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Reciprocal Formulae on Binomial Convolutions of Hagen-Rothe Type

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Abstract. – *By means of duplicate inverse series relations, we investigate dual relations of four binomial convolution identities. Four classes of reciprocal formulae on binomial convolutions of Hagen-Rothe type are established. They reflect certain "reciprocity" on the Hagen-Rothe-like convolutions in the sense that each binomial summation involved has no closed form in general, but their sum and difference in pairs do have simple expressions in a single term of binomial coefficients.*

1. – Introduction and Motivation

There are numerous binomial convolution formulae in mathematical literature. The most general ones may be the Hagen-Rothe identities:

$$\sum_{k=0}^{n}\frac{a}{a+bk}\binom{a+bk}{k}\binom{c-bk}{n-k}=\binom{a+c}{n}, \tag{1a}$$

$$\sum_{k=0}^{n}\binom{a+bk}{k}\frac{c-bn}{c-bk}\binom{c-bk}{n-k}=\binom{a+c}{n}, \tag{1b}$$

$$\sum_{k=0}^{n}\frac{a}{a+bk}\binom{a+bk}{k}\frac{c-bn}{c-bk}\binom{c-bk}{n-k}=\frac{a+c-bn}{a+c}\binom{a+c}{n}. \tag{1c}$$

The interested reader may consult Gould [10] and Graham *et al* [12, § 5.4] for different proofs as well as Merlini *et al* [13] and Sprugnoli [15] for their connections with Riordan arrays.

In their work on plane partition enumerations, Andrews and Burge [1, Eqs 3.1-3.2] discovered via hypergeometric transformations the two convolution formulae

$$\sum_{k\geq 0}\frac{2a}{a+k}\binom{a+k}{2k}\binom{c-k}{n-2k}=\binom{a+c}{n}+\binom{c-a}{n}, \tag{2a}$$

$$\sum_{k\geq 0}\frac{2a-1}{a+k}\binom{a+k}{1+2k}\binom{c-k}{n-2k}=\binom{a+c}{1+n}-\binom{1+c-a}{1+n}. \tag{2b}$$

By inverting the summation order, it is not hard to check that their reversals result in the following two further binomial identities

$$\sum_{k\geq 0}\binom{a+k}{2k}\frac{2c-n}{c-k}\binom{c-k}{n-2k}=\binom{a+c}{n}+\binom{c-a-1}{n}, \tag{3}$$

$$\sum_{k\geq 0}\binom{a+k}{1+2k}\frac{2c-n}{c-k}\binom{c-k}{n-2k}=\binom{a+c}{1+n}-\binom{c-a}{1+n}. \tag{4}$$

By examining the bisection series of the generating functions, the present author [4, 5] not only proved the last formulae, but also found two additional ones:

$$\sum_{k\geq 0}\frac{2a}{a+k}\binom{a+k}{2k}\frac{2c-n}{c-k}\binom{c-k}{n-2k} \tag{5a}$$

$$=\frac{2a+2c-n}{a+c}\binom{a+c}{n}+\frac{2c-2a-n}{c-a}\binom{c-a}{n}; \tag{5b}$$

$$\sum_{k\geq 0}\frac{2a-1}{a+k}\binom{a+k}{1+2k}\frac{2c-n}{c-k}\binom{c-k}{n-2k} \tag{6a}$$

$$=\frac{2a+2c-n-1}{a+c}\binom{a+c}{1+n}+\frac{n+2a-2c-1}{1+c-a}\binom{1+c-a}{1+n}. \tag{6b}$$

There are many inverse series relations in mathematical literatures (cf. Riordan's monograph [14, Chapters 2-3] and [2, 13]). Among them, Gould-Hsu [11] inversions have been shown to be efficient in dealing with binomial identities (cf. [3, 8, 9]). In order to prove the terminating balanced hypergeometric series identities, the same approach has been developed by Chu [6, 7] to the duplicate inverse series relations, which may be reproduced, for facilitating the subsequent use, as the following **Theorem**. For two complex variables x, y and four complex sequences $\{a_k, b_k, c_k, d_k\}_{k\geq 0}$, define two polynomial sequences by

$$\phi(x;0)\equiv 1 \quad\text{and}\quad \phi(x;m)=\prod_{i=0}^{m-1}(a_i+xb_i) \quad\text{for}\quad n=1,2,\cdots; \tag{7a}$$

$$\psi(y;0)\equiv 1 \quad\text{and}\quad \psi(y;n)=\prod_{j=0}^{n-1}(c_j+yd_j) \quad\text{for}\quad n=1,2,\cdots. \tag{7b}$$

Then the system of equations

$$\Omega_n=\sum_{k\geq 0}\binom{n}{2k}\frac{c_k+2kd_k}{\phi(n;k)\psi(n;k+1)}f(k) \tag{8a}$$

$$-\sum_{k\geq 0}\binom{n}{1+2k}\frac{a_k+(1+2k)b_k}{\phi(n;1+k)\psi(n;k+1)}g(k) \tag{8b}$$

is equivalent to the system of equations

$$f(n)=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\phi(k;n)\psi(k;n)\Omega_k, \tag{9a}$$

$$g(n)=\sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}\phi(k;n)\psi(k;n+1)\Omega_k. \tag{9b}$$

