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A Note on a Discrete Version of Borg's Theorem via Toeplitz–Laurent Operators with Matrix-Valued Symbols

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Abstract. – *Consider a one dimensional Schrödinger operator* $\tilde{A} = -\ddot{u} + V \cdot u$ *with a periodic potential* $V(\cdot)$*, defined on a suitable subspace of* $L^2(\mathbb{R})$*. Its spectrum is the union of closed intervals, and in general these intervals are separated by open intervals (spectral gaps). The Borg theorem states that we have no gaps if and only if the periodic potential* $V(\cdot)$ *is constant almost everywhere. In this paper we consider families of Finite Difference approximations of the operator* $\tilde{A}$*, which depend upon two parameters* n*, i.e., the number of periodicity intervals possibly infinite, and* p*, i.e., the precision of the approximation in each interval. We show that the approach, with fixed* p*, leads to families of sequences* $\{A_n(p)\}$*, where every matrix* $A_n(p)$ *can be interpreted as a block Toeplitz matrix generated by a* $p \times p$ *matrix-valued symbol* f*. In other words, every* $A_n(p)$ *with finite* n *is a finite section of the double infinite Toeplitz–Laurent operator* $A_\infty(p) = L(f)$*. The specific feature of the symbol* f*, which is a trigonometric polynomial of 1st degree, allows to identify the distribution of the collective spectra of the matrix-sequence* $\{A_n(p)\}$*, and, in particular, provide a simple way for proving a discrete version of Borg's theorem: the discrete operator* $L(f)$ *has no gaps if and only if the corresponding "potential" is constant. The result partly overlaps with known results by Flaschka from the operator theory. The main novelty here is the purely linear algebra approach.*

1. – Introduction and description of the problem

When considering self adjoint operators $\tilde{A}$ coming from Chemistry or Mathematical Physics [14], one is interested in the spectral gaps, because they represent the region of instability of the associated eigenvalue problem $\tilde{A}u = \lambda u$. An interval I is called a spectral gap if there exist real sets J_1, J_2 such that their union contains the spectrum of $\tilde{A}$, and $\sup J_1 \leq \inf I < \sup I \leq \inf J_2$. Historically, gap related problems have been studied with special attention for Schrödinger operators (see e.g. [7, 9, 10, 14]). We are interested in the one dimensional Schrödinger operator $\tilde{A} = -\ddot{u} + V \cdot u$ with the periodic potential, defined on a suitable subspace of $L^2(\mathbb{R})$. It can be proved (see, e.g., [14, Theorem XIII.90] that the spectrum is the union of closed intervals. In some cases these intervals may be separated by nonempty open intervals. By taking into account the above definition, it is evident that all these nonempty open sets are spectral gaps. For instance, for the Mathieu operator, which is defined by the potential

$V(x) = \beta \cos x$ with a certain nonzero constant β, it is known that all the spectral gaps are open; see [14, Example 1, p. 298]. A summary of general and elegant classical results regarding the Schrödinger operator with periodic potentials is reported below (see [14, Theorem XIII.91]).

THEOREM 1.1. – *Take the one dimensional Schrödinger operator*

$$\tilde{A} = -\ddot{u} + V \cdot u \tag{1}$$

with the periodic potential V, defined on a suitable subspace of $L^2(\mathbb{R})$.

- *There are no gaps in the spectrum if and only if the potential function reduces to a constant (Borg's theorem; see* [3], [11]).
- *If there exists exactly one gap, then the potential is an elliptic function.*
- *If there are finitely many gaps, then the potential is a real analytic function.*

In this paper we state the Borg type theorems in the case of discrete Schrödinger operator with a periodic potential. Moreover, we convert other results as those regarding the spectral distribution, in the spirit of the Szego theorems [13]. The main tools are the use of Finite Differences for identifying the analogous discrete operators and a formulation of the discrete problem in terms of block Toeplitz sequences with $p \times p$ matrix-valued symbols. We consider families of Finite Difference approximations of the operator $\tilde{A}$, which depend upon two parameters, n, the number of periodicity intervals, and p the precision of the approximation in each interval. We show that the approach, with fixed p, leads to families of sequences $\{A_n(p)\}$, where each matrix $A_n(p)$ can be interpreted as a block Toeplitz matrix generated by a $p \times p$ matrix-valued symbol. Indeed, the parameter p is the periodicity index, which appears on the diagonal of the approximating matrices, where the periodicity is induced by that of the potential V. In fact, the entries on the diagonal are, up to a proper scaling, related to the finesse discretization parameter $1/(p+1)$, exact samplings of the potential $V(\cdot)$ in equispaced points. The result partly overlaps with known results by Flaschka [12] from the operator theory (see, e.g., [8, Theorem 1.3]). In contrast, our approach relies on basic tools from linear algebra.

