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Abstract. – *In this paper we describe the groups of isometries acting transitively on the homogeneous tree of degree three. This description implies that the following three properties are equivalent: amenability, non-unimodularity and action without inversions. Moreover, we exhibit examples of non-unimodular transitive groups of isometries of a homogeneous tree of degree* $q + 1 > 3$ *which do not fix any point of the boundary of the tree.*

1. – Introduction

The purpose of this paper is to describe the closed groups of isometries of the homogeneous tree of degree three acting transitively on the vertices of the tree. The group of isometries of this particular tree is so small that it only has a few types of transitive subgroups. For example, in Lemma 3.3 below, we prove that a closed subgroup acting transitively on the set of oriented edges either is discrete or it acts transitively on the tree boundary. In this paper we prove that a closed non-discrete group of isometries acting transitively on the vertices of the homogeneous tree of degree three either acts transitively on the boundary of the tree, or fixes a point of the boundary and it acts transitively on the complementary of that point, or else it stabilizes a suitable set of non-oriented edges E (the set E is required to satisfy only the following condition: for every vertex v there exists one and only one edge of E containing v). Also the discrete transitive subgroups can be partitioned into three disjoint classes: the discrete groups acting transitively on the set of oriented edges (*i.e.* the transitive and locally transitive subgroups), the simply transitive subgroups (*i.e.* the transitive subgroup acting faithfully on X) and the discrete groups which stabilizes a set E of non-oriented edges as above. This description implies that the following three properties are equivalent for $q + 1 = 3$: *amenability*, *non-unimodularity* and *action without inversions*. In particular, every transitive non-unimodular subgroup fixes an end of the boundary. For a general homogeneous tree of degree $q + 1 \geq 3$, the prototype (and the most cited example) of a transitive non-unimodular group of isometries is just the stabilizer of an end of the boundary. For this reason, when

$q+1>3$, a question that arises is if there are other examples of transitive non-unimodular groups. In this paper we exhibit for every $q+1>3$, examples of closed transitive non-unimodular subgroups which are non-amenable (and so which do not fix any end of the tree). More precisely, we give examples of such a groups containing inversions (for every $q+1\geq 4$) and without inversions (for every $q+1\geq 5$; in fact in the special case $q+1=4$ every transitive non-unimodular non-amenable group contains inversions). Such groups, which we will denote by $Aut(X_f)$, are associated with a function f defined on the vertices of X. The definition of the groups $Aut(X_f)$ is suggested to us by the groups introduced by J. Tits in [4, p. 200].

2. – Preliminaries

Let X be a homogeneous tree of finite degree $q+1\geq 3$ and let Ω be the boundary of the tree X. We denote by $Aut(X)$ the locally compact group of all isometries of X with respect to the natural distance $d(x,y)$ of X, where $d(x,y)$ is the length of the unique geodesic connecting x to y. The group $Aut(X)$ is a separable totally disconnected locally compact group and the subgroups $Aut(X)_o$ and $Aut(X)^{\sim}_{[a,b]}$, respectively, the stabilizer of a vertex O and the stabilizer of an edge $[a,b]$, are compact open subgroups of $Aut(X)$. For undefined notions and terminology, we refer the reader to [1]; we recall only that an isometry of the tree is called a rotation if it fixes a vertex of X, an inversion if it interchanges the two adjacent vertices of an edge, and g is called a translation along a doubly infinite geodesic $\gamma=\{\dots x_{-n},\dots,x_{-1},x_0,x_1,x_2,\dots,x_n,\dots\}$ if there exist an integer $k\neq 0$ such that $g(x_n)=x_{n+k}$ for every n (the positive integer $|k|$ is called the step of the translation g). A theorem of Tits [4] implies that any isometry belongs to one and only one of the three classes above. In [3, Th. 1] we characterize the closed amenable subgroups of $Aut(X)$ for a locally finite tree X. In particular we prove that a closed subgroup G acting transitively on the vertices of a homogeneous tree X is amenable if and only if G fixes an end of the boundary Ω [3, Th. 2]. A transitive amenable group G fixing $\omega\in\Omega$ is a semidirect product $\langle h_0\rangle B^G_\omega$, where h_0 is a translation of step-1 along a doubly infinite geodesic directed towards ω and B^G_ω is the group of all rotations of G. B^G_ω acts transitively on each horocycle associated to ω (because G acts transitively on X) and so it acts transitively on $\Omega/\{\omega\}$ (for more details see [1, pp. 24-25]).