The purpose of this paper is to derive four classes of reciprocal formulae on binomial convolutions of Hagen-Rothe type by employing this duplicate inversion theorem to the binomial summation formulae displayed from (3) to (6). As an exemplification, we anticipate the following reciprocal relations (see Theorem 1)

$$\begin{aligned} A_0(x;a,b,n)+A_0(-x;a,b,n) &= \binom{x+n-1/2}{2n}, \\ A_1(x;a,b,n)+A_1(-x;a,b,n) &= 0; \end{aligned}$$

where the A-function is defined by the binomial convolution

$$A_\delta(x;a,b,n):=\sum_{k=0}^{\delta+2n}\binom{a+x+bk}{k}\binom{n-a-bk-1/2}{\delta+2n-k}\frac{a+2bn-n+1/2}{1+2a+2bk-k}.$$

The surprising fact is that the formulae obtained in this paper reflect certain "reciprocity" on the Hagen-Rothe-like convolutions; i.e., even though each binomial summation involved has no closed form in general, their sum and difference in pairs do have simple expressions in a single term of binomial coefficients. Considering the importance of Hagen-Rothe identities in combinatorial computation and enumeration, it is believed that similar applications will come out also for the reciprocal relations obtained here.

The rest of the paper will be organized as follows. From the second section to the fifth section, the eight reciprocity theorems will be shown, through the duplicate inversion theorem, to be dual relations to the four identities displayed from (3) to (6). Further reciprocal formulae will be presented in the sixth section, that result from linear combinations of four reciprocal relations established in the four preceding sections.

2. – Reciprocal Formulae via Identity (3)

This section will show how to utilize the inversion techniques to derive binomial sum identities. It will consist of three steps: reformulating first a known binomial equation, such as (3), (4), (5a-5b) and (6a-6b), in terms of (8a-8b), then identifying three sequences $\{f(k), g(k), \Omega_n\}$ and finally writing down the corresponding dual relations (9a-9b).

Now we are going to illustrate this process by taking (3) as an example. For a complex x and a nonnegative integer n, the shifted-factorial is defined by the classical Pochhammer symbol

$$(10) \qquad (x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for} \quad n = 1, 2, \cdots.$$

Replacing a by $x - 1/2$ and c by $a + bn + 1/2$ in (3), we can restate the resulting binomial sum as follows:

$$(11a) \quad \sum_{k\geq 0} \binom{n}{2k} \frac{a+2bk-k+1/2}{(1/2-a-bn)_k(a+bn-n+1/2)_{k+1}} \frac{(1/2+x)_k(1/2-x)_k}{a+2bk-k+1/2}$$

$$(11b) \quad = \frac{(-x-a-bn)_n}{(1+2a+2bn-n)(1/2-a-bn)_n} + \frac{(x-a-bn)_n}{(1+2a+2bn-n)(1/2-a-bn)_n}.$$

Comparing this equation with (8a-8b) specified under the settings

$$\phi(x;n) := (1/2-a-bx)_n, \quad \psi(y;n) := (a+by-y+1/2)_n;$$

$$f(n) := \frac{(1/2-x)_n(1/2+x)_n}{a+2bn-n+1/2}, \qquad g(n) := 0;$$

$$\Omega_n := \frac{(-x-a-bn)_n}{(1+2a+2bn-n)(1/2-a-bn)_n} + \frac{(x-a-bn)_n}{(1+2a+2bn-n)(1/2-a-bn)_n};$$

we get the following dual relations corresponding to (9a) and (9b) respectively

$$f(n) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} (1/2-a-bk)_n (a+bk-k+1/2)_n \Omega_k,$$

$$g(n) = \sum_{k=0}^{1+2n} (-1)^k \binom{1+2n}{k} (1/2-a-bk)_n (a+bk-k+1/2)_{n+1} \Omega_k .$$

They can be expressed as the following reciprocal convolution formulae.

THEOREM 1. – *For the A-function defined by the binomial convolution*

$$A_\delta(x;a,b,n) := \sum_{k=0}^{\delta+2n} \binom{a+x+bk}{k} \binom{n-a-bk-1/2}{\delta+2n-k} \frac{a+2bn-n+1/2}{1+2a+2bk-k}$$

there hold the following reciprocal relations:

$$A_0(x;a,b,n) + A_0(-x;a,b,n) = \binom{x+n-1/2}{2n},$$

$$A_1(x;a,b,n) + A_1(-x;a,b,n) = 0.$$

The two formulae are said to be "reciprocal" because $A_\delta(x;a,b,n) + A_\delta(-x;a,b,n)$ for $\delta = 0$ and 1 has simple expressions even though the binomial sum $A_\delta(x;a,b,n)$ does not have a closed form (which can easily be checked by

Mathematica). The reader will see that all the theorems proved in this paper reflect the "reciprocity" in this sense.