The paper is organized as follows. In Section 2 we describe the process of approximation of the Schrödinger operator, that leads to the families of matrix-sequences $\{A_n(p)\}$ and to the Toeplitz–Laurent operator $A_\infty(p) = L(f)$. Section 3 deals with basic notions, definitions and preliminary results. Section 4 contains the main results on the discrete Borg's theorem via block Toeplitz–Laurent operators. Section 5 is devoted to remarks, conclusions, and future lines of investigation.

2. – From continuous to discrete

In this section we propose a simple (in fact the simplest) Finite Differences approximation for the Schrödinger operator with a periodic potential $V(\cdot)$. Without loss of generality, we assume that the periodicity width is 1, that is, $V(x+1) = V(x)$ for every $x \in \mathbb{R}$. We approximate equation (1) in the interval $[-n, n]$ with $n \in \mathbb{N} \cup \{\infty\}$ by the standard difference

$$\frac{-u(x_{i+1,(j)}) + 2u(x_{i,(j)}) - u(x_{i-1,(j)})}{h^2}$$

with $h = h(p) = 1/p$, $x_{s,(j)} = j + sh(p)$, $j = -n, \ldots, n-1$, $s = 0, \ldots, p$ by using p equispaced points in each interval $[j, j+1] \subset [-n, n]$. In this way, letting $n = \infty$, we find a tridiagonal (Jacobi) matrix that, up to the scaling factor h^2, coincides with

$$A_n(p) = \begin{bmatrix} \ddots & \ddots & & & \\ \ddots & \ddots & \ddots & & \\ & -1 & 2+w_s & -1 & \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots \end{bmatrix}, \quad w_j = h^2 V(x_{s,(j)}), \tag{2}$$

$j = 0, \ldots, p-1$. Given the periodicity this matrix can be re-written as

$$A_n(p) = \begin{bmatrix} \ddots & \ddots & & & & & \\ \ddots & \ddots & \ddots & & & & \\ & -1 & 2+w_0 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 2+w_{p-1} & -1 & \\ & & & & -1 & 2+w_0 & -1 \\ & & & & & \ddots & \ddots & \ddots \\ & & & & & & \ddots & \ddots \end{bmatrix}. \tag{3}$$

When n is finite, the resulting matrix of the size np is just a truncation of the double infinite matrix reported above.

Along the same lines, we may consider the variable coefficient one dimensional Schrödinger operator $\tilde{A}(a) = -\dfrac{d}{dx}\left(a \cdot \dfrac{d}{dx} u\right) + V \cdot u$, with a positive and periodic function $a(\cdot)$, having the same period as that of $V(\cdot)$. In that case, the very same type of finite difference approximation will lead to a bi-infinite symmetric matrix of the form

$$
(4) \qquad A_{n,a}(p) = \begin{bmatrix}
\ddots & \ddots & & & & & & \\
\ddots & \ddots & \ddots & & & & & \\
& -\alpha_{p-1} & \gamma_0 & -\alpha_0 & & & & \\
& & \ddots & \ddots & \ddots & & & \\
& & & -\alpha_{p-2} & \gamma_{p-1} & -\alpha_{p-1} & & \\
& & & & -\alpha_{p-1} & \gamma_0 & -\alpha_0 & \\
& & & & & \ddots & \ddots & \ddots \\
& & & & & & \ddots & \ddots
\end{bmatrix},
$$

with $\gamma_s = \alpha_s + \alpha_{(s-1)\, mod\, p} + h^2 V(x_{s,(j)})$, $\alpha_s = a(x_{s+1/2,(j)})$, $x_{s+1/2,(j)} = j + h(p)(s + 1/2)$. We observe that resulting structure, up to the sign, represents the case of general p-periodic Jacobi matrices.

