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MATTEO PENEGINI

On the Classification of Surfaces of General Type with $p_g = q = 2$

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MATTEO PENEGINI

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0. – Introduction

The following is an extended version of the talk which I gave at the “XIX Congresso dell’UMI” in Bologna in September 2011. The aim of this paper is twofold: first, to give an overview on the recent development in the classification of surfaces of general type with $p_g = q = 2$; second, to point out some of the problems that are still open. Some of the results that will appear in this article have been proven in collaboration with Francesco Polizzi.

We shall use the standard notation from the theory of complex algebraic surfaces. Let S be a smooth, complex, projective *minimal* surface S of *general type*; this means that the canonical divisor K_S of S is *big* and *nef*.

The principal numerical invariants for the study of minimal surfaces of general type are

- the *geometric genus* $p_g(S) := h^0(S, \Omega_S^2) = h^0(S, \mathcal{O}_S(K_S))$,
- the *irregularity* $q(S) := h^0(S, \Omega_S^1)$, and
- the *self intersection of the canonical divisor* K_S^2 .

As a matter of fact, these determine all the other classical invariants, as

- the *Euler-Poincaré characteristic* $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$,

- the topological Euler number $e(S) = 12\chi(S) - K_S^2$, and
- the plurigenera $P_n(S) = \chi(S) + \binom{n}{2}K_S^2$.

By a theorem of Bombieri, a minimal surface of general type S with fixed invariants is birationally mapped to a normal surface X in a fixed projective space of dimension $P_5(S) - 1$. Moreover, X is uniquely determined and is called the *canonical model* of S . Let us recall Gieseker's Theorem (see [Gie77]).

THEOREM 0.1. – *There exists a quasi-projective coarse moduli space $\mathcal{M}_{K_S^2, \chi}$ for canonical models of surfaces of general type S with fixed invariants K_S^2 and χ .*

In particular, we can also consider the subscheme $\mathcal{M}_{K_S^2, p_g, q} \subset \mathcal{M}_{K_S^2, \chi}$ of surfaces with fixed K_S^2 , $p_g(S) = p_g$ and $q(S) = q$, which is a union of connected components of $\mathcal{M}_{K_S^2, \chi}$. The first basic question we would like to answer is

QUESTION 0.2. – *For which values of (K_S^2, p_g, q) is $\mathcal{M}_{K_S^2, p_g, q}$ not empty?*

Much is known about this question; indeed, we have the following classical inequalities:

- $K_S^2 \geq 1$ and $\chi \geq 1$.
- $K_S^2 \leq 9\chi(S)$ (Bogomolov-Miyaoka-Yau).
- $K_S^2 \geq 2p_g$, if $q > 0$, (Debarre).
- $K_S^2 \geq 2\chi - 6$ (Noether).

Once we have established that $\mathcal{M}_{K_S^2, p_g, q} \neq \emptyset$, the second basic question is

QUESTION 0.3. – *Can we describe $\mathcal{M}_{K_S^2, p_g, q}$?*

For example, can we find out the number of connected components it consists and their dimensions?

We are interested in surfaces with small invariants, and in particular in surfaces with $\chi(S) = 1$, which is the smallest possible value for a surface of general type. This implies that $p_g = q$ and the above inequalities yield

$$0 \leq p_g \leq 4.$$

A partial classification of these surfaces has already been accomplished. In the case $\mathbf{p}_g = \mathbf{q} = 4$ we have a full classification theorem.

THEOREM 0.4 [B82]. – *If S is a minimal surface of general type with $p_g = q = 4$, then S is a product of two curves of genus 2 and $K_S^2 = 8$. Moreover, $\mathcal{M}_{8,4,4}$ consists of exactly one connected component of dimension 6.*

Also, in the case $\mathbf{p}_g = \mathbf{q} = 3$ we have a full classification theorem due to several authors: [CCML98], [HP02], and [Pi02].

THEOREM 0.5. – *Let S be a minimal surface of general type with $p_g = q = 3$, then there are only two possibilities:*

(1) $K_S^2 = 8$, and $S = (C \times F)/G$ where C is a curve of genus 2, F is a curve of genus 3 and $G \cong \mathbb{Z}/2\mathbb{Z}$. Here G acts freely and diagonally on the product $C \times F$, on C as a hyperelliptic involution, and on F as a fixed point free involution. Moreover, $\mathcal{M}_{8,3,3}$ consists of exactly one connected component of dimension 5.

(2) $K_S^2 = 6$, and S is the symmetric square of a genus 3 curve. Moreover, $\mathcal{M}_{6,3,3}$ consists of exactly one connected component of dimension 6.