For three indeterminate λ, u, v, it is trivial to check the following linear relation

$$u + 2kv - k = \frac{u - \lambda v - k}{\lambda + n}(n - 2k) - \frac{\lambda + 2k}{\lambda + n}(k - u - nv). \tag{12}$$

Applying its special case corresponding to $u \to a + 1/2$ and $v \to b$, we can alternatively reformulate (11a-11b) as the following equation

$$\begin{aligned}
&\sum_{k\geq 0}\binom{n}{2k}\frac{a+2bk-k+1/2}{(-a-bn-1/2)_k(a+bn-n+1/2)_{k+1}}\frac{(1/2+x)_k(1/2-x)_k}{a+2bk-k+1/2}\frac{\lambda+2k}{a+2bk-k+1/2}\\
&+\sum_{k\geq 0}\binom{n}{1+2k}\frac{-a-b-2bk+k-1/2}{(-a-bn-1/2)_{k+1}(a+bn-n+1/2)_{k+1}}\frac{(1/2+x)_k(1/2-x)_k}{a+2bk-k+1/2}\frac{(1+2k)(a-b\lambda-k+1/2)}{a+b+2bk-k+1/2}\\
&=\frac{\lambda+n}{(a+bn+1/2)(1+2a+2bn-n)}\frac{(-x-a-bn)_n}{(1/2-a-bn)_n}\\
&+\frac{\lambda+n}{(a+bn+1/2)(1+2a+2bn-n)}\frac{(x-a-bn)_n}{(1/2-a-bn)_n}.
\end{aligned}$$

This equation matches with (8a-8b) perfectly under the following specifications:

$$\begin{aligned}
&\phi(x;n) := (-a-bx-1/2)_n, \quad \psi(y;n) := (a+by-y+1/2)_n;\\
&f(n) := \frac{(1/2+x)_n(1/2-x)_n}{a+2bn-n+1/2}\frac{\lambda+2n}{a+2bn-n+1/2},\\
&g(n) := -\frac{(1/2+x)_n(1/2-x)_n}{a+2bn-n+1/2}\frac{(1+2n)(a-b\lambda+n+1/2)}{a+b+2bn-n+1/2};\\
&\Omega_n := \frac{(-x-a-bn)_n}{(1/2-a-bn)_n}\frac{\lambda+n}{(a+bn+1/2)(1+2a+2bn-n)}\\
&\qquad + \frac{(x-a-bn)_n}{(1/2-a-bn)_n}\frac{\lambda+n}{(a+bn+1/2)(1+2a+2bn-n)}.
\end{aligned}$$

Then the dual relations corresponding to (9a) and (9b) lead, respectively, to the following equations:

$$\begin{aligned}
f(n) &= \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}(-a-bk-1/2)_n(a+bk-k+1/2)_n\Omega_k,\\
g(n) &= \sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}(-a-bk-1/2)_n(a+bk-k+1/2)_{n+1}\Omega_k.
\end{aligned}$$

Writing them explicitly in terms of binomial convolutions, we obtain the following reciprocal formulae.

THEOREM 2. – *For the $\mathcal{A}$-function defined by the binomial convolution*

$$\mathcal{A}_\delta(x,\lambda;a,b,n) := \sum_{k=0}^{\delta+2n}\binom{a+x+bk}{k}\binom{n-a-bk-1/2}{\delta+2n-k} \times\frac{(\lambda+k)(a+2bn-n+1/2)}{(1+2a+2bk-k)(a-n+bk+1/2)}$$

there hold the following reciprocal relations:

$$\mathcal{A}_0(x,\lambda;a,b,n)+\mathcal{A}_0(-x,\lambda;a,b,n) = \binom{x+n-1/2}{2n}\frac{\lambda+2n}{a+2bn-n+1/2},$$

$$\mathcal{A}_1(x,\lambda;a,b,n)+\mathcal{A}_1(-x,\lambda;a,b,n) = \binom{x+n-1/2}{2n}\frac{a-b\lambda-n+1/2}{a+b+2bn-n+1/2}.$$

For $\lambda=(a-n+1/2)/b$, this theorem reduces to Theorem 1. When $\lambda=\frac{2a+1}{2b-1}$, these reciprocal relations follow directly from (1b) because in this case, the binomial convolution $\mathcal{A}_\delta(x,\lambda;a,b,n)$ admits a closed form. However this is not possible in general due to the denominator factor $(1+2a+2bk-k)$ in the binomial sums.