3. – Preliminary results and notation

This section is divided into two parts. In the first one we briefly recall few results concerning the spectra of Toeplitz–Laurent operators with matrix-valued symbols, while in the second one we give a definition of the spectral distribution and provide the results regarding again the case of Toeplitz sequences coming from sections of infinite Toeplitz operators.

The connections among these ingredients will become evident in Section 4, since the approximation of the Schrödinger operator with periodic potential by using Finite Differences as in Section 2 leads to matrix-sequences $\{A_n(p)\}$, where p is a parameter associated with the precision of the approximation. For every p the sequence $\{A_n(p)\}$ can be interpreted as sections of a Toeplitz operator with a $p \times p$ matrix-valued symbol of polymomial type. The results in Section 3.1 allow us to prove the main results on the discrete version of Borg's theorem in Section 4, while the results in Section 3.2 are of interest for the distributional analysis.

3.1 – *Toeplitz operators and sequences*

Given a $p \times p$ matrix-valued integrable function f on the unit circle $\mathbb{T}$, the $p \times p$ matrices f_j, $j \in \mathbb{Z}$, represent the Fourier coefficients of f defined as

$$
f_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-ij\theta} d\theta, \quad j = 0, \pm 1, \pm 2, \dots .
$$

Then for n being a nonnegative integer number or ∞ we define $T_n(f)$ the Toeplitz matrix or operator of size n generated by f via the relations

$$(T_n(f))_{i,j} = f_{i-j}, \qquad i,j = 1,\dots,n.$$

When $n = \infty$, the Toeplitz operator $T_n(f)$ is simply written as $T(f)$. Similarly, $L(f)$ denotes the double infinite Toeplitz matrix with $(L(f))_{i,j} = f_{i-j}$, $i,j \in \mathbb{Z}$. Furthermore, by $\{T_n(f)\}$ we indicate the Toeplitz matrix-sequence generated by f, with $T_n(f)$ of finite order. The function f is referred to as the symbol.

Given a Hermitian $p \times p$ matrix A, let $\lambda_1(A) \geq \cdots \geq \lambda_p(A)$ denote its eigenvalues labelled in the decreasing order. Let f be a continuous and Hermitian symbol. It is well known that the essential spectra of both $L(f)$ and $T(f)$ coincide with the union of the ranges of the eigenvalues $\lambda_1(f(\cdot)) \geq \cdots \lambda_p(f(\cdot))$, that is,

$$\sigma_{ess}(L(f)) = \sigma_{ess}(T(f)) = \bigcup_{j=1}^{p} \left[\inf_{\theta} (\lambda_j(f(e^{i\theta}))), \sup_{\theta} (\lambda_j(f(e^{i\theta}))) \right]. \tag{5}$$

For the latter result, which is crucial for our approach to the discrete version of the Borg theorem, see [4, Proposition 2.29(a)].

3.2 – *Spectral Distributions*

Here we give the definition of spectral distribution concerning matrix-sequences of increasing size and report a distribution result for block Toeplitz sequences in the spirit of Weyl.

DEFINITION 3.1. – *Let $\mathcal{C}_0(\mathbb{C})$ be the set of continuous functions with bounded support defined over the complex plane, N a positive integer, and ψ a $p \times p$ matrix-valued measurable function defined on a set $G \subset \mathbb{R}^N$ of finite and positive Lebesgue measure $\mu(G)$. A matrix-sequence $\{A_n\}$ is said to be* distributed (in the sense of the eigenvalues) as the pair (ψ, G), *or to* have the eigenvalue distribution function ψ *($\{A_n\} \sim_\lambda (\psi, G)$), if, $\forall F \in \mathcal{C}_0(\mathbb{C})$, the following limit relation holds*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F\big(\lambda_j(A_n)\big) = \frac{1}{\mu(G)} \int_G \frac{1}{p} \sum_{s=1}^{p} F(\lambda_s(\psi(t)))\, dt, \qquad t = (t_1,\dots,t_N). \tag{6}$$

Concerning the spectral distribution of Toeplitz matrix-sequences, the main result is the Theorem of Szegö (see [5]), that was reported in its most general version due to Tilli [16].