Recall that, for every locally compact group, the modular function of G, denoted by Δ_G, is the unique function such that the following equality holds for every $g\in G$ and for every continuous function f with compact support:

$$\int_G f(xg)\,dx=\Delta_G(g)\int_G f(x)\,dx$$

(here dx denotes a left-invariant Haar measure on G). The modular function Δ_G is a continuous homomorphism of G into the multiplicative group of the positive real numbers which is independent of the choice of left-invariant Haar measure dx. If $E \subset G$ is a measurable set then the measure of a right-translate of E is $m(Eg^{-1}) = \Delta_G(g)m(E)$. In particular, if C is a compact open subset or a compact open subgroup of G then $m(C)$ is finite and positive and $m(gCg^{-1}) = m(C)$ if and only if $\Delta_G(g) = 1$.

A locally compact group is called unimodular if $\Delta_G \equiv 1$ or, equivalently, if the left-invariant Haar measure dx is also right-invariant. This means that every left-invariant Haar measure is biinvariant. For more details, we refer the reader to [2].

For a subgroup G of $Aut(X)$, we denote by $|G_a(b)|$ the cardinality of the finite orbit of the vertex b, where G_a is the stabilizer subgroup of the vertex a.

LEMMA 2.1. – *Assume that G is a closed subgroup of $Aut(X)$, then G is unimodular if and only if $|G_a(b)| = |G_b(a)|$ for every pair of vertices a and b in the same orbit of G on X. If in addition G acts transitively on X, then G is unimodular if and only if $|G_a(b)| = |G_b(a)|$ for every pair of adjacent vertices a and b.*

PROOF. – Let a be a vertex of X. If $g \in G$, then $g(G_a)g^{-1} = G_{g(a)}$ and $m(G_{g(a)}) = \Delta_G(g)m(G_a)$. Hence $\Delta_G(g) = 1$ if and only if $m(G_{g(a)}) = m(G_a)$. This proves that G is unimodular if and only if the function $\phi(x) = m(G_x)$ is constant on one (and hence on every) orbit of G on X. Since the intersection $G_a \cap G_{g(a)}$ has finite index in both subgroups G_a and $G_{g(a)}$, then $m(G_{g(a)}) = m(G_a)$ if and only if the index $[G_a : G_a \cap G_{g(a)}] = [G_{g(a)} : G_a \cap G_{g(a)}]$. But $[G_a : G_a \cap G_{g(a)}] = |G_a(g(a))|$ and $[G_{g(a)} : G_a \cap G_{g(a)}] = \left|G_{g(a)}(a)\right|$. This proves that G is unimodular if and only if $|G_a(b)| = |G_b(a)|$ for every pair of vertices a and b in the same orbit on X. If in addition G acts transitively on X then G is unimodular if and only if $m(G_x)$ is constant on X and so G is unimodular if and only if $m(G_a) = m(G_b)$ (that is $|G_a(b)| = |G_b(a)|$) for every pair of adjacent vertices a and b. □

The following two definitions are taken from [3].

DEFINITION 2.1. – *We say that a subgroup G of $Aut(X)$ has property $(*)$ if for every vertex x there exists an adjacent vertex $x^{\sim}$ such that:*

(1) $G_x \subsetneq G_{x^{\sim}}$
(2) G_x *acts transitively on the set* $\{y \in X \ : d(x, y) = 1\} \setminus \{x^{\sim}\}$.

By Definition 2.1 we have that $(x^{\sim})^{\sim} \neq x$ for every x and that the vertex $x^{\sim}$ is unique. Hence, if a group G has property $(*)$, then for every vertex x we can define an infinite geodesic γ_x starting at x as follows: $\gamma_x = \{y_1, y_2, \ldots, y_n, \ldots\}$ with $y_1 = x$ and $y_{n+1} = y_n^{\sim}$ for every n. Let ω_x be the end of Ω identified by the geodesic γ_x, that is $\gamma_x = [x, \omega_x)$. If $g \in G_x$ then g fixes all vertices of γ_x and $g(\omega_x) = \omega_x$. If G is a closed subgroup of $Aut(X)$ acting transitively on X, then the stabilizers G_x for $x \in X$ are conjugate to each other. Therefore, if the two conditions of the property $(*)$ are satisfied at a vertex x then every stabilizer satisfies the conditions 1) and 2) and G has property $(*)$. In Theorem 2 of [3], we prove also that property $(*)$ is equivalent to amenability for every transitive closed subgroup of $Aut(X)$ and for every $q + 1 \geq 3$.

A transitive group with property $(*)$ is not unimodular; in fact $G_x(x^{\sim}) = \{x^{\sim}\}$ and so $|G_x(x^{\sim})| = 1$ while $|G_{x^{\sim}}(x)| = q > 1$ because $(x^{\sim})^{\sim} \neq x$.

DEFINITION 2.2. – *We say that a subgroup G of $Aut(X)$ has property $(**)$ if for every vertex x there exists an adjacent vertex $x^{\sim}$ such that:*

(1) $G_x = G_{x^{\sim}}$
(2) G_x *acts transitively on the set* $\{y \in X \, : d(x, y) = 1\} \setminus \{x^{\sim}\}$.

