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On Nonlinear Systems of BVPs with Positive Green's Functions

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Abstract. – *This paper provides some existence and uniqueness theorems for nonlinear systems of BVPs where the Green's functions for the linearization have constant sign (hence these results apply, e.g., to Dirichlet problems for elliptic PDEs as well as to various multipoint BVPs for higher order ODEs). Proofs are based on an original way of using the Linear Functional Analysis of ordered Banach spaces in connection with the traditional topological methods of Nonlinear Functional Analysis.*

1. – Introduction

The aim of this paper is to prove some general existence and uniqueness theorems for systems of BVPs

$$\begin{cases} L_i u_i = f_i(x,u) \\ B_i u_i = 0 \end{cases} \qquad (i = 1, \dots, m),$$

such that, for each i, the Green function associated to the scalar BVP

$$\begin{cases} L_i v = h(x) \\ B_i v = 0 \end{cases}$$

is non-negative.

The starting point of this research was our desire to extend to elliptic systems some results on BVPs for ODEs obtained in the sixties by A. Lasota and collaborators. It appeared that our methods apply to all BVPs whose Green function is non-negative, independently of the type of equation as well as of the order of the differential operator L_i. In fact, we introduce an abstract framework made of several small items which clarifies what is behind some popular ideas and unifies some results related to different types of BVPs. Roughly speaking, it is based on a nonlinear map that plays in function spaces the role of the absolute value in the real line, leading to a strong interaction between the Linear Functional Analysis of ordered Banach spaces and the traditional topological methods of Nonlinear Functional Analysis. To limit the size of this paper, we shall apply it only to systems of second order elliptic equations subjected to Dirichlet boundary conditions, as well as to a broad class of conjugate multipoint BVPs for higher order ODEs.

The positivity of the Green's function allows us to employ the techniques of ordered Banach spaces in two ways: either by applying the Ahmad-Lazer consequences of the Krasnosel'skii version of the Krein-Rutman Theorem in connection with the Leray-Schauder topological degree, or by re-norming the function space in order to apply the Banach-Caccioppoli Contraction Fixed Point Theorem.

These methods lead us to prove the existence or uniqueness of solutions to the mentioned systems of BVPs when, e.g., the Jacobian matrix of $f := (f_1, \dots, f_m)$ is "bounded above" by a matrix enjoying suitable properties (such as that its spectral radius is sufficiently small). Our results may be viewed as the counterpart of those in AMANN [3] for systems "below the first eigenvalue" with the key feature of avoiding any assumption of symmetry on the involved matrices (e.g., no Jacobian matrix is requested to be symmetric). A remarkable point: the constant μ appearing in the last theorems of §§ 3 and 4 is independent of all norms on $\mathbb{R}^m$ which are monotone with respect to the standard (coordinatewise) order on $\mathbb{R}^m$.

Note: To avoid tedious repetitions and to focus on the key points of the statements, standing notations and assumptions are collected at the beginning of each section (and maintained thereafter).

2. – The abstract frame

In this section we prove some existence and uniqueness theorems in the context of Hammerstein type operator systems in ordered Banach spaces. In the next two sections we shall apply them to elliptic and to conjugate BVPs respectively. Their "abstract generality" is motivated by the desire to avoid repetitions of similar proofs in different contexts.

Perhaps the best way to explain the role of the abstract scheme below is to notice that in the applications to Dirichlet problems, we shall choose $X_i = C_0^1$, $Y_i = C^{0,\alpha}$, Z_i either L^2 or C^0 and T_i will be the solution operator generated by the Green's function, while $\left| \cdot \right|_i$ the map that to each real-valued function u associates the function $\left| u \right|_i$ defined by $x \rightsquigarrow |u(x)|$. Thus $\left| u \right|_i$ is not a scalar but a member of a Banach space, so that operators may act on it (and exactly this is the key of our arguments).

Our terminology related to ordered Banach spaces is based on AMANN [4].