THEOREM 3.2 (Szegö-Tilli). – *Let f be a $p \times p$ matrix-valued integrable function defined on $\mathbb{T}$, and let $\{T_n(f)\}$ be the block Toeplitz sequence generated* *by f. Assume that f is Hermitian almost everywhere on its definition set. Then*

$$\{T_n(f)\} \sim_\lambda (f, \mathbb{T}),$$

that is, for every function F continuous with bounded support we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(\lambda_j(T_n(f))) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{p} \sum_{s=1}^{p} F(\lambda_s(f(e^{i\theta})))d\,\theta.$$

where $\lambda_j(A)$ are the eigenvalues of a square matrix A.

4. – The discrete Borg Theorem

Our aim is to suggest a pure linear algebraic approach to the discrete version of the celebrated Borg theorem.

4.1 – *Periodic Jacobi matrices with all gaps closed*

We begin with a double infinite, p-periodic, $p \geq 2$, real Jacobi matrix

$$J = \begin{bmatrix} \ddots & \ddots & & & & & & \\ \ddots & b_1 & a_1 & & & & & \\ & a_1 & b_2 & a_2 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & \ddots & b_p & a_p & & \\ & & & & a_p & b_1 & a_1 & \\ & & & & & & \ddots & \ddots \end{bmatrix}, \quad a_{n+p} = a_n > 0, \quad b_{n+p} = b_n.$$

We follow the standard convention $a_n > 0$, which differs by sign from (4): indeed the bi-infinite matrix reported in (4) is easily converted into a Jacobi matrix multiplying it by -1.

An important observation is that $J = L(f_0)$, where in the case $p \geq 3$ the symbols are given by

$$(7)\quad f_k(e^{i\theta}) = \begin{bmatrix} b_{k+1} & a_{k+1} & 0 & \dots & e^{-i\theta}a_{k+p} \\ a_{k+1} & b_{k+2} & a_{k+2} & \dots & 0 \\ 0 & a_{k+2} & b_{k+3} & \dots & \\ \vdots & \vdots & \vdots & \ddots & a_{k+p-1} \\ e^{i\theta}a_{k+p} & 0 & \dots & a_{k+p-1} & b_{k+p} \end{bmatrix}, \quad k = 0, 1, \dots, p-1,$$

(for the case $p = 2$ see Example 4.1 below).

Denote by $\lambda_j(f_k)$, $j = 1, 2, \ldots, p$ the eigenvalues of f_k, arranged in the decreasing order

$$\lambda_1(f_l(\theta)) \geq \lambda_2(f_l(\theta)) \geq \ldots \geq \lambda_p(f_l(\theta)),$$

and put

$$\lambda^+_{j,k} := \max_\theta \lambda_j(f_k(\theta)), \qquad \lambda^-_{j,k} := \min_\theta \lambda_j(f_k(\theta)).$$

The spectrum of the original matrix J is (see Section 3.1 above)

$$\sigma(J) = \bigcup_{j=1}^{p} [\lambda^-_{j,k}, \lambda^+_{j,k}],$$

and the right hand side does not depend on k.

DEFINITION 4.1. – *We say that J has no gaps in the spectrum (in other words, all gaps are closed) if $\lambda^-_{j,k} = \lambda^+_{j+1,k}$ for all $j = 1, 2, \ldots, p-1$.*

If $p = 1$ (Jacobi matrices with constant entries) the spectrum is an interval, so the gaps are automatically closed.

Consider the matrices of order $p - 1$

$$J_k := \begin{bmatrix} b_{k+1} & a_{k+1} & 0 & \ldots & \\ a_{k+1} & b_{k+2} & a_{k+2} & \ldots & \\ 0 & a_{k+2} & b_{k+3} & \ldots & \\ \vdots & \vdots & \vdots & \ddots & a_{k+p-2} \\ & & \ldots & a_{k+p-2} & b_{k+p-1} \end{bmatrix}, \quad k = 0, 1, \ldots, p-1,$$

and put

$$\sigma(J_k) = \{\mu_{1,k} > \mu_{2,k} > \ldots > \mu_{p-1,k}\}.$$

The Cauchy interlacing properties for eigenvalues of Hermitian matrices lead to the following inequalities

$$\lambda_1(f_k(\theta)) \geq \mu_{1,k} \geq \lambda_2(f_k(\theta)) \geq \ldots \geq \lambda_{p-1}(f_k(\theta)) \geq \mu_{p-1,k} \geq \lambda_p(f_k(\theta)),$$

$$\lambda_1(f_k(\theta)) \geq \mu_{1,k+1} \geq \lambda_2(f_k(\theta)) \geq \ldots \geq \lambda_{p-1}(f_k(\theta)) \geq \mu_{p-1,k+1} \geq \lambda_p(f_k(\theta)).$$