On the other hand, in the case $p_g \leq 2$ a complete classification theorem is still missing. It seems that the classification becomes more complicated as the value of p_g decreases. In this paper, we address the case $p_g = q = 2$.

Since we are dealing with irregular surfaces, i.e., with $q > 0$, a useful tool that we can use is the Albanese map. The *Albanese variety* of S is defined as $\text{Alb}(S) := H^0(\Omega_S^1)^\vee / H_1(S, \mathbb{Z})$. By Hodge theory, $\text{Alb}(S)$ is an abelian variety. For a fixed base point $x_0 \in S$, we define the *Albanese morphism*

$$\alpha_{x_0}: S \longrightarrow \text{Alb}(S), \quad x \mapsto \int_{x_0}^x .$$

If we choose a different base point in S , the Albanese morphism changes by a translation of $\text{Alb}(S)$; so we often ignore the base point and write α .

The dimension of $\alpha(S)$ is called the *Albanese dimension* of S and it is denoted by $\text{Albdim}(S)$. If $\text{Albdim}(S) = 2$, we say that S has *maximal Albanese dimension*. If, moreover, $q(S) > 2$, we say that S is of *Albanese general type*.

For surfaces with $q(S) = 2$, we have two possibilities:

- (1) either $\alpha(S)$ is a curve, or
- (2) α is a generically finite cover of an abelian surface.

In the latter case, the degree of α plays a crucial role. Indeed, if $q(S) = 2$, then the degree of the Albanese map α is a topological invariant, see e.g. [Ca11, Section 5].

In this paper, we want to present the up-to-date list of the known surfaces of general type with $\mathbf{p}_g = \mathbf{q} = 2$, which can be shortly summarized in the following table.

TABLE 0.

n.	K_S^2	$Albdim(S)$	$\deg \alpha$	‡ Families	dim	Name
1	8	1	–	24	$3^{15}, 4^6, 5^2, 6$	Isog. to a Prod.
2	8	2	≤ 6	4	$3^3, 4$	Isog. to a Prod.
3	6	2	4	1	4	
4	6	2	2	3	$4^2, 3$	
5	5	2	3	1	4	Chen-Hacon surf.
6	4	2	2	1	4	Catanese surf.

In the table we used the notation 3^{15} to mean that there are 15 families of dimensions 3.

The paper is organized as follows. We shall describe the families of our surfaces following the numeration of Table 0.

Indeed, in the first section we shall treat the case where $\alpha(S)$ is a curve. Here we have a complete classification theorem. We introduce the surfaces isogenous to a product of curves and some of their properties. The end of the section is dedicated to product-quotient surfaces, which are a generalization of surfaces isogenous to a product.

In the second section, we describe the four families of surfaces with $p_g = q = 2$ and $K_S^2 = 6$. Three families are characterized by the fact that their elements have Albanese map of degree 2, while the last one has Albanese map of degree 4.

The third section is devoted to describe *Chen-Hacon surfaces*. These surfaces were introduced by Chen and Hacon in [CH06] and together with F. Polizzi were able to give a description of the connected component of the moduli space of surfaces of general type they belong to.

The last section is devoted to the last cases and to give an account of the open problems on this topic.

1. – Product-Quotient Surfaces with $p_g = q = 2$

A surface S is said to be *isogenous to a (higher) product of curves* if and only if S is a quotient $S := (C \times F)/G$, where C and F are curves of genus at least two, and G is a finite group acting freely on $C \times F$.

Let S be a surface isogenous to a higher product, and $G^\circ := G \cap (Aut(C) \times Aut(F))$. Then G° acts on the two factors C and F and diagonally on the product $C \times F$. If G° acts faithfully on both curves, we say that $S = (C \times F)/G$ is a *minimal realization* of S . In [Ca00], the author proves that any surface iso-

genous to a higher product admits a unique minimal realization. From now on we shall work only with minimal realizations.

There are two cases: the *mixed* case where the action of G exchanges the two factors (in this case C and F are isomorphic and $G^\circ \neq G$); the *unmixed* case (where $G = G^\circ$, and therefore it acts diagonally).

Moreover, we observe that a surface isogenous to a product of curves is of general type. It is always minimal and its numerical invariants are explicitly given in terms of the genera of the curves and the order of the group. Indeed, we have the following proposition.

