{t_2-h} (\nabla \dot{u}, \zeta^2(E_t^s E_i^h - I)\dot{\sigma})_\Omega dt ds \end{aligned}$$

for all t_1, t_2 with $0 \leq t_1 < t_2 - h \leq T - h$. To estimate the term $\mathcal{R}_\mu(t_1, t_2)$, we use the following argument: If $\omega \in L^\infty(L^1)$, then for all t_1, t_2 such that $0 \leq t_1 \leq t_2 - h \leq T - h$ it follows

$$(5.16) \quad \left| \int_{t_1}^{t_2-h} \int_\Omega \zeta^2(E_t^s E_i^h - I)\omega dx dt \right| \leq Ch \|\omega\|_{L^\infty(L^1)}.$$

⁽³⁾ Like in the proof of Theorem 2.1, in the first term it suffices to use the weak convergence and lower semi-continuity.

Indeed, rewriting the integral we find

$$\begin{aligned} & \left| \int_{t_1}^{t_2-h} \int_{\Omega} \zeta^2 (E_t^s E_i^h - I) \omega \, dx \, dt \right| = \left| \int_{t_1}^{t_2-h} \int_{\Omega} \zeta^2 (E_t^s - I) E_i^h \omega + \zeta^2 (E_i^h - I) \omega \, dx \, dt \right| \\ & \leq \left| \int_{t_1+s}^{t_2-h+s} \int_{\Omega} \zeta^2 E_i^h \omega \, dx \, dt - \int_{t_1}^{t_2-h} \int_{\Omega} \zeta^2 E_i^h \omega \, dx \, dt \right| + \left| \int_{t_1}^{t_2-h} \int_{\Omega} \zeta^2 (E_i^h - I) \omega \, dx \, dt \right| \\ & \leq \left| \int_{t_1}^{t_1+s} \int_{\Omega} \zeta^2 E_i^h \omega \, dx \, dt \right| + \left| \int_{t_2-h}^{t_2-h+s} \int_{\Omega} \zeta^2 E_i^h \omega \, dx \, dt \right| + \left| \int_{t_1}^{t_2-h} \int_{\Omega} \zeta^2 (E_i^h - I) \omega \, dx \, dt \right| \\ & \leq C h \|\omega\|_{L^\infty(L^1)}, \end{aligned}$$

here we also used $|(E_i^h - I)\zeta| \leq ch$.

Hence, taking also the integration over s into account, it follows

$$|\mathcal{R}_\mu(t_1, t_2)| \leq C h^2 (\|\dot{\sigma}_\mu\|_{L^\infty(L^2)}^2 + \|\dot{\xi}_\mu\|_{L^\infty(L^2)}^2),$$

where C is independent of μ and h . With Lemma 5.2 it follows now for almost all t_1, t_2 , such that $0 \leq t_1 \leq t_2 - h \leq T - h$:

$$\mathcal{L}(t_1, t_2) \leq \limsup_{\mu \rightarrow 0} \mathcal{T}_{i\mu}(t_1, t_2) + \limsup_{\mu \rightarrow 0} |\mathcal{R}_\mu(t_1, t_2)| - \mathcal{S}(t_1, t_2) \leq Ch^2 - \mathcal{S}(t_1, t_2),$$

where the constant depends neither on h nor on t_1, t_2 . Since \mathcal{L} and \mathcal{S} depend continuously on t_1 and t_2 , we obtain

$$(5.17) \quad \mathcal{L}(0, T) \leq Ch^2 + |\mathcal{S}(0, T)|.$$

Due to the positivity assumptions on A and H we have

$$\mathcal{L}(0, T) \geq C \int_0^h \int_0^{T-h} \int_{\Omega} (|(E_t^s E_i^h - I)\dot{\sigma}|^2 + |(E_t^s E_i^h - I)\dot{\xi}|^2) \zeta^2 \, dx \, dt \, ds.$$