PROPOSITION 4.2. – *Suppose that J has no gaps. Then all J_k have the same spectrum.*

PROOF. – Assuming the contrary, we would have $\mu_{j,k'} > \mu_{j,k''}$, so by the interlacing properties there is a gap in the spectrum. $\square$

REMARK 4.1. – Let $b_j = 0$, $a_1 = a_3 = \ldots$, $a_2 = a_4 = \ldots$, but $a_1 \neq a_2$. Put $p = 4$. Then

$$\det(J_0 + \lambda) = \det(J_1 + \lambda) = \begin{vmatrix} \lambda & a_1 & 0 \\ a_1 & \lambda & a_2 \\ 0 & a_2 & \lambda \end{vmatrix} = \lambda^3 - (a_1^2 + a_2^2)\lambda,$$

so $\sigma(J_0) = \sigma(J_1)$, and the converse to Proposition 4.2 is false.

PROPOSITION 4.3. – *Suppose that J has no gaps. Then $b_1 = b_2 = \ldots = b_p$.*

PROOF. – By Proposition 4.2 $\mathrm{tr}(J_0) = \mathrm{tr}(J_1) = \ldots = \mathrm{tr}(J_{p-1})$ and so

$$\sum_{j=1}^{p-1} b_j = \sum_{j=2}^{p} b_j = \ldots = \sum_{j=p}^{2p-2} b_j,$$

which implies $b_1 = b_p$, $b_2 = b_{p+1} = b_1$, $b_3 = b_{p+2} = b_2$ etc. as claimed. □

With no loss of generality we put $b_j = 0$. Assume also that the period p is an even number (otherwise take $2p$ as the period, see also Remark 4.2 below).

We proceed with the simple case of $p = 2$.

EXAMPLE 4.1. – Let $p = 2$ and $b_j = 0$. The symbol now is

$$f(\theta) = \begin{bmatrix} 0 & a_1 + e^{-i\theta} a_2 \\ a_1 + e^{i\theta} a_2 & 0 \end{bmatrix}, \quad \lambda_{1,2}(f(\theta)) = \pm |a_1 + e^{-i\theta} a_2|.$$

All gaps are closed if and only if $\min_\theta |a_1 + e^{-i\theta} a_2| = 0$, or equivalently, $a_1 = a_2$, so $p = 1$.

The following result is a version of the celebrated Borg theorem for Jacobi matrices.

THEOREM 4.4. – *Let $p = 2m + 2$ and J has all its gaps closed. Then $a_1 = a_2 = \ldots = a_p$, so the actual period is $p = 1$.*

PROOF. – Denote

$$D(\lambda; a_1, a_2, \ldots, a_n) := \begin{vmatrix} \lambda & a_1 & 0 & \ldots & \\ a_1 & \lambda & a_2 & \ldots & \\ 0 & a_2 & \lambda & \ldots & \\ \vdots & \vdots & \vdots & \ddots & a_n \\ & & \ldots & a_n & \lambda \end{vmatrix},$$

By expanding over the last row and induction we see that

$$D(\lambda; a_1, a_2, \ldots, a_n) = \lambda^{n+1} - \lambda^{n-1} \sum_{j=1}^{n} a_j^2 + \ldots.$$

By Proposition 4.2

$$\det(J_k + \lambda) = D(\lambda; a_{k+1}, \ldots, a_{k+2m}) = \lambda^{2m+1} - \lambda^{2m-1} \sum_{j=k+1}^{k+2m} a_j^2 + \ldots$$

does not depend on k so

$$\sum_{j=k+1}^{k+2m} a_j^2 = \sum_{j=k+2}^{k+2m+1} a_j^2 \quad \Rightarrow \quad a_{k+1} = a_{k+2m+1}, \quad k = 0, 1, \ldots, p-1,$$