PROPOSITION 1.1. – *Let $S = (C \times F)/G$ be a surface isogenous to a higher product of curves, then:*

$$\chi(S) = \frac{(g(C) - 1)(g(F) - 1)}{|G|},$$

$$e(S) = \frac{4(g(C) - 1)(g(F) - 1)}{|G|},$$

$$K_S^2 = \frac{8(g(C) - 1)(g(F) - 1)}{|G|}.$$

In the unmixed case G acts separately on C and F , and the two projections $\pi_C: C \times F \rightarrow C$, $\pi_F: C \times F \rightarrow F$ induce two isotrivial fibrations $\alpha: S \rightarrow C/G$, $\beta: S \rightarrow F/G$, whose smooth fibres are isomorphic to F and C , respectively.

A surface isogenous to a product of unmixed type $S := (C \times F)/G$ is said to be of *generalized hyperelliptic type* if:

- (1) the Galois covering $\pi_1: C \rightarrow C/G$ is unramified, and
- (2) the quotient curve F/G is isomorphic to \mathbb{P}^1 .

Surfaces of generalized hyperelliptic type play a crucial role among surfaces with $p_g = q = 2$ because of the following proposition.

PROPOSITION 1.2 ([Z03] Prop. 4.2). – *If S is a surface of general type with $p_g = q = 2$ and $\text{Albdim}(S) = 1$. Then S is of generalized hyperelliptic type.*

By this proposition, once we classify all the surfaces isogenous to a product with $p_g = q = 2$, we also achieve the classification of all the ones such that $\alpha(S)$ is a curve. This work was started by Zucconi in [Z03] and completed in [Pe11]. We can summarize the results in the following theorem.

THEOREM 1.3 ([Pe11] Theorem 1.1). – *Let S be a surface isogenous to a higher product of curves with $p_g = q = 2$. Then we have the following possibilities:*

(1) *If $\text{Albdim}(S) = 1$, then $S \cong (C \times F)/G$ and it is of generalized hyperelliptic type. The classification of these surfaces is given by the cases labelled with *GH* in Table 1, where we specify the possibilities for the genera of the two curves C and F , and for the group G .*

(2) *If $\text{Albdim}(S) = 2$, then there are two cases:*

- *S is isogenous to product of curves of unmixed type $(C \times F)/G$, and the classification of these surfaces is given by the cases labelled with *UnMix* in Table 1;*
- *S is isogenous to a product of curves of mixed type $(C \times C)/G$, there is only one such case and it is labelled with *Mix* in Table 1;*

TABLE 1.

Type	K_S^2	$g(F)$	$g(C)$	G	IdSmallGroup	m	dim	n
GH	8	2	3	$\mathbb{Z}/2\mathbb{Z}$	$G(2,1)$	(2^6)	6	1
GH	8	2	4	$\mathbb{Z}/3\mathbb{Z}$	$G(3,1)$	(3^4)	4	1
GH	8	2	5	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$G(4,2)$	(2^5)	5	2
GH	8	2	5	$\mathbb{Z}/4\mathbb{Z}$	$G(4,1)$	$(2^2, 4^2)$	4	1
GH	8	2	6	$\mathbb{Z}/5\mathbb{Z}$	$G(5,1)$	(5^3)	3	1
GH	8	2	7	$\mathbb{Z}/6\mathbb{Z}$	$G(6,2)$	$(2^2, 3^2)$	4	1
GH	8	2	7	$\mathbb{Z}/6\mathbb{Z}$	$G(6,2)$	$(3, 6^2)$	3	1
GH	8	2	9	$\mathbb{Z}/8\mathbb{Z}$	$G(8,1)$	$(2, 8^2)$	3	1
GH	8	2	11	$\mathbb{Z}/10\mathbb{Z}$	$G(10,2)$	$(2, 5, 10)$	3	1
GH	8	2	13	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	$G(12,5)$	$(2, 6^2)$	3	2
GH	8	2	7	\mathfrak{S}_3	$G(6,1)$	$(2^2, 3^2)$	4	1
GH	8	2	9	Q_8	$G(8,4)$	(4^3)	3	1
GH	8	2	9	D_4	$G(8,3)$	$(2^3, 4)$	4	2
GH	8	2	13	D_6	$G(12,4)$	$(2^3, 3)$	3	2
GH	8	2	13	$D_{4,3,-1}$	$G(12,1)$	$(3, 4^2)$	3	1
GH	8	2	17	$D_{2,8,3}$	$G(16,8)$	$(2, 4, 8)$	3	1
GH	8	2	25	$\mathbb{Z}/2\mathbb{Z} \times ((\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z})$	$G(24,8)$	$(2, 4, 6)$	3	2
GH	8	2	25	$\text{SL}(2, \mathbb{F}_3)$	$G(24,3)$	$(3^2, 4)$	3	1
GH	8	2	49	$\text{GL}(2, \mathbb{F}_3)$	$G(48,29)$	$(2, 3, 8)$	3	1
UnMix	8	3	3	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$G(4,2)$	$(2^2), (2^2)$	4	1
UnMix	8	3	4	\mathfrak{S}_3	$G(6,1)$	$(3), (2^2)$	3	1
UnMix	8	3	5	D_4	$G(8,3)$	$(2), (2^2)$	3	1
Mix	8	3	3	$\mathbb{Z}/4\mathbb{Z}$	$G(4,1)$	–	3	1