3. – Reciprocal Formulae via Identity (4)

Replacing a by x and c by $a+bn$ in (4), we can rewrite the binomial sum as

$$\sum_{k\geq 0}\binom{n}{2k}\frac{a+2bk-k}{(1-a-bn)_k(a+bn-n)_{k+1}}\frac{x(1+x)_k(1-x)_k}{(1+2k)(a+2bk-k)} \tag{13a}$$

$$=\frac{a+x+bn}{(1+n)(2a+2bn-n)}\frac{(1-x-a-bn)_n}{(1-a-bn)_n} \tag{13b}$$

$$-\frac{a-x+bn}{(1+n)(2a+2bn-n)}\frac{(1+x-a-bn)_n}{(1-a-bn)_n}. \tag{13c}$$

Comparing this equation with (8a-8b) specified under the settings

$$\phi(x;n) := (1-a-bx)_n, \quad \psi(y;n) := (a+by-y)_n;$$

$$f(n) := \frac{(1+x)_n(1-x)_n}{a+2bn-n}\frac{x}{1+2n}, \qquad g(n) := 0;$$

$$\Omega_n := \frac{(1-x-a-bn)_n}{(1-a-bn)_n}\frac{a+x+bn}{(1+n)(2a+2bn-n)} - \frac{(1+x-a-bn)_n}{(1-a-bn)_n}\frac{a-x+bn}{(1+n)(2a+2bn-n)};$$

we have the following dual relations corresponding to (9a) and (9b) respectively

$$f(n)=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}(1-a-bk)_n(a+bk-k)_n\Omega_k,$$
$$g(n)=\sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}(1-a-bk)_n(a+bk-k)_{n+1}\Omega_k.$$

For the last two equations, performing the parameter replacements $a\to a+b$ and $k\to k-1$, then $n\to n-1$ for the latter, we derive, after elementary simplifications, the following reciprocal theorem.

THEOREM 3. – *For the B-function defined by the binomial convolution*

$$B_\delta(x;a,b,n):=\sum_{k=0}^{\delta+2n}\binom{a+x+bk}{k}\binom{\delta+n-a-bk-1}{\delta+2n-k}\frac{1+a+2bn-n+b\delta-\delta}{1+2a+2bk-k}$$

there hold the following reciprocal relations:

$$B_0(x;a,b,n)-B_0(-x;a,b,n)=0,$$
$$B_1(x;a,b,n)-B_1(-x;a,b,n)=\binom{x+n}{1+2n}.$$

Applying the special case of (12) corresponding to $u\to a$ and $v\to b$, we can alternatively reformulate (13a-13b-13c) as the following equation

$$\sum_{k\geq 0}\binom{n}{2k}\frac{a+2bk-k}{(-a-bn)_k(a+bn-n)_{k+1}}\frac{(1+x)_k(1-x)_k}{a+2bk-k}\frac{x(1+\lambda+2k)}{(1+2k)(a+2bk-k)}$$
$$-\sum_{k\geq 0}\binom{n}{1+2k}\frac{-a-b-2bk+k}{(-a-bn)_{k+1}(a+bn-n)_{k+1}}\frac{(1+x)_k(1-x)_k}{a+2bk-k}\frac{x(b\lambda-a+b+k)}{a+b+2bk-k}$$
$$=\frac{(1+\lambda+n)(a+x+bn)}{(1+n)(a+bn)(2a+2bn-n)}\frac{(1-x-a-bn)_n}{(1-a-bn)_n}$$
$$-\frac{(1+\lambda+n)(a-x+bn)}{(1+n)(a+bn)(2a+2bn-n)}\frac{(1+x-a-bn)_n}{(1-a-bn)_n}.$$

This equation matches with (8a-8b) perfectly under the following specifications:

$$\phi(x;n):=(-a-bx)_n,\quad \psi(y;n):=(a+by-y)_n;$$
$$f(n):=\frac{(1+x)_n(1-x)_n}{a+2bn-n}\frac{x(1+\lambda+2n)}{(1+2n)(a+2bn-n)},$$
$$g(n):=\frac{(1+x)_n(1-x)_n}{a+2bn-n}\frac{x(b\lambda-a+b+n)}{a+b+2bn-n};$$
$$\Omega_n:=\frac{(1-x-a+b-n-bn)_n}{(1-a-bn)_n}\frac{(1+\lambda+n)(a+x+bn)}{(1+n)(a+bn)(2a+2bn-n)}$$
$$-\frac{(1+x-a-bn)_n}{(1-a-bn)_n}\frac{(1+\lambda+n)(a-x+bn)}{(1+n)(a+bn)(2a+2bn-n)}.$$

Then the dual relations corresponding to (9a) and (9b) give, respectively, the following equations:

$$f(n) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} (-a-bk)_n (a+bk-k)_n \Omega_k,$$

$$g(n) = \sum_{k=0}^{1+2n} (-1)^k \binom{1+2n}{k} (-a-bk)_n (a+bk-k)_{n+1} \Omega_k.$$

For the last two equations, performing the parameter replacements $a \to a+b$ and $k \to k-1$, then $n \to n-1$ for the latter, we derive, after elementary simplifications, the following reciprocal theorem.