Now we apply the argument

$$\begin{aligned} |(E_t^s E_i^h - I)\dot{\sigma}|^2 &= |(E_t^s E_i^h - E_i^h + E_i^h - I)\dot{\sigma}|^2 \geq \frac{7}{8} |A_i^h \dot{\sigma}|^2 - \frac{1}{8} |A_t^s E_i^h \dot{\sigma}|^2, \\ |(E_t^s E_i^h - I)\dot{\xi}|^2 &\geq \frac{7}{8} |A_i^h \dot{\xi}|^2 - \frac{1}{8} |A_t^s E_i^h \dot{\xi}|^2. \end{aligned}$$

We combine this with (5.17), use the translation invariance of integrals and

arrive at

$$\int_0^h \int_0^{T-h} \int_{\Omega} (|\mathcal{A}_i^h \dot{\sigma}|^2 + |\mathcal{A}_i^h \dot{\xi}|^2) \zeta^2 dx dt ds = h \int_0^{T-h} \int_{\Omega} (|\mathcal{A}_i^h \dot{\sigma}|^2 + |\mathcal{A}_i^h \dot{\xi}|^2) \zeta^2 dx dt$$

$$\leq ch^2 + |\mathcal{S}(0, T)| + \tilde{C} \int_0^h \int_0^{T-h} \int_{\Omega} (|\mathcal{A}_i^s \dot{\sigma}|^2 + |\mathcal{A}_i^s \dot{\xi}|^2) \zeta^2 dx dt ds.$$

Finally we multiply this inequality by h^{-2} , then the assertion follows from Lemma 5.3 and Theorem 2.1. □

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REFERENCES

- [1] H.-D. ALBER, *Materials with memory*, volume 1682 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1998. Initial-boundary value problems for constitutive equations with internal variables.
- [2] H.-D. ALBER - S. NESENEKO, *Local H^1 -regularity and $H^{1/3-\delta}$ -regularity up to the boundary in time dependent viscoplasticity*. *Asymptot. Anal.*, **63** (3) (2009),151-187.
- [3] J. FREHSE - D. LÖBACH, *Hölder continuity for the displacements in isotropic and kinematic hardening with von Mises yield criterion*. *ZAMM Z. Angew. Math. Mech.*, **88** (8) (2008), 617-629.
- [4] J. FREHSE - D. LÖBACH, *Regularity results for three-dimensional isotropic and kinematic hardening including boundary differentiability*. *Math. Models Methods Appl. Sci.*, **19** (12) (2009), 2231-2262.
- [5] J. FREHSE - D. LÖBACH, *Improved L^p -estimates for the strain velocities in hardening problems*. *GAMM-Mitteilungen*, **34** (2011), 10-20.
- [6] J. FREHSE - M. SPECOVIVUS-NEUGEBAUER, *Fractional differentiability for the stress velocities to the solution of the Prandtl-Reuss problem*. To appear in *ZAMM Z. Angew. Math. Mech.*
- [7] P. GRUBER - D. KNEES - S. NESENEKO - M. THOMAS, *Analytical and numerical aspects of time-dependent models with internal variables*. *ZAMM Z. Angew. Math. Mech.*, **90** (10-11, Special Issue: On the Occasion of Zenon Mroz’ 80th Birthday) (2010), 861-902.
- [8] C. JOHNSON, *Existence theorems for plasticity problems*. *J. Math. Pures Appl.* (9), **55** (4) (1976), 431-444.
- [9] C. JOHNSON, *On plasticity with hardening*. *J. Math. Anal. Appl.*, **62**(2) (1978), 325-336.
- [10] D. KNEES, *On global spatial regularity and convergence rates for time dependent elasto-plasticity*. *M3AS*, **20** (2010), 1823-1858.

- [11] J.-L. LIONS - LIONS - E. MAGENES, *Non-homogeneous boundary value problems and applications*. Vol. I. Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.
- [12] D. LÖBACH, *On regularity for plasticity with hardening*. Technical Report 388, Bonner Mathematische Schriften, 2008.
- [13] G. A. SERÉGIN, *Differential properties of solutions of evolution variational inequalities in the theory of plasticity*. J. Math. Sci., **72** (6) (1994), 3449-3458.

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