In Table 1 `ldSmallGroup` denotes the label of the group G in the GAP4 database of small groups, \mathbf{m} is the branching data, i.e., the number of points (exponent) of given branching order (base) of the covering $F \rightarrow \mathbb{P}^1$ (e.g. 2^6 means that there are six points with ramification order two). Moreover, each item provides a union of connected components of the moduli space of surfaces of general type, their dimension is listed in the column *dim*, and n is the number of connected components.

The proof of the theorem involves techniques from both geometry and combinatorial group theory developed in [BCG08] and [CP09].

An idea of the proof of the theorem is the following. There are two cases. In the first case S is not of Albanese maximal dimension; it is of generalized hyperelliptic type, hence one classifies all possible finite groups G which induce a G -covering $\pi_F: F \rightarrow \mathbb{P}^1$ with $g(F) = 2$ (see [Bol88]). Then we need to check whether such groups G induce an unramified G -covering $\pi_C: C \rightarrow B \cong C/G$, where the genus of B is 2 and the genus of C is determined by the Riemann-Hurwitz formula. We notice that the action of G on the product $C \times F$ is always free, because the action on C is free.

In the second case, S is of maximal Albanese dimension and it is slightly more difficult. In this case, both projections $C \rightarrow C/G$ and $C \rightarrow F/G$ are ramified covers of elliptic curves. By [Z03, Corollary 2.4] the genera of C and F can be at most five. Therefore, we can proceed as in the previous case by analyzing all possible combinations of genera. This time we also take into account the fact that, since we do not have an étale cover, we have to check whether the action of G on the product of the two curves is free or not.

REMARK 1.4. – By the above theorem, we know that $\mathcal{M}_{8,2,2}$ consists of at least 28 connected components of dimensions between 3 and 6.

An interesting and still unanswered question is the following one

QUESTION 1.5. – *Is there a surface with $p_g = q = 2$ and $K_S^2 = 8$ which is not isogenous to a product?*

It is worth noticing that up to now the only known examples of surfaces of general type with $\chi(S) = 1$ and $K_S^2 = 8$ are all isogenous to a higher product of curves. These surfaces have all been classified, in particular for $p_g = q = 1$ see [CP09], and for $p_g = q = 0$ see [BCG08].

The study of surfaces of general type with $p_g = q = 2$ and $\text{Albdim}(S) = 2$ started with a generalization of the definition of surface isogenous to a product. A surface S is said to be a *product-quotient* surface if S is the minimal model of the minimal desingularization of the quotient surface $T := (C_1 \times C_2)/G$ where C_1, C_2 are curves of genus at least two, and G is a finite group acting diagonally but not necessarily freely on the product.

THEOREM 1.6 ([Pe11] Theorem 1.1). – *Let S be a product-quotient surface of general type with $p_g = q = 2$. In particular S is the minimal desingularization of $T := (C_1 \times C_2)/G$, and these surfaces are classified in Table 2.*

TABLE 2.

K_S^2	$g(C_1)$	$g(C_2)$	G	IdSmallGroup	m	Type	Num. Sing.	dim	n
4	2	2	$\mathbb{Z}/2\mathbb{Z}$	$G(2,1)$	$(2^2)(2^2)$	$\frac{1}{2}(1, 1)$	4	4	1
4	3	3	D_4	$G(8,3)$	$(2)(2)$	$\frac{1}{2}(1, 1)$	4	2	1
4	3	3	Q_8	$G(8,4)$	$(2)(2)$	$\frac{1}{2}(1, 1)$	4	2	1
5	3	3	\mathfrak{S}_3	$G(6,1)$	$(3)(3)$	$\frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$	2	2	1
6	4	4	\mathfrak{A}_4	$G(12,3)$	$(2)(2)$	$\frac{1}{2}(1, 1)$	2	2	1

In Table 2 each item provides a union of irreducible subvarieties of the moduli space of surfaces of general type, their dimension is listed in the column *dim* and n is the number of subvarieties. Moreover the columns of Table 2 labelled with *Type* and *Num.Sing.* indicate the types and the number of singularities of T .