By the definition of property $(**)$, it follows that the vertex $x^{\sim}$ associated to x is unique and $x = (x^{\sim})^{\sim}$ for every x. The fact that $x = (x^{\sim})^{\sim}$ for every x is the only difference between the two properties $(*)$ and $(**)$. Even in this case, for a transitive subgroup G, it is enough that the two conditions of property $(**)$ are satisfied in a vertex x.

We give now examples of groups with property $(**)$. Let E be a set of (non-oriented) edges such that for every vertex x there exists one and only one edge of E which contains the vertex x. Let $Aut(X_E)$ be the subgroup of $Aut(X)$ consisting of all isometries stabilizing the set E, that is the group of all isometries g such that for every edge $[a, b]$ of E the edge $g([a, b])$ belongs to E (we recall that g is bijective then if $[a, b] \notin E$ then also $g([a, b]) \notin E$). An example of a set E in a tree of degree 4 is described by Fig. 1 where the set E is the set consisting of all marked edges. It is easy to see that $Aut(X_E)$ is a closed subgroup acting transitively on X. Such a group has property $(**)$, indeed the vertex $x^{\sim}$ is the other vertex different from x which belongs to the unique edge of E containing x. Conversely, if G is transitive on X and it has property $(**)$, then we can define E as the set of edges $[x, x^{\sim}]$ for every x; hence $G \subset Aut(X_E)$. For every set E, the group $Aut(X_E)$ is not discrete because the stabilizer of a finite set of vertices in $Aut(X_E)$ is not trivial. Later (see the section below) we will provide examples of discrete transitive subgroups with property $(**)$.

We observe that the closed transitive subgroups with property $(**)$ are all unimodular. In fact it is enough to prove that $|G_a(b)| = |G_b(a)|$ for every pair of adjacent vertices a and b. If a and b is a pair of the type x and $x^{\sim}$ we have that $G_x(x^{\sim}) = \{x^{\sim}\}$ and $G_{x^{\sim}}(x) = \{x\}$, therefore $|G_x(x^{\sim})| = |G_{x^{\sim}}(x)| = 1$; otherwise $|G_a(b)| = |G_b(a)| = q$.

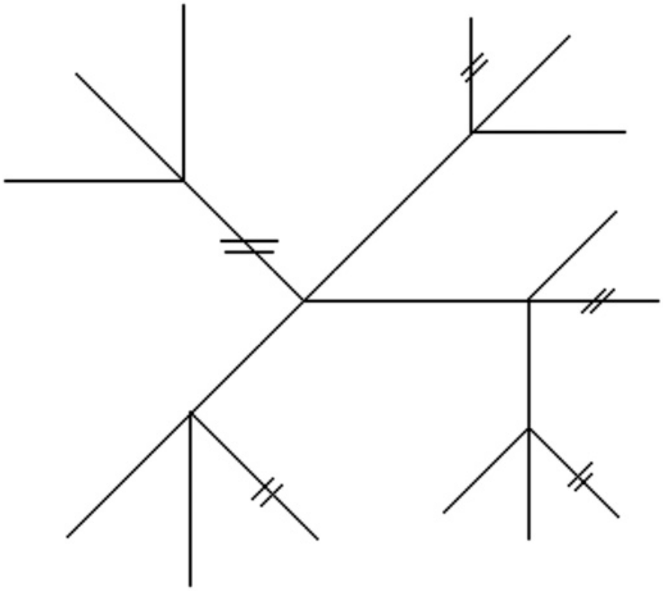

Fig. 1. – The edges of the set E have been marked with a double slash.

Finally, we observe that a transitive group with property $(*)$ contains no inversions (because an inversion cannot fix any point of Ω) while a transitive group with property $(**)$ contains inversions (surely on the edges of E because the group acts transitively on X and, by definition, E does not contain adjacent edges).

DEFINITION 2.3. – *We say that a subgroup G of $Aut(X)$ acts locally transitively on X if, for every vertex a, the stabilizer G_a acts transitively on the set of adjacent vertices of a.*

The Definition 2.3 means that if a, b and c are three distinct vertices such that $d(a,b) = d(a,c) = 1$ (and so $d(b,c) = 2$) then there exists a rotation g fixing a and such that $g(b) = c$. Because every geodesic of length $2h$ can be viewed as a union of h geodesics of length 2, if x and y are two vertices such that $d(x,y)$ is even, then there exist g in G such that $g(x) = y$. This implies that, for every vertex x, the orbit $G(x)$ on X contains all the vertices v such that $d(x,v)$ is even. Therefore if O and O' are two adjacent vertices, then $G(O) \cup G(O') = X$ and so either $G(O) = G(O')$ and G acts transitively on X or $G(O) \neq G(O')$ and G has two orbits. If G acts transitively and locally transitively on X then G acts transitively on the set of oriented edges. If G has two orbits, then $G(O)$ and $G(O')$ are the two equivalence classes (called X^+ and X^- in [1, p. 27]) of the relation "$d(x,y)$ is an even number".