THEOREM 4. – *For the $\mathcal{B}$-function defined by the binomial convolution*

$$\mathcal{B}_\delta(x, \lambda; a, b, n) := \sum_{k=0}^{\delta+2n} \binom{a+x+bk}{k} \binom{\delta+n-a-bk-1}{\delta+2n-k} \times \frac{(\lambda+k)(1+a+2bn-n+b\delta-\delta)}{(1+2a+2bk-k)(1+a+bk-n-\delta)}$$

there hold the following reciprocal relations:

$$\mathcal{B}_0(x, \lambda; a, b, n) - \mathcal{B}_0(-x, \lambda; a, b, n) = \frac{2n}{x+n} \binom{x+n}{2n} \frac{1+a-b\lambda-n}{1+a-b+2bn-n},$$

$$\mathcal{B}_1(x, \lambda; a, b, n) - \mathcal{B}_1(-x, \lambda; a, b, n) = \binom{x+n}{1+2n} \frac{1+\lambda+2n}{a+b+2bn-n}.$$

Similar to the observation after Theorem 4, this theorem contains Theorem 3 as a special case for $\lambda = (1+a-n-\delta)/b$. When $\lambda = \dfrac{2a+1}{2b-1}$, the reciprocal relations can be derived directly from (1b).

4. – Reciprocal Formulae via Identity (5)

Replacing a by x and c by $a+bn$ in (5a-5b), we can restate the resulting binomial identity as

$$\sum_{k \geq 0} \binom{n}{2k} \frac{a+2bk-k}{(1-a-bn)_k (a+bn-n)_{k+1}} \frac{2(x)_k(-x)_k}{a+2bk-k} \tag{14a}$$

$$= \frac{(2a+2x+2bn-n)}{(a+x+bn)(2a+2bn-n)} \frac{(-a-x-bn)_n}{(1-a-bn)_n} \tag{14b}$$

$$+ \frac{(2a-2x+2bn-n)}{(a-x+bn)(2a+2bn-n)} \frac{(-a+x-bn)_n}{(1-a-bn)_n}. \tag{14c}$$

Comparing this equation with (8a-8b) under the following settings

$$\begin{aligned}
&\phi(x;n) := (1-a-bx)_n, \quad \psi(y;n) := (a+by-y)_n;\\
&f(n) := \frac{2(x)_n(-x)_n}{a+2bn-n}, \qquad g(n) := 0;\\
&\Omega_n := \frac{(-a-x-bn)_n}{(1-a-bn)_n}\frac{(2a+2x+2bn-n)}{(a+x+bn)(2a+2bn-n)}\\
&\qquad + \frac{(-a+x-bn)_n}{(1-a-bn)_n}\frac{(2a-2x+2bn-n)}{(a-x+bn)(2a+2bn-n)};
\end{aligned}$$

we obtain the dual relations corresponding to (9a) and (9b) respectively

$$f(n) = \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}(1-a-bk)_n(a+bk-k)_n\Omega_k,$$

$$g(n) = \sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}(1-a-bk)_n(a+bk-k)_{n+1}\Omega_k.$$

They are equivalent to the following reciprocal convolution formulae.

THEOREM 5. – *For the C-function defined by the binomial convolution*

$$C_\delta(x;a,b,n) := \sum_{k=0}^{\delta+2n}\binom{a+x+bk}{k}\binom{n-a-bk}{\delta+2n-k}\frac{(2a+2x+2bk-k)(a+2bn-n)}{(a+x+bk)(2a+2bk-k)}$$

there hold the following reciprocal relations:

$$C_0(x;a,b,n) + C_0(-x;a,b,n) = \frac{2x}{x+n}\binom{x+n}{2n},$$

$$C_1(x;a,b,n) + C_1(-x;a,b,n) = 0.$$

In view of the special case of (12) corresponding to $u \to a$ and $v \to b$, we can alternatively reformulate (14a-14b-14c) as the following equation

$$\begin{aligned}
&\sum_{k\geq 0}\binom{n}{2k}\frac{a+2bk-k}{(-a-bn)_k(a+bn-n)_{k+1}}\frac{2(x)_k(-x)_k}{a+2bk-k}\frac{\lambda+2k}{a+2bk-k}\\
&+\sum_{k\geq 0}\binom{n}{1+2k}\frac{-a-b-2bk+k}{(-a-bn)_{k+1}(a+bn-n)_{k+1}}\frac{2(x)_k(-x)_k}{a+2bk-k}\frac{(1+2k)(a-b\lambda-k)}{a+b+2bk-k}\\
&=\frac{(\lambda+n)(2a+2x+2bn-n)}{(a+bn)(a+x+bn)(2a+2bn-n)}\frac{(-a-x-bn)_n}{(1-a-bn)_n}\\
&+\frac{(\lambda+n)(2a-2x+2bn-n)}{(a+bn)(a-x+bn)(2a+2bn-n)}\frac{(-a+x-bn)_n}{(1-a-bn)_n}.
\end{aligned}$$