The following is our collection of notations and assumptions for the present section (note that the order of the spaces X_i, Y_i, Z_i is the same, hence there is no need of distinctions):

- $\mathcal{L}(V, W)$ denotes the space of bounded linear operators $V \to W$, V and W being Banach spaces;
- $\mathcal{L}(V) := \mathcal{L}(V, V)$;
- m is a given positive integer;
- i ranges in $\{1, \dots, m\}$;

- $X_i := (X_i, \|\cdot\|_{X_i}, P_i)$ is an ordered Banach space such that $\overset{\circ}{P}_i \neq \emptyset$;
- $Y_i := (Y_i, \|\cdot\|_{Y_i}, Q_i)$ is an ordered Banach space such that $X_i \subseteq Y_i$ with continuous immersion and $P_i \subseteq Q_i$;
- $Z_i := (Z_i, \|\cdot\|_{Z_i}, R_i)$ is an ordered Banach space such that $Y_i \subseteq Z_i$ with continuous immersion and $Q_i \subseteq R_i$;
- $T_i \in \mathcal{L}(Z_i)$ is positive, compact and has the following properties: $T_i(Y_i) \subseteq X_i$ and $T_i\big|_{Y_i} : Y_i \to X_i$ is compact and strongly positive;
- $X := X_1 \times \cdots \times X_m$;
- $\|\cdot\|_X$ is a norm on X whose topology is the product topology;
- $P := P_1 \times \cdots \times P_m$;
- $Y := Y_1 \times \cdots \times Y_m$;
- $\|\cdot\|_Y$ is a norm on Y whose topology is the product topology;
- $Q := Q_1 \times \cdots \times Q_m$;
- $Z := Z_1 \times \cdots \times Z_m$;
- $\|\cdot\|_Z$ is a norm on Z whose topology is the product topology;
- $R := R_1 \times \cdots \times R_m$;
- T is the operator in $\mathcal{L}(Z)$ defined by

$$Tz := (T_1 z_1, \ldots, T_m z_m) \qquad (z \in Z);$$

- • I is the identity map;
- $\left|\,\cdot\,\right|_i$ is a map $Z_i \to R_i$ with the following properties:

$$\begin{aligned} &\text{(i)}\ \ 0 < \left| z \right|_i \ \text{ whenever } \ z \neq 0,\\ &\text{(ii)}\ \ \left| T_i z \right|_i \leqslant T_i \left| z \right|_i,\\ &\text{(iii)}\ \ \left| z \right|_i \in Y_i \ \text{ whenever } \ z \in Y_i, \end{aligned}$$

  for all i and $z \in Z_i$;
- $\mathcal{S}^+$ is the set of all $S \in \mathcal{L}(Y)$ such that

$$x \in Q \setminus \{0\} \quad \Rightarrow \quad \begin{cases} (Sx)_i \in Q_i \setminus \{0\} \\ \\ \left| (Sx)_i \right|_i \leqslant \Big(S(\left| x_1 \right|_1, \ldots, \left| x_m \right|_m)\Big)_i \end{cases} \qquad (i = 1, \ldots, m).$$

In this context, we prove three fixed point theorems, two by the help of the topological degree and the other by the Banach-Caccioppoli theorem on contractions. However, each of the three theorems has its proof founded on linear techniques from ordered Banach spaces.

The following lemma is needed in the proof of Theorem 1 and provides a concrete example for the hypothesis "$\lambda^+(S) < 1$" of Theorem 1 as explained below.

LEMMA 1. – *For each $S \in \mathcal{S}^+$ there is a unique positive eigenvalue $\lambda^+(S)$ of*

$$(T \circ S)\big|_X x = \lambda x$$

which has an eigenvector $x \in \overset{\circ}{P}$. Also, this eigenvalue $\lambda^+(S)$ is larger than the *absolute values of all eigenvalues of* $T \circ S|_X$ *when* $|\cdot|_i$ *satisfies*

$$|\lambda z|_i = |\lambda|\, |z|_i$$

for all i *and* $z \in Z_i$.

Moreover: if $S_1, S_2 \in \mathcal{S}^+$ *satisfy the condition*

$$x \in \overset{\circ}{P} \qquad \Rightarrow \qquad S_1 x < S_2 x$$

for the order on Y *induced by* Q, *then* $\lambda^+(S_1) < \lambda^+(S_2)$.

PROOF. – The proof is based on some results of AHMAD-LAZER [1] which are a follow up of the Krasnosel'skii version of the Krein-Rutman Theorem.

Let $\leqslant$ be the order on $X := (X, \|\cdot\|_X, P)$. Clearly $T|_X : X \to X$ is a positive and compact linear operator.