3. – Groups acting transitively on the homogeneous tree of degree three

In this section we will give a description of closed transitive subgroups of isometries of the homogeneous tree of degree three and we will discuss some characterizations which hold only if $q+1=3$.

LEMMA 3.1. – *Let X_3 be an homogeneous tree of degree three, G a closed transitive subgroup of $Aut(X_3)$ and G_O the stabilizer of the vertex O in G. Then*

(1) *If G_O is trivial then G is isomorphic either to $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ or to $\mathbb{Z} * \mathbb{Z}_2$.*
(2) *If G_O is not trivial then G is either amenable or unimodular, depending on whether G satisfies $(*)$ or not.*

PROOF. – Let X_3 be the homogeneous tree of degree three; let G be a closed subgroup of $Aut(X_3)$ acting transitively on X_3. The stabilizer G_O of a vertex O identifies a permutation group on the set of the three adjacent vertices of O. The stabilizers G_O are conjugates of each other because G acts transitively on X_3, hence the groups of permutations induced by the stabilizers G_O are all isomorphic to each other. Therefore, there are only three possibilities:

i) for every vertex O, the stabilizer G_O fixes the three adjacent vertices of O.
ii) for every O, the stabilizer G_O fixes one adjacent vertex and it interchanges the other two.
iii) for every O, the stabilizer G_O acts transitively on the set of adjacent vertices of O.

If i) holds, then every rotation around the vertex O fixes all the adjacent vertices of O and this is true for every vertex. In particular this is true for the three adjacent vertices of O and so on. Hence $G_O=\{\mathbf{1}\}$ for every O (*i.e.* G acts faithfully on X) and so G is a simply transitive subgroup. Moreover, G is discrete and it is isomorphic either to $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ or to $\mathbb{Z} * \mathbb{Z}_2$ (see [1, p. 15]).

If ii) holds, then the group G has either property $(*)$ or property $(**)$. In section 2 (Preliminaries), we have discussed the groups with these properties for every $q+1 \geq 3$. Therefore G is either amenable or unimodular and it is amenable if and only if G satisfies $(*)$.

Finally, if iii) holds, then G is locally transitive and so it is unimodular because $|G_a(b)| = |G_b(a)| = q+1$ for every pair of vertices a and b such that $d(a,b)=1$. □

Observe that every simply transitive subgroup is unimodular, non-amenable and it contains inversions (in fact the generator of $\mathbb{Z}_2$ is an inversion).

If $q+1=3$ and G is a transitive subgroup with property $(**)$ then G is non-amenable, but it has infinitely many non-compact open amenable subgroups. In fact, for every vertex x, there exists one and only one doubly infinite geodesic γ_x containing x and such that γ_x contains no edge of the set E. Because G acts transitively on X_3, by [3, Lemma 1 p. 378], G contains a step-1 translation along a doubly infinite geodesic containing x, which is nothing but γ_x because the step-1 translation stabilizes E. The stabilizer $G^{\sim}_{\gamma_x} = \{g \in G : g(\gamma_x) = \gamma_x\}$ of the geodesic γ_x in G is an open amenable non-compact subgroup of G for every vertex x because $G^{\sim}_{\gamma_x}$ contains the compact open subgroup $G_a \cap G_b$, where $[a, b]$ is an edge of γ_x. The groups $G^{\sim}_{\gamma_x}$ are conjugate to each other in G.

If G is locally transitive then it is transitive on the set of oriented edges. In particular G is non-amenable and it contains inversions. The transitive action on the set of oriented edges implies that there are only two possibilities as illustrated by Lemma 3.3 below which is true only if $q + 1 = 3$.

LEMMA 3.2. – *Let G be a closed subgroup of Aut(X). We suppose that there exists a positive integer n_0 such that, for every geodesic $[a, b]$ of length n_0, the subgroup $G_{[a,b]}$ of G fixing all vertices of $[a, b]$ also fixes all the adjacent vertices of a and all the adjacent vertices of b. Then G is discrete.*