This equation matches with (8a-8b) perfectly under the following specifications:

$$\phi(x;n) := (-a-bx)_n, \quad \psi(y;n) := (a+by-y)_n;$$

$$f(n) := \frac{2(x)_n(-x)_n}{a+2bn-n}\,\frac{\lambda+2n}{a+2bn-n},$$

$$g(n) := -\frac{2(x)_n(-x)_n}{a+2bn-n}\,\frac{(1+2n)(a-b\lambda-n)}{a+b+2bn-n};$$

$$\Omega_n := \frac{(-a-x-bn)_n}{(1-a-bn)_n}\,\frac{(\lambda+n)(2a+2x+2bn-n)}{(a+bn)(a+x+bn)(2a+2bn-n)}$$
$$+\frac{(-a+x-bn)_n}{(1-a-bn)_n}\,\frac{(\lambda+n)(2a-2x+2bn-n)}{(a+bn)(a-x+bn)(2a+2bn-n)}.$$

Then the dual relations corresponding to (9a) and (9b) read, respectively, as the following equations:

$$f(n) = \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}(-a-bk)_n(a+bk-k)_n\Omega_k,$$

$$g(n) = \sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}(-a-bk)_n(a+bk-k)_{n+1}\Omega_k.$$

In terms of binomial convolutions, they are the following reciprocal formulae.

THEOREM 6. – *For the $\mathcal{C}$-function defined by the binomial convolution*

$$\mathcal{C}_\delta(x,\lambda;a,b,n) := \sum_{k=0}^{\delta+2n}\binom{a+x+bk}{k}\binom{n-a-bk}{\delta+2n-k}$$
$$\times\frac{(\lambda+k)(2a+2x+2bk-k)(a+2bn-n)}{(a+x+bk)(2a+2bk-k)(a+bk-n)}$$

there hold the following reciprocal relations:

$$\mathcal{C}_0(x,\lambda;a,b,n)+\mathcal{C}_0(-x,\lambda;a,b,n) = \frac{2x}{x+n}\binom{x+n}{2n}\frac{\lambda+2n}{a+2bn-n},$$

$$\mathcal{C}_1(x,\lambda;a,b,n)+\mathcal{C}_1(-x,\lambda;a,b,n) = \frac{2x}{x+n}\binom{x+n}{2n}\frac{a-b\lambda-n}{a+b+2bn-n}.$$

For $\lambda=(a-n)/b$, this theorem reduces to Theorem 5. When $\lambda=\frac{2a}{2b-1}$, the reciprocal relations can also be derived directly from (1c), as observed previously.

5. – Reciprocal Formulae via Identity (6)

Replacing a by $x+1/2$ and c by $a+bn-1/2$ in (6a-6b), we can express the resulting binomial identity as

$$\sum_{k\geq 0}\binom{n}{2k}\frac{a+2bk-k-1/2}{(3/2-a-bn)_k(a-1/2+bn-n)_{k+1}}\frac{(1/2+x)_k(1/2-x)_k}{a+2bk-k-1/2}\frac{2x}{1+2k} \tag{15a}$$

$$=\frac{(2a+2x+2bn-n-1)}{(1+n)(2a+2bn-n-1)}\frac{(1-x-a-bn)_n}{(3/2-a-bn)_n} \tag{15b}$$

$$-\frac{(2a-2x+2bn-n-1)}{(1+n)(2a+2bn-n-1)}\frac{(1+x-a-bn)_n}{(3/2-a-bn)_n}. \tag{15c}$$

Comparing this equation with (8a-8b) under the following settings

$$\phi(x;n):=(3/2-a-bx)_n,\quad \psi(y;n):=(a+by-y-1/2)_n;$$

$$f(n):=\frac{(1/2+x)_n(1/2-x)_n}{a+2bn-n-1/2}\frac{2x}{1+2n},\qquad g(n):=0;$$

$$\Omega_n:=\frac{(1-x-a-bn)_n}{(3/2-a-bn)_n}\frac{(2a+2x+2bn-n-1)}{(1+n)(2a+2bn-n-1)}$$
$$-\frac{(1+x-a-bn)_n}{(3/2-a-bn)_n}\frac{(2a-2x+2bn-n-1)}{(1+n)(2a+2bn-n-1)};$$

we derive from the dual relations corresponding to (9a) and (9b) respectively

$$f(n)=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}(3/2-a-bk)_n(a+bk-k-1/2)_n\Omega_k,$$
$$g(n)=\sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}(3/2-a-bk)_n(a+bk-k-1/2)_{n+1}\Omega_k.$$

For the last two equations, performing the parameter replacements $a\to a+b$ and $k\to k-1$, then $n\to n-1$ for the latter, we establish, after elementary simplifications, the following reciprocal theorem.