Fix $S \in \mathcal{S}^+$. For every $x \in (Q \cap X) \setminus \{0\}$ we have $(Sx)_i \in Q_i \setminus \{0\}$ by virtue of the definition of $\mathcal{S}^+$. Consequently $T_i\big((Sx)_i\big) \in \overset{\circ}{P}_i$ for every i, since by hypothesis $T_i|_{Y_i} : Y_i \to X_i$ is strongly positive. Then $(T \circ S)x \in \overset{\circ}{P}$ whenever $x \in (Q \cap X) \setminus \{0\}$ because $\overset{\circ}{P} = \overset{\circ}{P}_1 \times \cdots \times \overset{\circ}{P}_m$. Since the immersions $X_i \hookrightarrow Y_i$ are positive, it follows that $T \circ S|_X$ is strongly positive in X. Therefore Theorem 2 of AHMAD-LAZER [1] implies that $T \circ S|_X$ has a unique positive eigenvalue $\lambda^+(S)$ with an eigenvector in $\overset{\circ}{P}$, while Theorem 3 and Corollary 3.1 of [1] imply respectively that

$$\lambda^+(S) = \max\big\{\gamma \geqslant 0 : \ \gamma x \leqslant T\big(S(x)\big) \text{ for some } x \in P \setminus \{0\}\big\}$$

and that $\lambda^+(S) > \gamma$ whenever $\gamma x < T\big(S(x)\big)$ for some $x \in P \setminus \{0\}$, in the order of X generated by P. When $\lambda x = T\big(Sx\big)$, then

$$\begin{aligned} |\lambda|\ |x_i|_i = |\lambda x_i|_i &= |T_i(Sx)_i|_i \\ &\leqslant T_i\, |(Sx)_i|_i \\ &\quad \text{[by property (ii) of } |\cdot|_i] \\ &\leqslant T_i\big(S(|x_1|_1, \ldots, |x_m|_m)\big)_i \\ &\quad \text{[by the definition of } \mathcal{S}^+ \text{ and the positivity of } T_i]. \end{aligned}$$

Therefore, setting

$$x^0 := (|x_1|_1, \ldots, |x_m|_m) \in Y,$$

we have proved that

$$|\lambda|\, x^0 \leqslant T(Sx^0)$$

in the ordered Banach space $(Y, \|\cdot\|_Y, Q)$. Since $x \neq 0$, there is i such that $x_i \neq 0$. Then $|x_i^0|_i > 0$ in Y_i by virtue of property (i) of $|\cdot|_i$. Consequently $x^0 > 0$ and so $x_0 := T(Sx^0) \in \overset{\circ}{P}$ by the above reasons. Applying the positive operator $T \circ S$ to both sides of the inequality $|\lambda|\, x^0 \leqslant T(Sx^0)$, we get $|\lambda|\, x_0 \leqslant T(Sx_0)$ in the ordered Banach space $(X, \|\cdot\|_X, P)$, hence (again by the above) $|\lambda| \leqslant \lambda^+(S)$. Thus the first statement of the lemma is proved.

To state the second part, consider any $S_1,\ S_2 \in \mathcal{S}^+$ satisfying $S_1\big|_{\mathring{P}} < S_2\big|_{\mathring{P}}$. Let $x_0 \in \mathring{P}$ be an eigenvector of $\lambda^+(S_1)$. As $S_1 x_0 < S_2 x_0$, there is at least one i such that $(S_1 x_0)_i < (S_2 x_0)_i$. Consequently $T_i(S_1 x_0)_i < T_i(S_2 x_0)_i$, hence $T(S_1 x_0) < T(S_2 x_0)$ in X. Then we have

$$\lambda^+(S_1)\, x_0 = T(S_1 x_0) < T(S_2 x_0)$$

in X, so that the above remarks based on Corollary 3.1 of AHMAD-LAZER [1] imply the desired conclusion, i.e. that $\lambda^+(S_2) > \lambda^+(S_1)$. □

THEOREM 1. – *Suppose that the linear operator $S_0 \in \mathcal{S}^+$ and the subset $\mathcal{S} \subseteq \mathcal{L}(Z)$ both satisfy the following conditions:*

- *$\lambda^+(S_0) < 1$, $\lambda^+(S_0)$ being the largest eigenvalue of $(T \circ S_0)\big|_X$,*
- *$\mathcal{S}$ is weakly sequentially compact in $\mathcal{L}(Z)$ and contains the weak limit of every convergent sequence in it,*
- *$\big| (Sx)_i \big|_i \leqslant \big| (S_0 x)_i \big|_i$ for all $S \in \mathcal{S}$, $x \in Y$ and $i \in \{1, \ldots, m\}$,*
- *if $x = T(Sx)$ with $x \in Z$ and $S \in \mathcal{S}$, then $x \in Y$.*