PROOF. – Let $I(O, n_0) = \{v \in X : d(O, v) \leq n_0\}$, the ball of center O and radius n_0. Let $G_{I(O,n_0)}$ be the subgroup of G fixing all vertices of the ball $I(O, n_0)$ and let $h \in G_{I(O,n_0)}$. For every vertex x at a distance $n_0 + 1$ from O, let $[O, x]$ be the geodesic of length $n_0 + 1$ joining O to x and let y be the vertex in $[O, x]$ at a distance 1 from x. Then the geodesic $[O, y]$ is contained in $I(O, n_0)$ and it is fixed by h. Because $[O, y]$ has length n_0, then, by hypothesis, $h(x) = x$. This is true for every x at a distance $n_0 + 1$ from O, and so h fixes every vertex of the ball $I(O, n_0 + 1)$. We proceed in the same manner and we prove that h fixes every vertex of $I(O, n)$ for every n and so $h = \mathbf{1}$. This means that $G_{I(O,n_0)} = \{\mathbf{1}\}$ and G is discrete because $I(O, n_0)$ is finite and $G_{I(O,n_0)}$ is a compact open subgroup of G. □

LEMMA 3.3. – *Let G be a closed subgroup of Aut(X_3), where X_3 is the homogeneous tree of degree three. If G acts transitively and locally transitively on X_3, then either G is discrete or G acts transitively on the tree boundary Ω.*

PROOF. – Suppose G is non-discrete. To prove Lemma 3.3, it is enough to show that G acts transitively on the set of oriented geodesics of length n, for every n. We prove this by induction on n. The case $n = 1$ is true by hypothesis. We suppose that G acts transitively on the set of oriented geodesics of length n. Let $G_{[a,b]}$ be the group fixing each vertex of a geodesic $[a, b]$ of length n. Then if the group $G_{[a,b]}$ also fixes the other two adjacent vertices of b not contained in $[a, b]$, then this would be true for every oriented geodesic of length n because the stabilizers of the oriented geodesics of length n are conjugate to each other. Therefore, Lemma 3.2 with $n_0 = n$ would imply the discreetness of G. Hence $G_{[a,b]}$ interchanges the two adjacent vertices of b not contained in $[a,b]$, for every oriented geodesic $[a,b]$ of length n. We prove now that G acts transitively on the set of oriented geodesics of length $n+1$. Let $[a,b]$ and $[a',b']$ be two geodesics of length $n+1$; let x and x' be the vertices at a distance 1 from b and b' in the geodesics $[a,b]$ and $[a',b']$, respectively. Because the geodesics $[a,x]$ and $[a',x']$ have length n, then there exists $g \in G$ such that g maps $[a,x]$ onto $[a',x']$ (with $g(a)=a'$ and $g(x)=x'$). The vertex $g(b)$ has a distance 1 from x' and it is not in $[a',x']$, therefore there exists h in G fixing each vertex of $[a',x']$ and such that $h(g(b))=b'$. Hence, the isometry hg maps the geodesic $[a,b]$ onto $[a',b']$ with $hg(a)=a'$ and $hg(b)=b'$. □

We can now summarize the results obtained.

PROPOSITION 3.1. – *Let X_3 be the homogeneous tree of degree three. Let G be a closed non-discrete subgroup of $Aut(X_3)$ acting transitively on X_3. Then one and only one of the following occurs.*

(1) *G acts transitively on the tree boundary Ω.*
(2) *G fixes one end ω of Ω and it acts transitively on $\Omega \setminus \{\omega\}$.*
(3) *G leaves invariant a set E of non-oriented edges of X_3 such that for every vertex x there exists one and only one edge in E containing x.*

In both cases 1) and 3), G is non-amenable, unimodular and it contains inversions while, in the case 2), G is amenable, non-unimodular and it does not contain inversions. The groups of the class 2) have the property $(*)$ and the groups of the class 3) have the property $(**)$.

The groups of both classes 1) and 2) are non-discrete because any discrete subgroup of $Aut(X)$ is countable while the boundary Ω and $\Omega \setminus \{\omega\}$ are continuous sets. Therefore, a discrete transitive subgroup of $Aut(X)$ does not act transitively neither on Ω nor on $\Omega \setminus \{\omega\}$ and it is non-amenable.

PROPOSITION 3.2. – *Let X_3 be the homogeneous tree of degree three. Let Γ be a discrete subgroup of $Aut(X_3)$ acting transitively on X_3. Then one and only one of the following occurs.*

(1) *Γ acts locally transitively on X_3 (i.e. Γ acts transitively on the set of oriented edges of X_3).*
(2) *Γ acts simply transitively on X_3 (and Γ is isomorphic either to $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ or to $\mathbb{Z} * \mathbb{Z}_2$).*
(3) *Γ leaves invariant a set E of non-oriented edges of X_3 such that for every vertex x there exists one and only one edge in E containing x.*