We observe that a complete classification of product-quotient surfaces of general type with $\chi(S) = 1$ and $p_g \leq 1$ is still missing. A partial classification for the ones with $p_g = q = 1$ is given in [MP10] and for the ones with $p_g = q = 0$ in [BP11].

2. – Surfaces with $p_g = q = 2$ and $K_S^2 = 6$

Albanese map of degree 4

We briefly recall the construction of the surface with $p_g = q = 2$ and $K_S^2 = 6$ given in Theorem 1.6. Let us denote by \mathfrak{A}_4 the alternating group on four symbols and by V_4 its Klein subgroup, namely

$$V_4 = \langle id, (12)(34), (13)(24), (14)(23) \rangle \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

V_4 is normal in \mathfrak{A}_4 and the quotient $H := \mathfrak{A}_4/V_4$ is a cyclic group of order 3. By Riemann’s existence theorem, it is possible to construct two smooth curves C_1, C_2 of genus 4 endowed with an action of \mathfrak{A}_4 such that the only non-trivial stabilizers are the elements of V_4 . Then

- $E'_i := C_i/\mathfrak{A}_4$ is an elliptic curve;
- the \mathfrak{A}_4 -cover $f_i: C_i \rightarrow E'_i$ is branched at exactly one point of E'_i , with branching order 2.

It follows that the surface

$$\widehat{X} := (C_1 \times C_2)/\mathfrak{A}_4,$$

where \mathfrak{A}_4 acts diagonally, has two rational double points of type $\frac{1}{2}(1, 1)$ and no other singularities. It is straightforward to check that the desingularization \widehat{S} of \widehat{X} is a minimal surface of general type with $p_g = q = 2$, $K_S^2 = 6$ and that \widehat{X} is the canonical model of \widehat{S} .

The \mathfrak{A}_4 -cover $f_i: C_i \rightarrow E'_i$ factors through the bidouble cover $g_i: C_i \rightarrow E_i$, where $E_i := C_i/V_4$. Note that E_i is again an elliptic curve, so there is an isogeny $E_i \rightarrow E'_i$, which is a triple Galois cover with Galois group H . Consequently, we have an isogeny

$$p: E_1 \times E_2 \rightarrow \widehat{A} := (E_1 \times E_2)/H,$$

where the group H acts diagonally, and a commutative diagram

$$(1) \quad \begin{array}{ccc} C_1 \times C_2 & \xrightarrow{\tilde{p}} & (C_1 \times C_2)/\mathfrak{A}_4 = \widehat{X} \\ g_1 \times g_2 \downarrow & & \downarrow \hat{\alpha} \\ E_1 \times E_2 & \xrightarrow{p} & (E_1 \times E_2)/H = \widehat{A}. \end{array}$$

The morphism $\hat{\alpha}: \widehat{X} \rightarrow \widehat{A}$ is the Albanese map of \widehat{X} ; it is a finite, non-Galois quadruple cover.

This last observation led us to study surfaces \widehat{S} with $p_g = q = 2$ and $K_S^2 = 6$ whose Albanese map $\hat{\alpha}: \widehat{S} \rightarrow \widehat{A} := \text{Alb}(\widehat{S})$ is a quadruple cover of an abelian surface \widehat{A} , in order to understand and describe the deformations of \widehat{S} .

The main result in the theory of quadruple cover for algebraic varieties is the following.

THEOREM 2.1 ([HM99, Theorem 1.2]). – *Let Y be a smooth algebraic variety. Any quadruple cover $f: X \rightarrow Y$ is determined by a locally free \mathcal{O}_Y -module \mathcal{E}^\vee of rank 3 and a totally decomposable section $\eta \in H^0(Y, \bigwedge^2 S^2 \mathcal{E}^\vee \otimes \bigwedge^3 \mathcal{E})$.*

The vector bundle \mathcal{E}^\vee is called the *Tschirnhausen bundle* of the cover.

We have $f_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{E}^\vee$.

In the situation above, we can prove that the Tschirnhausen bundle \mathcal{E}^\vee associated with the Albanese cover is of the form $\Phi^P(\mathcal{L}^{-1})^\vee$, where \mathcal{L} is a polarization on A (the dual abelian variety of \widehat{A}) and Φ^P denotes the Fourier-Mukai transform with kernel the normalized Poincaré bundle \mathcal{P} i.e.,

$$\mathcal{E} = \Phi^P(\mathcal{L}^{-1}) := R^i \pi_{A*} (\mathcal{P} \otimes \pi_A^* \mathcal{L}^{-1}).$$

More precisely, the bundle \mathcal{E}^\vee has rank 3 and \mathcal{L} is a polarization of type (1, 3).

We are able to prove the following theorem.