If $\{A(x) :\ x \in Z\} \subseteq \mathcal{S}$ and $G : Y \to Y$ satisfy the following properties:

- *$\{A(x)\big|_Y :\ x \in Y\}$ is a bounded subset of $\mathcal{L}(Y)$,*
- *$\tilde{A} : x \rightsquigarrow A(x)\, x$ is a continuous map $Y \to Y$,*
- *G is continuous, bounded on bounded sets and*

$$\lim_{\|x\|_Y \to \infty} \frac{\|G(x)\|_Y}{\|x\|_Y} = 0,$$

then the fixed point equation

$$x = T\big(A(x)\, x + G(x)\big)$$

has at least one solution in X.

Note that $\lambda^+(S_0) < 1$ whenever there is $S_1 \in \mathcal{S}^+$ such that

$$S_0 x < \lambda^+(S_1)\, S_1 \quad \text{on } \mathring{P},$$

because obviously $\lambda^+\big(\lambda^+(S_1)\, S_1\big) = 1$, while $\lambda^+(S_0) < \lambda^+\big(\lambda^+(S_1)\, S_1\big)$ by virtue of the second part of Lemma 1.

If we require only that $\{A(x) :\ \|x\|_Z \geqslant \rho\} \subseteq \mathcal{S}$ for a given $\rho > 0$, then we get nothing other than a statement included in Theorem 1, as can be deduced from the proof of Theorem 4 below.

PROOF. – The proof is divided into three steps:

STEP 1. – *The fixed point equation $x = (T \circ S)\,x$ has only the trivial solution in Z (i.e., $\lambda = 1$ is not an eigenvalue of $T \circ S$) when $S \in \mathcal{S}$.* For the sake of a contradiction, assume that $x \in Z \setminus \{0\}$ satisfies

$$x = (T \circ S)\,x$$

with $S \in \mathcal{S}$. Consequently for every i we have $x_i \in Y_i$ by the properties of $\mathcal{S}$, hence also $\left| x_i \right|_i \in Y_i$ by property (iii) of $\left| \cdot \right|_i$. Moreover, in Z_i we have

$$\begin{aligned} \left| x_i \right|_i &= \left| T_i(Sx)_i \right|_i \\ &\leqslant T_i \left| (Sx)_i \right|_i \\ &\qquad [\text{by property } (ii) \text{ of } \left| \cdot \right|_i] \\ &\leqslant T_i \left| (S_0 x)_i \right|_i \\ &\qquad [\text{as } T_i \text{ is positive}] \\ &\leqslant T_i\big(S_0(\left| x_1 \right|_1, \ldots, \left| x_m \right|_m)\big)_i \\ &\qquad [\text{by the definition of } \mathcal{S}^+ \text{ and the positivity of } T_i]. \end{aligned}$$

Therefore, setting

$$x^0 := (\left| x_1 \right|_1, \ldots, \left| x_m \right|_m) \in Y,$$

we have proved that

$$x^0 \leqslant T(S_0 x^0) \tag{1}$$

in the ordered Banach space $(Y, \|\cdot\|_Y, Q)$. Since $x \neq 0$, there is at least one i such that $x_i \neq 0$. Then $\left| x_i^0 \right|_i > 0$ in Y_i by virtue of property (i) of $\left| \cdot \right|_i$. Consequently $x^0 > 0$ in the order of Y. As seen in the proof of Lemma 1 for the composition of members of $\mathcal{S}^+$ with T, $T \circ S_0$ sends $Q \setminus \{0\}$ into $\mathring{P}$. Thus $x_0 := T(S_0 x^0) \in \mathring{P}$. Applying the positive operator $T \circ S_0$ to both sides of (1) we get $x_0 \leqslant T(S_0 x_0)$ in the ordered Banach space $(X, \|\cdot\|_X, P)$. Then from Theorem 3 of AHMAD-LAZER [1] we deduce that $\lambda^+(S_0) \geqslant 1$, contradicting the hypotheses. We conclude that the claimed uniqueness of fixed points holds.