Now we provide examples of discrete groups of the classes 1) and 3) of the Proposition 3.2. We consider the general case $q+1 \geq 3$. The first example is a discrete transitive group which acts locally transitively on X, in fact it acts locally as the full permutation groups S_{q+1}. Let $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2 * \ldots * \mathbb{Z}_2$ be the free product of $q+1$ copies of $\mathbb{Z}_2$. Let $\{a_1, a_2, \ldots, a_{q+1}\}$ be a set of generators of Γ consisting of the $q+1$ generators of order 2 of the $q+1$ copies of $\mathbb{Z}_2$. The Cayley graph X associated to Γ and $\{a_1, a_2, \ldots, a_{q+1}\}$ is a homogeneous tree of degree $q+1$. Moreover, Γ acts on X by isometries and, as subgroup of $Aut(X)$, it is simply transitive. Let O be the vertex of X which corresponds with the identity of Γ. Therefore, the set $\{a_1, a_2, \ldots, a_{q+1}\}$ is the set of $q+1$ adjacent vertices of O. Let $N(\Gamma) = \{g \in Aut(X) : g\Gamma = \Gamma g\}$ be the normalizer of Γ in $Aut(X)$. By definition, Γ is a normal subgroup of $N(\Gamma)$ and so $N(\Gamma) = \Gamma N_O$, where N_O is the stabilizer of the vertex O in $N(\Gamma)$. It is easy to see that the rotations of N_O are, exactly, the group-automorphisms of $\Gamma \approx X$ which stabilize the set $\{a_1, a_2, \ldots, a_{q+1}\}$ of generators (recall that a group-automorphism of Γ is a bijective map θ from Γ onto Γ such that $\theta(xy) = \theta(x)\theta(y)$ for every x and y in Γ).

Vice versa, the group-automorphisms of Γ which stabilize the set of generators $\{a_1, a_2, \ldots, a_{q+1}\}$ can be regarded as rotations of $X \approx \Gamma$ normalizing Γ. Each permutation of the set of generators $\{a_1, a_2, \ldots, a_{q+1}\}$ can be uniquely extended to an automorphism of the group Γ, hence the group N_O is finite and it is isomorphic to the full permutation group on $q+1$ objects S_{q+1} (if the simply transitive group Γ is not of the type $\mathbb{Z}_2 * \mathbb{Z}_2 * \ldots * \mathbb{Z}_2$ then the situation is a little different because Γ contains generators of infinite order and $N_O \neq S_{q+1}$). Therefore, $N(\Gamma)$ is discrete because N_O is finite and it is transitive and locally transitive on X, in fact $N(\Gamma)$ acts locally as the full group S_{q+1}. In particular $N(\Gamma)$ is locally doubly transitive (*i.e.* for every vertex x the stabilizer of x in $N(\Gamma)$ acts doubly transitively on the set of $q+1$ adjacent vertices of x) and so it is transitive on the set of oriented geodesics of length 2 (recall that a group G acts transitively on the set of oriented geodesics of length 2 if and only if G is both transitive and locally doubly transitive on X). We observe also that it is not transitive on the set of oriented geodesics of length 3. In fact if a rotation θ of $N(\Gamma)$ fixes α, O and β, where α and β are two generators of Γ and, as said already, O is the identity of $\Gamma \approx X$, then θ is an automorphism of the group Γ and so θ also fixes the words $\alpha\beta$, $\alpha\beta\alpha$, $\alpha\beta\alpha\beta \ldots$ and the words β, $\beta\alpha$, $\beta\alpha\beta$, $\beta\alpha\beta\alpha \ldots$ etc. This means that θ fixes all vertices of a doubly infinite geodesic containing O, α and β and the action of $N(\Gamma)$ on the set of oriented geodesics of length 3 is not transitive, otherwise there would be a rotation θ in $N(\Gamma)$ fixing α, O and β such that $\theta(\alpha\beta) = \alpha\gamma \neq \alpha\beta$ where γ is a generator of Γ with $\alpha \neq \gamma \neq \beta$. Finally we observe that, in the special case $q+1 = 3$, the group $N(\Gamma)$ acts simply transitively on the set of oriented geodesics of length 2; in fact if, as above, θ is a rotation in $N(\Gamma)$ which fixes α, O and β then it also fixes the third generator of Γ and so θ is the identity because it is a group-automorphism of Γ fixing all generators of Γ (if $q+1 > 3$, then the group-automorphism θ fixing α, O and β can be chosen different from the identity and so $N(\Gamma)$ is not simply transitive on the set of oriented geodesics of length 2 when $q+1 > 3$).

The subgroups of $N(\Gamma)$ provide us with examples of discrete transitive groups with property $(**)$. Let θ be the automorphism of the group Γ such that $\theta(a_1) = a_1$, $\theta(a_2) = a_3$, $\theta(a_3) = a_4, \ldots$, $\theta(a_q) = a_{q+1}$ and $\theta(a_{q+1}) = a_2$. Because $\theta\Gamma = \Gamma\theta$, then $\Gamma' = \Gamma\langle\theta\rangle$ (*i.e.* the product of Γ and the subgroup generated by θ) is a group, in fact a discrete transitive subgroup of $N(\Gamma)$. Γ' has property $(**)$, indeed for every vertex x there exists one and only one edge containing both vertices x and xa_1. The set of these edges is the Γ'-invariant set E which appear in the definition of property $(**)$. Because $\Gamma'_{xa_1} = \Gamma'_x$, then Γ' has property $(**)$.