Hence $a_1 = a_3 = \ldots = a_{2m+1}$, $a_2 = a_4 = \ldots = a_{2m+2}$, so $p = 2$. By Example 4.1 $a_1 = a_2 = \ldots = a_p$, as claimed. $\square$

REMARK 4.2. – For the odd period $p = 2m + 1$ the argument is simple. Since

$$\begin{vmatrix} 0 & x_1 & 0 & \ldots & \\ x_1 & 0 & x_2 & \ldots & \\ 0 & x_2 & 0 & \ldots & \\ \vdots & \vdots & \vdots & \ddots & x_{2m-1} \\ & & \ldots & x_{2m-1} & 0 \end{vmatrix} = (-1)^m \prod_{j=1}^{m} x_{2j-1}^2,$$

then $\det J_k = (-1)^m \prod_{j=1}^{m} a_{k+2j-1}^2$. But the left hand side is independent from k, so $a_{k+1} = a_{k+2m+1} = a_{k+p} = a_k$.

4.2 – *A specific example: ω-circulants*

We consider $f_k(\theta)$ with all b_j equal to zero and all a_j equal to one, $j = 0, \ldots, p-1$. The related matrix structure is a ω circulant, independent of k, with $\omega = e^{-i\theta}$, so that all the eigenvalues and all singular values of $f(\theta) = f_k(\theta)$ are known in closed form

$$f(\theta) = \begin{bmatrix} 0 & 1 & 0 & \ldots & e^{-i\theta} \\ 1 & 0 & 1 & \ddots & 0 \\ 0 & 1 & 0 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & 1 \\ e^{i\theta} & 0 & \ldots & 1 & 0 \end{bmatrix}.$$

Now call Z the generator of circulants that is

$$Z = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \ddots & 0 \\ 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}.$$

Then it is well known that

$$Z = F_p D F_p *$$

where

$$F_p = \sqrt{\frac{1}{p}} \left(e^{-i\frac{2\pi jk}{p}} \right)_{j,k=0}^{p-1}$$

is the celebrated Fourier matrix of size p and

$$D = \mathrm{diag}_{j=0,\dots,p-1} \left(e^{i\frac{2\pi j}{p}} \right).$$

Now consider

$$Z_\omega = \begin{bmatrix} 0 & 0 & 0 & \dots & \omega \\ 1 & 0 & 0 & \ddots & 0 \\ 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 0 & 1 & 0 \end{bmatrix}.$$

Then a direct computation shows that

$$Z_\omega = \delta D_\delta^{-1} Z D_\delta, \qquad D_\delta = \mathrm{diag}_{j=0,\dots,p-1} \left(\delta^j \right)$$

with $\delta^p = \omega$. Therefore, taking into account that $f(\theta) = Z_{e^{-i\theta}} + Z^*_{e^{-i\theta}}$, we have a complete Jordan decomposition of our symbol as

$$\begin{aligned} f(\theta) &= D^*_{e^{-i\theta/p}} F_p \left[e^{-i\theta/p} D + e^{i\theta/p} D^* \right] F_p^* D_{e^{-i\theta/p}} \\ &= D^*_{e^{-i\theta/p}} F_p \, \mathrm{diag}_{j=0,\dots,p-1} \left(2\cos\left(\frac{2\pi j - \theta}{p} \right) \right) F_p^* D_{e^{-i\theta/p}}. \end{aligned}$$

With reference to the previous notations we observe that for fixed j and p large we have

$$\lambda_j^+ - \lambda_j^- = \lambda_{p-j}^+ - \lambda_{p-j}^- \sim p^{-2},$$

while, for indices j in a fixed neighborhood of $p/2$, with size independent of p, we have

$$\lambda_j^+ - \lambda_j^- \sim p^{-1}.$$

Here the symbol $\alpha_p \sim \beta_p$ for nonnegative sequences α_p, β_p is equivalent to say that simultaneously we have $\alpha_p = O(\beta_p)$ and $\beta_p = O(\alpha_p)$. Finally, for all indices j, we obtain $\lambda_j^+ - \lambda_j^- = O\big(p^{-1}\big)$.