STEP 2. – *There is an a priori bound for the solutions of the family*

$$x = T\big(A(\lambda\, x)\, x\big) + \lambda\, T\big(G(x)\big) \qquad (0 \leqslant \lambda \leqslant 1)$$

of fixed point equations in Y. To obtain a contradiction, assume the existence of $(\lambda_n)_n$ and $(x_n)_n$ such that $\lambda_n \to \lambda_\infty$, $\|x_n\|_Y \to \infty$ and

$$x_n = T\big(A(\lambda_n\, x_n)\, x_n\big) + \lambda_n\, T\big(G(x_n)\big) \qquad (n \geqslant 1).$$

Dividing this identity by $\|x_n\|_Y$ and setting $z_n := x_n / \|x_n\|_Y$ we get

$$z_n = T\left(A(\lambda_n\, x_n)\, z_n + \lambda_n \frac{G(x_n)}{\|x_n\|_Y}\right) \qquad (n \geqslant 1).$$

In view of the hypotheses of the theorem, the $A(\lambda_n x_n)\big|_Y$'s are uniformly bounded in $\mathcal{L}(Y)$ and

(2) $$\lim_n \frac{\|G(x_n)\|_Y}{\|x_n\|_Y} = 0.$$

Therefore the sequence

$$A(\lambda_n x_n) z_n + \frac{\lambda_n G(x_n)}{\|x_n\|_Y} \qquad (n \geqslant 1)$$

is bounded in Y. As $T\big|_Y : Y \to X$ is a compact operator by assumption, passing to a subsequence if necessary we get $z_n \to z_\infty$ in X for a suitable z_∞. In view of the continuity of $X \hookrightarrow Y$, we have $z_n \to z_\infty$ also in Y, hence $\|z_\infty\|_Y = 1$ and so $z_\infty \neq 0$.

Since $\mathcal{S}$ is weakly sequentially compact and contains its weak sequential limits by hypotheses, there is $A_\infty \in \mathcal{S}$ such that (passing to a subsequence if necessary) $A(\lambda_n x_n) \rightharpoonup A_\infty$ in $\mathcal{L}(Z)$. As $S \rightsquigarrow S z_\infty$ is a bounded linear map $\mathcal{L}(Z) \to Z$, it is also weakly continuous. Moreover, $\mathcal{S}$ is bounded in $\mathcal{L}(Z)$ (since, by Eberlein-Šmulian's theorem, $\mathcal{S}$ is relatively weakly compact, hence every linear functional is bounded on it and so Corollary II.3 of BREZIS [5] applies). Then from

$$A(\lambda_n x_n) z_n - A_\infty z_\infty = A(\lambda_n x_n) z_n - A(\lambda_n x_n) z_\infty + A(\lambda_n x_n) z_\infty - A_\infty z_\infty$$

it follows that

$$A(\lambda_n x_n) z_n \rightharpoonup A_\infty z_\infty$$

in Z. Then the compactness of T and (2) together with the continuity of $Y \hookrightarrow Z$ imply

$$T\left(A(\lambda_n x_n) z_n + \lambda_n \frac{G(x_n)}{\|x_n\|_Y}\right) \to T\big(A_\infty z_\infty\big)$$

in Z, so that

$$z_\infty = T\big(A_\infty z_\infty\big).$$

As $z_\infty \neq 0$, we have contradicted Step 1. Consequently the desired a priori bound does exist.

STEP 3. – *Conclusion*. Since $T\big|_Y : Y \to X$, hence $T\big|_Y : Y \to Y$, is a compact linear operator and $\tilde{A}$ and G are continuous and bounded on bounded sets, the map $x \rightsquigarrow T\big(A(\lambda x) x + \lambda G(x)\big)$ is completely continuous in Y. Then we apply the Homotopy Invariance of the Leray-Schauder topological degree in Y and we get

$$\deg\big(I - T \circ (\tilde{A} + G), B(0,\varepsilon), 0\big) = \deg\big(I - T \circ A(0), B(0,\varepsilon), 0\big)$$

where $B(0,\varepsilon)$ is the ball in Y having center at the origin and radius ε and ε is chosen larger than the a priori bound established by Step 2. As $A(0) \in \mathcal{S}$,

$$\ker\big(I - T \circ A(0)\big) = \{0\}$$

by Step 1 and so

$$\deg\big(I - T \circ A(0), B(0,\varepsilon), 0\big) \neq 0$$

by a theorem of Leray and Schauder. Then the Solution Property of the Leray-Schauder topological degree implies the existence of an $x_0 \in Y$ such that $x_0 = T\big(A(x_0)\,x_0 + G(x_0)\big)$. As $A(x_0)\,x_0 + G(x_0) \in Y$ and $T(Y) \subseteq X$ by the standing assumptions, $x_0 \in X$ and we are done. □

COROLLARY. – *Let S_0, $\mathcal{S}$ and $\{A(x):\ x \in Z\}$ be as in the statement of Theorem 1. If $F : Y \to Y$ admits the representation*

$$F(x) = A(x)\,x + F(0) \qquad (x \in Y)$$

and if to every pair x, $y \in Y$ there corresponds $S_{xy} \in \mathcal{S}$ such that

$$F(x) - F(y) = S_{xy}\,(x - y),$$

then the fixed point equation

$$x = TF(x)$$

has a unique solution in X.