THEOREM 2.2 ([PP12, Theorem 2.1]). – *There exists a 4-dimensional family \mathcal{M}_ϕ of minimal surfaces of general type with $p_g = q = 2$ and $K_S^2 = 6$ such that, for the general element $\widehat{S} \in \mathcal{M}_\phi$, the canonical class $K_{\widehat{S}}$ is \widehat{S} ample and the Albanese map $\hat{\alpha}: \widehat{S} \rightarrow \widehat{A}$ is a finite cover of degree 4.*

The Tschirnhausen bundle \mathcal{E}^\vee associated with $\hat{\alpha}$ is isomorphic to $\Phi^P(L^{-1})$, where \mathcal{L} is a polarization of type (1, 3) on A .

The family \mathcal{M}_ϕ provides an irreducible component of the moduli space $\mathcal{M}_{6,2,2}$ of canonical models of minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 6$. Such a component is generically smooth and contains the 2-dimensional family of product-quotient surfaces given in Theorem 1.6.

This theorem is obtained by extending the construction given in [CH06] to the much more complicated case of quadruple covers. More precisely, in order to build a quadruple cover $\hat{\alpha}: \widehat{S} \rightarrow \widehat{A}$ with Tschirnhausen bundle \mathcal{E}^\vee , we first build a quadruple cover $\alpha: S \rightarrow A$ with Tschirnhausen bundle $\phi_{\mathcal{L}}^* \mathcal{E}^\vee$ (here $\phi_{\mathcal{L}}: A \rightarrow \widehat{A}$ denotes the group homomorphism sending $x \in A$ to $t_x^* \mathcal{L}^{-1} \otimes \mathcal{L} \in \widehat{A}$) and then, by using the Schrödinger representation of the finite Heisenberg group \mathcal{H}_3 on $H^0(A, \mathcal{L})$, we identify the covers of this type which descend to a quadruple cover $\hat{\alpha}: \widehat{S} \rightarrow \widehat{A}$. For the general surface \widehat{S} , the branch divisor $\widehat{B} \subset \widehat{A}$ of $\alpha: \widehat{S} \rightarrow \widehat{A}$ is a curve in the linear system $|\widehat{\mathcal{L}}^{\otimes 2}|$, where $\widehat{\mathcal{L}}$ is a polarization of type (1, 3) on \widehat{A} . \widehat{B} has six ordinary cusps and no other singularities; such a curve is \mathcal{H}_3 -equivariant and can be associated with the dual of a member of the Hesse pencil of plane cubics in \mathbb{P}^2 . Once we have established these facts with the help of a MAGMA script, we can show that \widehat{S} is smooth in the general case.

Albanese map of degree 2

In [PP11] we give other families of surfaces with $p_g = q = 2$, $K_S^2 = 6$, and with Albanese map of degree equal to two.

Let us introduce some notation. Let (A, \mathcal{L}) be a (1, 2)-polarized abelian surface and let us denote by $\phi_2: A[2] \rightarrow \widehat{A}[2]$ the restriction of the canonical homomorphism $\phi_{\mathcal{L}}: A \rightarrow \widehat{A}$ to the subgroup of 2-division points. Then $\text{im } \phi_2$ consists of four line bundles $\{\mathcal{O}_A, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3\}$. Let us denote by $\text{im } \phi_2^\times$ the set $\{\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3\}$.

THEOREM 2.3 ([PP11, Theorem A]). – *Given an abelian surface A with a symmetric polarization \mathcal{L} of type (1, 2), not of product type, for any $\mathcal{Q} \in \text{im } \phi_2$ there exists a curve $D \in |\mathcal{L}^2 \otimes \mathcal{Q}|$ whose unique singularity is an ordinary quadruple point at the origin $o \in A$. Let $\mathcal{Q}^{1/2}$ be a square root of \mathcal{Q} , and if $\mathcal{Q} = \mathcal{O}_A$*

assume moreover $\mathcal{Q}^{1/2} \neq \mathcal{O}_A$. Then the minimal desingularization S of the double cover of A branched over D and defined by $\mathcal{L} \otimes \mathcal{Q}^{1/2}$ is a minimal surface of general type with $p_g = q = 2$, $K_S^2 = 6$ and Albanese map of degree 2.

Conversely, every minimal surface of general type with $p_g = q = 2$, $K_S^2 = 6$ and Albanese map of degree 2 can be constructed in this way.

When $\mathcal{Q} = \mathcal{Q}^{1/2} = \mathcal{O}_A$, we obtain a minimal surface with $p_g = q = 3$, which correspond to the second case of Theorem 0.5.