4.3 – *A specific example: essential period p implies p disjoint spectral intervals*

We say that J has essential period p if p is the minimal positive integer for which $a_{n+p} = a_n > 0$, $b_{n+p} = b_n$, for all integer n. Here we give a specific example that support the general statement that J of essential period p implies p disjoint spectral intervals, that is exactly $p-1$ gaps. Consider $f_0(\theta)$ with $a_1 = \cdots = a_p = 1$ and $b_1 = \cdots = b_{p-1} = 0$, $b_p = 1$. The last relation implies that the essential period is p. Now following the argument in the previous section, we deduce that the eigenvalues of $f_0(\theta)$ are separated by those of

$$H_q = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}.$$

and those of

$$\hat{H}_q = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \dots & 0 & 1 & 1 \end{bmatrix}.$$

with $q = p-1$ and H_q, $\hat{H}_q$ being principal minors of $f_0(\theta)$. In order to prove that there exist exactly $q = p-1$ gaps, it is enough to prove that $\lambda_j(\hat{H}_q) > \lambda_j(H_q)$, for every $j = 1, \dots, q$. The matrices H_q and $\hat{H}_q$ are the generators of sine-transform algebras with different boundary conditions and their Jordan form can be explicitly computed.

In both case we observe that the matrices are real symmetric and irreducible so that the use of the first and of the third Gershgorin theorem implies that the eigenvalues belong to the open interval $(-2, 2)$. Let $\lambda(X)$ be a generic eigenvalue of $X \in \{H_q, \hat{H}_q\}$ and let $v(X)$ the corresponding normalized eigenvector. Setting $\lambda(X)/2 = \cos(\psi)$, we have

$$v_{i-1}(X) + v_{i-1}(X) = 2\cos(\psi)v_i(X), \qquad i = 1, \ldots, q$$

with boundary conditions given by $v_0(H_q) = v_q(H_q) = 0$ and $v_0(\hat{H}_q) = 0$, $v_q(\hat{H}_q) = v_p(\hat{H}_q)$. The general solution of the latter linear difference equation is then given in both cases by $v_j(X) = A_X e^{ij\psi} + B_X e^{-ij\psi}$ so that the use of the boundary conditions shows that

$$\lambda_j(\hat{H}_q) = 2\cos\left(\frac{\pi(2j-1)}{2q+1}\right) > \lambda_j(H_q) = 2\cos\left(\frac{\pi j}{q+1}\right), \qquad j = 1, \ldots, q,$$

where the strict inequality

$$2\cos\left(\frac{\pi(2j-1)}{2q+1}\right) > 2\cos\left(\frac{\pi j}{q+1}\right)$$

is true simply because $\dfrac{\pi(2j-1)}{2q+1} < \dfrac{\pi j}{q+1}$ for every $j = 1, \ldots, q$.

5. – Conclusions and Remarks

We start with some observations on the sequence $\{A_n(p)\}$. Clearly each $A_n(p)$ can be viewed as $T_n(f_k) + R_{n,k}$, where f_k, $k = 0, \ldots, p-1$, are those reported in (7), and where the correction term $R_{n,k}$ is Hermitian for every n and k, and has rank bounded uniformly by p. Therefore in the light of general perturbation results (see, e.g., [15, Proposition 2.3]), and by Theorem 3.2, we have

$$\{A_n(p)\} \sim_\lambda (f_k, \mathbb{T}), \qquad k = 0, \ldots, p-1, \tag{8}$$

which clearly implies that, for every j, the range of $\lambda_j(g_s)$ coincides with the range of $\lambda_j(g_t)$, for $s, t = 0, \ldots, p-1$. Indeed, we know even more. By (8), for each continuous function F with a bounded support, we obtain that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{p}\sum_{k=1}^{p} F(\lambda_k(f_s(e^{i\theta})))d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{p}\sum_{k=1}^{p} F(\lambda_k(f_t(e^{i\theta})))d\theta, \tag{9}$$

$s, t = 0, \ldots, p-1$, which means that the eigenvalues of each f_s induce the same measure on the real line, $s = 0, \ldots, p-1$.

Concerning future work, there are some interesting issues that should be addressed. These operators could be considered in multidimensional domains and in the case of systems of equations (as in [11]). A further intriguing issue could be the following: how to relate the number of gaps to the periodicity index p of the diagonal periodic sequence, the latter question being supported by the fact that no gaps is equivalent to have all equal diagonal entries and by the example reported in Section 4.3.

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