As observed, all the groups belonging to the class 2) of Proposition 3.1 are amenable, non-unimodular and they contain no inversions. On the contrary, all other transitive subgroups are non-amenable, unimodular and they contain inversions. Therefore, Propositions 3.1 and 3.2 imply the following Corollary.

COROLLARY 3.1. – *Let G be a closed transitive subgroup of Aut*(X_3)*. Then the following are equivalent:*

(1) *G is amenable.*
(2) *G is not unimodular.*
(3) *G acts without inversions.*

These properties are equivalent only if $q + 1 = 3$ (see the section below).

4. – Examples of transitive non-unimodular non-amenable groups

In this section we describe a class of groups denoted $Aut(X_f)$ which contains examples of non-unimodular non-amenable transitive subgroups for $q + 1 > 3$. The definition of the groups $Aut(X_f)$ is very close to that of the groups $Aut_f X$ introduced by J. Tits in [4, p. 200].

Let f be a positive function on the vertices of X, so $f: X \to \mathbb{R}^+$ (the condition of positivity of f is not really necessary but in all examples considered in this paper f is positive). We define $Aut(X_f)$ as the set of all isometries of X such that:

$$Aut(X_f) = \{g \in Aut(X) : \exists \delta(g) \in \mathbb{R}^+ \text{ such that } f(g(x)) = \delta(g) f(x)\, \forall x \in X\}$$

The group $Aut_f X$ of Tits is the subgroup of $Aut(X_f)$ consisting of all isometries g such that $\delta(g) = 1$. The group $Aut_f X$ acts transitively on X only if f is the constant function but, in this case, $Aut_f X$ is the full group $Aut(X)$.

For every f, it is easy to see that the set $Aut(X_f)$ is a closed subgroup of $Aut(X)$ and δ is a continuous homomorphism of $Aut(X_f)$ into the multiplicative group of positive real numbers $\mathbb{R}^+$ whose kernel is the group $Aut_f X$ of Tits. We observe that the kernel of δ contains all rotations and all inversions of $Aut(X_f)$ (and so it contains also the subgroup generated by them). In fact if k is a rotation of $Aut(X_f)$ and O is a vertex such that $k(O) = O$ then $f(O) = f(k(O)) = \delta(k) f(O)$

and so $\delta(k) = 1$. If u is an inversion of $Aut(X_f)$ then u^2 is a rotation and $(\delta(u))^2 = \delta(u^2) = 1$, that is $\delta(u) = 1$. For particular functions f, the groups $Aut(X_f)$ are transitive, non-amenable and non-unimodular.

First we suppose $q+1 \geq 5$. We define a positive function f on the vertices of X according to the following rule: for every vertex v, the function f has the value of $2f(v)$ twice and the value of $f(v)/2$ $(q-1)$-times on the set of $q+1$ adjacent vertices of v. In fact, we can define $f(O) = 1$ on a vertex O; thus we proceed respecting this rule at each vertex. If $f = 1$ on a vertex, then f assumes only the values 2^n for every integer n. If a and b are the two vertices of the edge $[a, b]$, then the value $f(a)$ is two times $f(b)$ or vice versa. Hence $Aut(X_f)$ acts without inversions because $\delta(u) = 1$ for every inversion.

Moreover, if $f(x) = 1$ and $f(y) = 2^n$, then the tree whose vertices are labeled with the values of the function f viewed from the vertex y is the same as the tree whose vertices are labeled with the values of the function $2^n f$ viewed from the vertex x. In other words, there exists an isometry g in $Aut(X_f)$ with $\delta(g) = 2^n$ and such that $g(x) = y$. This means that $Aut(X_f)$ is a closed transitive subgroup of $Aut(X)$ acting without inversions.

The fact that $\delta \equiv 1$ on the set of rotations implies that the group $Aut(X_f)_O$ has two orbits on the set of adjacent vertices of O, one orbit containing the two vertices where f has the value of $2f(O)$ and the other containing the $q-1$ vertices on which f has the value of $f(O)/2$. This means that $Aut(X_f)$ is non-amenable because it does not fix any point of Ω. Moreover, if $d(a, b) = 1$ and $f(b) = f(a)/2$, then

$$\left|Aut(X_f)_a(b)\right| = (q-1) \geq 3 \quad \text{and} \quad |Aut(X)_b(a)| = 2$$

This proves that $Aut(X_f)$ is not unimodular.