THEOREM 7. – *For the D-function defined by the binomial convolution*

$$D_\delta(x;a,b,n):=\sum_{k=0}^{\delta+2n}\binom{a+x+bk}{k}\binom{\delta+n-a-bk-1/2}{\delta+2n-k}$$
$$\times\frac{(2a+2x+2bk-k)(a+2bn-n+b\delta-\delta+1/2)}{(a+x+bk)(2a+2bk-k)}$$

there hold the following reciprocal relations:

$$D_0(x;a,b,n)-D_0(-x;a,b,n)=0,$$
$$D_1(x;a,b,n)-D_1(-x;a,b,n)=\frac{4x}{1+2x+2n}\binom{x+n+1/2}{1+2n}.$$

By means of the special case corresponding to $u\to a-1/2$ and $v\to b$ of (12), we can alternatively reformulate (15a-15b-15c) as the following equation

$$\begin{aligned}&\sum_{k\ge0}\binom{n}{2k}\frac{a+2bk-k-1/2}{(1/2-a-bn)_k(a+bn-n-1/2)_{k+1}}\frac{(1/2+x)_k(1/2-x)_k}{a+2bk-k-1/2}\frac{2x(1+\lambda+2k)}{(a+2bk-k-1/2)(1+2k)}\\
-&\sum_{k\ge0}\binom{n}{1+2k}\frac{1/2-a-b-2bk+k}{(1/2-a-bn)_{k+1}(a+bn-n-1/2)_{k+1}}\frac{(1/2+x)_k(1/2-x)_k}{a+2bk-k-1/2}\frac{2x(1/2-a+b+b\lambda+k)}{a+b+2bk-k-1/2}\\
=&\frac{(1+\lambda+n)(2a+2x+2bn-n-1)}{(1+n)(a+bn-1/2)(2a+2bn-n-1)}\frac{(1-x-a-bn)_n}{(3/2-a-bn)_n}\\
-&\frac{(1+\lambda+n)(2a-2x+2bn-n-1)}{(1+n)(a+bn-1/2)(2a+2bn-n-1)}\frac{(1+x-a-bn)_n}{(3/2-a-bn)_n}.\end{aligned}$$

This equation matches with (8a-8b) perfectly under the following specifications:

$$\begin{aligned}&\phi(x;n):=(1/2-a-bx)_n,\quad \psi(y;n):=(a+by-y-1/2)_n;\\
&f(n):=\frac{(1/2+x)_n(1/2-x)_n}{a+2bn-n-1/2}\frac{2x(1+\lambda+2n)}{(a+2bn-n-1/2)(1+2n)},\\
&g(n):=\frac{(1/2+x)_n(1/2-x)_n}{a+2bn-n-1/2}\frac{2x(1/2-a+b+b\lambda+n)}{a+b+2bn-n-1/2};\\
&\Omega_n:=\frac{(1-x-a-bn)_n}{(3/2-a-bn)_n}\frac{(1+\lambda+n)(2a+2x+2bn-n-1)}{(1+n)(a+bn-1/2)(2a+2bn-n-1)}\\
&\qquad-\frac{(1+x-a-bn)_n}{(3/2-a-bn)_n}\frac{(1+\lambda+n)(2a-2x+2bn-n-1)}{(1+n)(a+bn-1/2)(2a+2bn-n-1)}.\end{aligned}$$

Then the dual relations corresponding to (9a) and (9b) yield, respectively, to the following equations:

$$f(n)=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}(1/2-a-bk)_n(a+bk-k-1/2)_n\Omega_k,$$
$$g(n)=\sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}(1/2-a-bk)_n(a+bk-k-1/2)_{n+1}\Omega_k.$$

For the last two equations, performing the parameter replacements $a\to a+b$ and $k\to k-1$, then $n\to n-1$ for the latter, we derive, after elementary simplifications, the following reciprocal theorem.

THEOREM 8. – *For the $\mathcal{D}$-function defined by the binomial convolution*

$$\mathcal{D}_\delta(x,\lambda;a,b,n):=\sum_{k=0}^{\delta+2n}\binom{a+x+bk}{k}\binom{\delta+n-a-bk-1/2}{\delta+2n-k}$$
$$\times\frac{(\lambda+k)(2a+2x+2bk-k)(a+2bn-n+b\delta-\delta+1/2)}{(a+x+bk)(2a+2bk-k)(a-n+bk-\delta+1/2)}$$

there hold the following reciprocal relations:

$$\mathcal{D}_0(x,\lambda;a,b,n)-\mathcal{D}_0(-x,\lambda;a,b,n) = \frac{4nx}{(x+n-1/2)_2}\binom{x+n+1/2}{2n}\frac{a-b\lambda-n+1/2}{a-b+2bn-n+1/2},$$
$$\mathcal{D}_1(x,\lambda;a,b,n)-\mathcal{D}_1(-x,\lambda;a,b,n) = \frac{4x}{1+2x+2n}\binom{x+n+1/2}{1+2n}\frac{1+\lambda+2n}{a+b+2bn-n-1/2}.$$

This theorem reduces to Theorem 7 for $\lambda=(a-n-\delta+1/2)/b$. In particular when $\lambda=\dfrac{2a}{2b-1}$, these reciprocal relations can be deduced directly from the binomial convolution formula displayed in (1c).