We use the following terminology:

- if $\mathcal{Q} = \mathcal{O}_A$ we say that S is a *surface of type I*. Furthermore, if $\mathcal{Q}^{1/2} \notin \text{im } \phi_2$ we say that S is of type Ia , whereas if $\mathcal{Q}^{1/2} \in \text{im } \phi_2^\times$ we say that S is of type Ib ;
- if $\mathcal{Q} \in \text{im } \phi_2^\times$ we say that S is a *surface of type II*.

Recall that the degree of the Albanese map in this case is a topological invariant since $q(S) = 2$. Therefore, let us consider the moduli space \mathcal{M} of minimal surfaces of general type with $p_g = q = 2$, $K_S^2 = 6$ and Albanese map of degree 2. Let \mathcal{M}_{Ia} , \mathcal{M}_{Ib} , \mathcal{M}_{II} be the algebraic subsets whose points parametrize isomorphism classes of surfaces of type Ia , Ib , II , respectively. Therefore \mathcal{M} can be written as the disjoint union

$$\mathcal{M} = \mathcal{M}_{Ia} \sqcup \mathcal{M}_{Ib} \sqcup \mathcal{M}_{II}.$$

Our result on the moduli space is

THEOREM 2.4 ([PP11, Theorem B]). – *The following holds:*

- (i) \mathcal{M}_{Ia} , \mathcal{M}_{Ib} , \mathcal{M}_{II} are the connected components of \mathcal{M} ;
- (ii) these are also irreducible components of the moduli space of minimal surfaces of general type;
- (iii) \mathcal{M}_{Ia} , \mathcal{M}_{Ib} , \mathcal{M}_{II} are generically smooth, of dimension 4, 4, 3, respectively;
- (iv) the general surface in \mathcal{M}_{Ia} and \mathcal{M}_{Ib} has ample canonical class; all surfaces in \mathcal{M}_{II} have ample canonical class.

Some interesting questions remain open also for surfaces with $p_g = q = 2$ and $K_S^2 = 6$ namely

QUESTION 2.5. – • Is \mathcal{M}_ϕ a *connected* component of $\mathcal{M}_{6,2,2}$?

- What are the possible degrees for the Albanese map of a minimal surface with $p_g = q = 2$ and $K^2 = 6$?
- What are the irreducible/connected components of $\mathcal{M}_{6,2,2}$?

3. – Surfaces with $p_g = q = 2$ and $K_S^2 = 5$

A similar calculation as the one in the previous section shows that any element of the family of surfaces with $p_g = q = 2$ and $K_S^2 = 5$ given in Theorem 1.6 has Albanese map of degree 3. Moreover, for such element S the image $\alpha(S)$ is an abelian surface with a polarization of type (1, 2). In order to understand the connected components of the moduli space that contains this family, we have to study triple covers in detail.

Let us introduce some terminology. Let S be a minimal surface of general type with $p_g = q = 2$ and $K_S^2 = 5$, such that its Albanese map $\alpha: S \rightarrow \text{Alb}(S)$ is a generically finite morphism of degree 3. If we consider the Stein factorization of α , i.e.,

$$S \xrightarrow{p} \widehat{X} \xrightarrow{\hat{f}} \text{Alb}(S),$$

the map $\hat{f}: \widehat{X} \rightarrow \text{Alb}(S)$ is a flat triple cover, which can be studied by applying the techniques developed in [M85]. In particular, \hat{f} is determined by a rank 2 vector bundle \mathcal{E}^\vee on $\text{Alb}(S)$, called the *Tschirnhausen bundle* of the cover, and by a global section $\eta \in H^0(\text{Alb}(S), S^3\mathcal{E}^\vee \otimes \wedge^2 \mathcal{E})$. In the examples of [CH06] and [Pe11], the surface \widehat{X} is singular; nevertheless, in both cases the numerical invariants of \mathcal{E} are the same predicted by the formulae of [M85], as if \widehat{X} were smooth. This leads us to introduce the definition of *negligible singularity* for a triple cover, which is similar to the corresponding well-known definition for double covers. Then, inspired by the construction in [CH06], we say that S is a *Chen-Hacon surface* if there exists a polarization \mathcal{L} of type (1, 2) on $\text{Pic}^0(S) = \widehat{\text{Alb}}(S)$ such that \mathcal{E}^\vee is the Fourier-Mukai transform of the line bundle \mathcal{L}^{-1} .

Our first result is the following characterization of Chen-Hacon surfaces.