More precisely, the group $Aut(X_f)_a \cap Aut(X_f)_b$ has index $(q-1)$ in $Aut(X_f)_a$ and it has index 2 in $Aut(X_f)_b$. Hence

$$m(Aut(X_f)_a) = \frac{q-1}{2} m(Aut(X_f)_b)$$

Let $t \in Aut(X_f)$ be such that $t(b) = a$, then $Aut(X_f)_a = t(Aut(X_f)_b)t^{-1}$ and so

$$m(Aut(X_f)_a) = \Delta_{Aut(X_f)}(t) m(Aut(X_f)_b)$$

that is $\Delta_{Aut(X_f)}(t) = (q+1)/2$ while $\delta(t) = 2$; hence $\Delta_{Aut(X_f)} \neq \delta$.

On the other hand, if γ is a doubly infinite geodesic on which f takes only two values, say α and 2α, which repeat alternately, then there exists a step-2 translation v in $Aut(X_f)$ along γ and such that $\delta(v) = \Delta_{Aut(X_f)}(v) = 1$.

If $q+1 = 4$, then the same construction gives us a unimodular group. To obtain a non-unimodular group when $q+1 = 4$, it is enough to modify the definition of f as follows: for every vertex v, the function f has the value of $f(v)$ once, the value of $2f(v)$ once and the value of $f(v)/2$ twice on the four adjacent vertices of v. In this way, for $q+1=4$, we have a transitive non-amenable non-unimodular group which contains inversions, precisely, on those edges $[a,b]$ such that $f(a)=f(b)$.

It cannot be otherwise: if $q+1=4$, then every transitive non-unimodular non-amenable group contains always inversions. In fact, as in the case of the tree X_3, if G is a closed transitive subgroup of $Aut(X_4)$, then the stabilizer G_O of a vertex O identifies a permutation group on the set of the four adjacent vertices of O and the groups of permutations induced by the stabilizers G_O are all isomorphic to each other. It is easy to see that here there are five possibilities for the action of the stabilizer G_O on the set of the adjacent vertices of O, all independent from O because G is transitive. We recall that the groups with property $(*)$ are amenable and the groups with property $(**)$ are unimodular. Also we recall that if the orbits of the action of G_O on the set of the adjacent vertices have all the same number of vertices then G is unimodular because $|G_a(b)| = |G_b(a)|$ for every pair of adjacent vertices a and b (Lemma 2.1). Therefore every closed transitive non-unimodular non-amenable subgroup of $Aut(X_4)$ has the following local property (as the group $Aut(X_f)$ just defined for $q+1=4$):

The stabilizer of a vertex O fixes two adjacent vertices of O and it interchanges the other two.

This means that, for every vertex O there exist two distinct adjacent vertices, say O^+ and O^-, such that:

$$G_O \subset G_{O^+} \cap G_{O^-}$$

and there exists a rotation k in G such that $k(O)=O$, $k(O^+)=O^+$, $k(O^-)=O^-$, $k(a)=b$ and $k(b)=a$ where a and b are the other two adjacent vertices of O. This fact implies easily that the group G always contains inversions.

The construction for $q+1=4$ can be extended, mutatis mutandis, to every $q+1\geq 4$ to obtain non-unimodular non-amenable transitive subgroups with inversions.

Finally we observe that for every $q+1\geq 3$, $Aut(X)_\omega$, the stabilizer of an end $\omega\in\Omega$, can be regarded as a group $Aut(X_f)$ for a suitable function f. In fact we define f in the following way: for every vertex v, the function f has the value of $f(v)/2$ on each adjacent vertex of v except the unique adjacent vertex in the direction of ω on which f has the value of $2f(v)$ (we recall that the vertex in the direction of ω is the unique adjacent vertex of v which belongs to the geodesic $[v,\omega)$). Therefore, for every vertex x, there is one and only one infinite geodesic starting at x, say $\{x,x_1,x_2,x_3,\ldots\}$, such that $\lim f(x_n)=+\infty$ as n tends to infinity. The geodesic $\{x,x_1,x_2,x_3,\ldots\}$ is precisely the geodesic $[x,\omega)$. If $g\in Aut(X_f)$, then $f(g(x_n))=\delta(g)f(x_n)$ for every n and so $f(g(x_n))$ also tends to infinity as n tends to infinity. This means that $g\omega=\omega$ and $Aut(X_f)\subset (Aut(X))_\omega$.

Moreover, the function f is constant on each horocycle associated to ω and so $B_\omega \subset Aut(X_f)$. It is easy to see that the group $Aut(X_f)$ contains a step-1 translation t with $\delta(t) = 2$ along a doubly infinite geodesic containing $\{x, x_1, x_2, x_3, \ldots\}$. Hence $Aut(X_f) = Aut(X)_\omega$ because $Aut(X)_\omega = \langle t \rangle B_\omega$.

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