6. – Further Reciprocal Formulae

According to Chu [6, 7], for the inverse series relations determined by (8a-8b) and (9a-9b), there exists also a third relation

$$h(n) = \sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}\phi(k;1+n)\psi(k;n)\Omega_k \tag{16a}$$

where $h(n)$ is defined through the linear combination

$$h(n):=f(n)\frac{(1+2n)\{a_nd_n-b_nc_n\}}{c_n+d_n(1+2n)}+g(n)\frac{a_n+b_n(1+2n)}{c_n+d_n(1+2n)}. \tag{16b}$$

Following the proofs of four Theorems 2, 4, 6, 8, we can further establish four reciprocal relations on binomial convolutions. They are summarized as follows.

THEOREM 9. – *There hold the following reciprocal relations*

$$\mathbb{A}(x,\lambda;a,b,n)+\mathbb{A}(-x,\lambda;a,b,n) = \binom{x+n-1/2}{2n}\frac{a-b\lambda+\lambda+n+1/2}{a+b+2bn-n-1/2}, \tag{17a}$$

$$\mathbb{B}(x,\lambda;a,b,n)-\mathbb{B}(-x,\lambda;a,b,n) = \frac{2n}{x+n}\binom{x+n}{2n}\frac{a-b\lambda+\lambda+n}{1+a-b+2bn-n}, \tag{17b}$$

$$\mathbb{C}(x,\lambda;a,b,n)+\mathbb{C}(-x,\lambda;a,b,n) = \frac{2x}{x+n}\binom{x+n}{2n}\frac{a-b\lambda+\lambda+n}{a+b+2bn-n-1}, \tag{17c}$$

$$\mathbb{D}(x,\lambda;a,b,n)-\mathbb{D}(-x,\lambda;a,b,n) = \frac{4nx}{(x+n-1/2)_2}\binom{x+n+1/2}{2n}\frac{a-b\lambda+\lambda+n-1/2}{a-b+2bn-n+1/2}; \tag{17d}$$

where the functions $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$ *are defined by the binomial convolutions*

$$\text{(18a)} \quad \mathbb{A}(x,\lambda;a,b,n) := \sum_{k=0}^{1+2n}\binom{a+x+bk}{k}\binom{n-a-bk+1/2}{1+2n-k}$$

$$\text{(18b)} \quad \times \frac{(\lambda+k)(a+2bn-n+1/2)}{(1+2a+2bk-k)(a+bk-n-1/2)},$$

$$\text{(19a)} \quad \mathbb{B}(x,\lambda;a,b,n) := \sum_{k=0}^{2n}\binom{a+x+bk}{k}\binom{n-a-bk}{2n-k}$$

$$\text{(19b)} \quad \times \frac{(\lambda+k)(a+2bn-n)}{(1+2a+2bk-k)(a+bk-n)},$$

$$\text{(20a)} \quad \mathbb{C}(x,\lambda;a,b,n) := \sum_{k=0}^{1+2n}\binom{a+x+bk}{k}\binom{n-a-bk+1}{1+2n-k}$$

$$\text{(20b)} \quad \times \frac{(\lambda+k)(2a+2x+2bk-k)(a+2bn-n)}{(a+x+bk)(2a+2bk-k)(a+bk-n-1)},$$

$$\text{(21a)} \quad \mathbb{D}(x,\lambda;a,b,n) := \sum_{k=0}^{2n}\binom{a+x+bk}{k}\binom{n-a-bk+1/2}{2n-k}$$

$$\text{(21b)} \quad \times \frac{(\lambda+k)(2a+2x+2bk-k)(a+2bn-n-1/2)}{(a+x+bk)(2a+2bk-k)(a+bk-n-1/2)}.$$

Finally, we remark that the two reciprocities displayed in (17a) and (17b) can be obtained from (1b) when $\lambda = \frac{2a+1}{2b-1}$. The same is true for (17c) and (17d) from (1c) when $\lambda = \frac{2a}{2b-1}$. Similar to the other binomial convolutions treated in this paper, these four binomial sums have no closed forms in general.

Concluding comments This last section suggests that there may exist a large class of reciprocal formulae on binomial convolutions of Hagen-Rothe type which can also be justified by the existence of binomial convolutions (see Chu [4, Theorems 1.6 and 1.7] for example) similar to those labeled from (3) to (6). However the corresponding reciprocal relations would be too complicated to produce.

Recall that the following implicit generating functions are well–known and are at the heart of the Hagen-Rothe identities (cf. Gould [10] and Riordan [14, § 4.5]:

$$\sum_{k\geq 0}\binom{\alpha+\beta k}{k}x^k = \frac{y^{1+\alpha}}{\beta+y-\beta y} \quad \text{and} \quad \sum_{k\geq 0}\frac{\alpha}{\alpha+\beta k}\binom{\alpha+\beta k}{k}x^k = y^\alpha$$

where $y = 1 + xy^\beta$. They have been used by the bisection series method in [4, 5] to derive binomial identities (5a-5b) and (6a-6b). The author believes that the reciprocal relations established in this paper should also be derivable by manipulating the above generating functions. The interested reader is encouraged to investigate them through this approach.

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