THEOREM 3.1 ([PP10, Theorem A]). – *Let S be a minimal surface of general type with $p_g = q = 2$ and $K_S^2 = 5$ such that the Albanese map $\alpha: S \rightarrow \text{Alb}(S)$ is a generically finite morphism of degree 3. Let*

$$S \xrightarrow{p} \widehat{X} \xrightarrow{\hat{f}} \text{Alb}(S)$$

be the Stein factorization of α . Then S is a Chen-Hacon surface if and only if \widehat{X} has only negligible singularities.

Moreover, we can completely describe all the possibilities for the singular locus of \widehat{X} . It follows that \widehat{X} is never smooth, and it always contains a cyclic quotient singularity of type $\frac{1}{3}(1, 1)$. Therefore S always contains a (-3) -curve, which turns out to be the fixed part of the canonical system $|K_S|$.

Now let $\mathcal{M}_{5,2,2}$ be the moduli space of surfaces with $p_g = q = 2$ and $K_S^2 = 5$, and let $\mathcal{M}^{CH} \subset \mathcal{M}_{5,2,2}$ be the subset of the points parametrizing (isomorphism classes of) Chen-Hacon surfaces. Our second result is the following.

THEOREM 3.2 ([PP10, Theorem B]). – \mathcal{M}^{CH} is an irreducible, connected, generically smooth component of $\mathcal{M}_{5,2,2}$ of dimension 4.

It is worth noticing that Theorem 3.2 shows that every small deformation of a Chen-Hacon surface is still a Chen-Hacon surface; in particular, no small deformation of S makes the (-3) -curve disappear. Moreover, since \mathcal{M}^{CH} is generically smooth, the same is true for the first-order deformations. By contrast, Burns and Wahl proved in [BW74] that first-order deformations always smooth all the (-2) -curves. Furthermore, Catanese used this fact in [Ca89] to produce examples of surfaces of general type with everywhere non-reduced moduli spaces. Theorem 3.2 shows rather strikingly that the results of Burns-Wahl and Catanese cannot be extended to the case of (-3) -curves and, as far as we know, provides the first explicit example of this situation.

QUESTION 3.3. – There are mainly three questions open for surfaces with $p_g = q = 2$ and $K_S^2 = 5$:

- Are there surfaces with these invariants whose Albanese map has degree different from 3?
- Are there surfaces with these invariants whose Albanese map has degree 3, but which are not Chen–Hacon surfaces?
- What are the irreducible/connected components of $\mathcal{M}_{5,2,2}$?

4. – Surfaces with $p_g = q = 2$ and $K_S^2 = 4, 7$ and 9

Surfaces with $p_g = q = 2$ and $K_S^2 = 4$ were studied by Ciliberto and Mendes Lopes in [CML02]. The authors classified all surfaces with $p_g = q = 2$ and non-birational bicanonical map (not presenting the standard case). Their result is the following.

THEOREM 4.1. – *If S is a minimal surface of general type with $p_g = q = 2$ and non-birational bicanonical map not presenting the standard case, then S is a double cover of a principally polarized abelian surface (A, Θ) , with Θ irreducible. The double cover $S \rightarrow A$ is branched along a divisor $B \in |\Theta|$, having at most double points. In particular $K_S^2 = 4$.*

Surfaces with $p_g = q = 2$ and $K_S^2 = 4$ were also considered by Manetti in his work on Severi’s conjecture ([M03]). In particular, Manetti proved that if

$p_g = q = 2$, K_S is ample, and $K_S^2 = 4$, then S is a double cover of its Albanese variety, which turns out to be a principally polarized abelian surface (A, Θ) . In our investigation on product-quotient surfaces in [Pe11], we constructed three families of surfaces with $p_g = q = 2$ and $K_S^2 = 4$: see Theorem 1.6. In all these cases, the canonical bundle is not ample; nevertheless, the Albanese map has degree two and the Albanese variety is a principally polarized abelian surface.

Eventually in [CMLP13], it is proven that all surfaces with $p_g = q = 2$ and $K_S^2 = 4$ are double covers of a principally polarized abelian surface. This also proves that there is only one connected component of the moduli space $\mathcal{M}_{4,2,2}$ of dimension 4.

The cases with $K_S^2 = 7$ and 9 are more mysterious. There are no examples known up to now.

As we have seen, the degree of the Albanese map plays a very important role in the classification of surfaces with $p_g = q = 2$, so it would be nice to have an upper bound on this number. We notice that the degree of the Albanese map of all the families given in Table 0 is always less or equal to $K_S^2 - 2$. Unfortunately we cannot prove that this is the actual upper bound.

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Matteo Penegini, Dipartimento di Matematica "F. Enriques"
Via Saldini 50, I-20100 Milano, Italy
E-mail: matteo.penegini@unimi.it