PROOF. – An application of Theorem 1 with $G \equiv F(0)$ provides the existence. If there exist two distinct solutions x and y, then

$$x - y = T\big(F(x) - F(y)\big) = T\big(S_{xy}\,(x - y)\big)$$

contradicting Step 1 of the proof of the previous theorem. □

The next theorem is related to the repeated product of a single space Z_i. Equivalently, we suppose that the spaces Z_i as well as the operators T_i are all equal. We have selected Z_1 simply to fix notation.

Note that $\rho(T_1) \geqslant \rho(T_1|_{X_1}) > 0$, the strict inequality being due to Lemma 1.

THEOREM 2. – *Let $Z_1^m := Z_1 \times \cdots \times Z_1$, $R_1^m := R_1 \times \cdots \times R_1$, $T_0 := (T_1, \ldots, T_1)$ and let $\|\cdot\|_{Z_1^m}$ be a monotone norm on Z_1^m whose topology is the product topology. Let $\rho(T_1)$ be the spectral radius of $T_1 \in \mathcal{L}(Z_1)$ and let $\boldsymbol{|}\cdot\boldsymbol{|} : Z_1^m \to R_1^m$ have the following properties:*

$$\text{(I)}\quad \boldsymbol{|}\,T_0\,x\,\boldsymbol{|} \leqslant T_0\,\boldsymbol{|}\,x\,\boldsymbol{|},$$

$$\text{(II)}\quad \boldsymbol{|}\,\lambda x\,\boldsymbol{|} = |\lambda|\,\boldsymbol{|}\,x\,\boldsymbol{|} \quad \textit{whenever } \lambda \in \mathbb{R},$$

$$\text{(III)}\quad \boldsymbol{|}\,x + y\,\boldsymbol{|} \leqslant \boldsymbol{|}\,x\,\boldsymbol{|} + \boldsymbol{|}\,y\,\boldsymbol{|},$$

$$\text{(IV)}\quad \|\,\boldsymbol{|}\,x\,\boldsymbol{|}\,\|_{Z_1^m} = \|x\|_{Z_1^m},$$

for all x, $y \in Z_1^m$. If $F : Z_1^m \to Z_1^m$ satisfies

$$\boldsymbol{|}\,F(x) - F(y)\,\boldsymbol{|} \leqslant \mu\,\boldsymbol{|}\,x - y\,\boldsymbol{|} \qquad (\textit{all } x,\, y)$$

with $0 < \mu < 1/\rho(T_1)$*, then the fixed point equation*

$$x = T_0\big(F(x)\big)$$

has a unique solution and it is the limit in Z_1^m *of the sequence of successive approximations* $x_{n+1} = T_0\big(F(x_n)\big)$ *with* x_0 *arbitrarily chosen.*

PROOF. – To prove the theorem it suffices to show that $T_0 \circ F$ is a contraction in Z_1^m. We shall do so by defining an equivalent norm on Z_1^m. To this aim we note that $\rho(T_1) = \rho(T_0)$ because T_1 and T_0 have the same eigenvalues, so that $\rho(T_0) > 0$. Fix $\nu \in]\mu, 1/\rho(T_0)[$. In view of the well-known formula $\rho(T_0) = \lim_n \|T_0^n\|^{1/n}$, there is n_ν such that

$$\|T_0^{n_\nu}\| < \frac{1}{\nu^{n_\nu}}.$$

With this n_ν we define

$$\|x\|_* := \|x\|_{Z_1^m} + \sum_{n=1}^{n_\nu - 1} \nu^n \, \big\|T_0^n \,|\, x \,|\, \big\|_{Z_1^m}$$

and we verify that it is a norm on Z_1^m. In view of (IV), $\|x\|_* = 0$ if and only if $x = 0$. From (II) and the linearity of T_0^n it follows that $\|\lambda x\|_* = |\lambda| \, \|x\|_*$. To get the triangle inequality, we note that from (III) and the positivity and linearity of T_0^n we obtain

$$0 \leqslant T_0^n \,|\, x + y \,| \leqslant T_0^n \,|\, x \,| \, + T_0^n \,|\, y \,|$$

in Z_1^m and so, by the monotonicity of $\|\cdot\|_{Z_1^m}$, we deduce

$$\big\|T_0^n \,|\, x+y \,|\, \big\|_{Z_1^m} \leqslant \big\|T_0^n \,|\, x \,| \, + T_0^n \,|\, y \,|\, \big\|_{Z_1^m} \leqslant \big\|T_0^n \,|\, x \,|\, \big\|_{Z_1^m} + \big\|T_0^n \,|\, y \,|\, \big\|_{Z_1^m}$$

from which the triangle inequality follows. Conclusion: $\|\cdot\|_*$ is a norm on Z_1^m. It is equivalent to $\|\cdot\|_{Z_1^m}$ because

$$\|x\|_{Z_1^m} \leqslant \|x\|_* \leqslant \left\{1 + \sum_{n=1}^{n_\nu - 1} \nu^n \, \|T_0^n\|\right\} \cdot \|x\|_{Z_1^m}$$

where the inequality at the right-hand side is a consequence of (IV).

Fix any x, $y \in Z_1^m$. We have

$$\begin{aligned} \big|\, T_0\big(F(x)\big) - T_0\big(F(y)\big) \,\big| &= \big|\, T_0\big(F(x) - F(y)\big) \,\big| \\ &\leqslant T_0 \,\big|\, F(x) - F(y) \,\big| \\ &\qquad \text{[by (I)]} \\ &\leqslant \mu \, T_0 \,|\, x - y \,| \\ &\qquad \text{[in view of the hypotheses and the positivity of } T_0\text{]} \end{aligned}$$

and consequently

$$T_0^n \,|\, T_0(F(x)) - T_0(F(y)) \,| \leqslant \mu\, T_0^{n+1} \,|\, x-y \,|$$

by virtue of the positivity of T_0^n in Z_1^m. Then the monotonicity of $\|\cdot\|_{Z_1^m}$ implies

$$\Big\| T_0^n \,|\, T_0(F(x)) - T_0(F(y)) \,| \Big\|_{Z_1^m} \leqslant \mu \Big\| T_0^{n+1} \,|\, x-y \,| \Big\|_{Z_1^m}. \tag{3}$$

From these preliminaries we deduce the following evaluation

$$\begin{aligned}
\|T_0(F(x)) - T_0(F(y))\|_* &= \|T_0(F(x)) - T_0(F(y))\|_{Z_1^m} \\
&\qquad + \sum_{n=1}^{n_\nu - 1} \nu^n \Big\| T_0^n \,|\, T_0(F(x)) - T_0(F(y)) \,| \Big\|_{Z_1^m} \\
&= \|\,|\, T_0(F(x)) - T_0(F(y)) \,|\,\|_{Z_1^m} \\
&\qquad + \sum_{n=1}^{n_\nu - 1} \nu^n \Big\| T_0^n \,|\, T_0(F(x)) - T_0(F(y)) \,| \Big\|_{Z_1^m} \\
&\quad \text{[by (IV)]} \\
&\leqslant \mu \|T_0 \,|\, x-y \,|\,\|_{Z_1^m} + \sum_{n=1}^{n_\nu - 1} \nu^n \mu \|T_0^{n+1} \,|\, x-y \,|\,\|_{Z_1^m} \\
&\quad \text{[by virtue of (3)]} \\
&= \frac{\mu\,\nu}{\nu} \|T_0 \,|\, x-y \,|\,\|_{Z_1^m} \\
&\qquad + \sum_{n=1}^{n_\nu - 1} \frac{\mu\,\nu^{n+1}}{\nu} \|T_0^{n+1} \,|\, x-y \,|\,\|_{Z_1^m} \\
&\qquad \pm \frac{\mu}{\nu} \|x-y\|_{Z_1^m} \\
&= \frac{\mu}{\nu} \|x-y\|_* + \frac{\mu}{\nu} \nu^{n_\nu} \|T_0^{n_\nu} \,|\, x-y \,|\,\|_{Z_1^m} \\
&\qquad - \frac{\mu}{\nu} \|x-y\|_{Z_1^m} \\
&\leqslant \frac{\mu}{\nu} \|x-y\|_* + \frac{\mu}{\nu} \nu^{n_\nu} \|T_0^{n_\nu}\| \, \|x-y\|_{Z_1^m} - \frac{\mu}{\nu} \|x-y\|_{Z_1^m} \\
&\quad \text{[by (IV)]} \\
&\leqslant \frac{\mu}{\nu} \|x-y\|_* \\
&\quad [\text{as } \nu^{n_\nu} \|T_0^{n_\nu}\| < 1].
\end{aligned}$$

This shows that $T_0 \circ F$ is a contraction in the Banach space $(Z_1^m, \|\cdot\|_*)$, thus we are done. □

Although the next theorem has an involved proof, it is essentially a consequence of the previous one.

THEOREM 3. – *Assume that* $\|\cdot\|_{Z_1}$, $\rho(T_1)$, Z_1^m, $\|\cdot\|_{Z_1^m}$, R_1^m, T_0, $\left|\,\cdot\,\right|$ *and* μ *are all as in Theorem* 2 *and further assume that*

- *the Banach space* Z_1 *is reflexive,*
- $\left|\,\cdot\,\right| : Z_1^m \to R_1^m$ *is continuous.*

If F, $G : Z_1^m \to Z_1^m$ *have the following properties:*

- F *is continuous and* G *is bounded on bounded sets;*
- $\left|F(x)\right| \leqslant \mu \left|x\right| + \left|G(x)\right|$ *for all* x*;*
- *to each* $\varepsilon > 0$ *there corresponds* $\delta_\varepsilon > 0$ *such that* $\left|G(x)\right| \leqslant \varepsilon \left|x\right|$ *whenever* $\|x\|_{Z_1^m} > \delta_\varepsilon$*,*

then the fixed point equation

$$x = T_0\big(F(x)\big)$$

has a solution.

To prove it we need the following lemma which is a more convenient version of the main theorem in LASOTA [18].

LEMMA 2. – *Let* X *be an arbitrary Banach space,* $F : X \to X$ *completely continuous, and for each* $x \in X$*, let* $H(x)$ *be a bounded convex subset of* X*. If*

- $H(x) = -H(-x)$ *for all* $x \in X$*,*
- *there exists* $\gamma > 0$ *such that* $\|x\| \leqslant \gamma \|y\|$ *whenever* $x + y \in H(x)$*,*
- $\displaystyle\lim_{\|x\|\to\infty} \frac{\operatorname{dist}\big(F(x), H(x)\big)}{\|x\|} = 0,$

then F *has a fixed point.*

Note: the assumption on γ implies that $x = 0$ whenever $x \in H(x)$.

PROOF. – In view of the hypotheses, there is $\varepsilon > 0$ such that

$$\frac{\operatorname{dist}\big(F(x), H(x)\big)}{\|x\|} < \frac{1}{2\gamma} \qquad (\|x\| \geqslant \varepsilon)$$

so that with $\delta := \varepsilon / 2\gamma$ we have

$$\operatorname{dist}\big(F(x), H(x)\big) + \delta < \frac{\varepsilon}{\gamma} \qquad (\|x\| = \varepsilon). \tag{4}$$

To determine that F has a fixed point on $B(0,\varepsilon)$, we plan to show that $I-F$ has a zero on $B(0,\varepsilon)$ by proving that $I-F$ fulfils the following condition

$$x - F(x) \neq \lambda \cdot \big(- x - F(- x)\big) \qquad (\|x\| = \varepsilon,\ 0 \leqslant \lambda \leqslant 1) \tag{5}$$

because, according to a result of KRASNOSELSKI [15] (whose proof, based on the Borsuk Antipodal Theorem, also appears in § 5.2.1 of VIDOSSICH [23]), it follows that $\deg\big(I-F, B(0,\varepsilon), 0\big)$ is an odd number, so that the Solution Property of the Leray-Schauder topological degree would guarantee the existence of the desired zero.

Assume that (5) fails. Then there exist x_0 and λ_0 such that $\|x_0\| = \varepsilon$, $\lambda_0 \in [0,1]$ and

$$x_0 - F(x_0) = \lambda_0 \cdot \big(- x_0 - F(- x_0)\big). \tag{6}$$

Choose $v_0 \in H(x_0)$ and $w_0 \in H(- x_0)$ such that

$$\|F(x_0) - v_0\| < \operatorname{dist}\big(F(x_0), H(x_0)\big) + \delta$$

and

$$\|F(- x_0) - w_0\| < \operatorname{dist}\big(F(- x_0), H(- x_0)\big) + \delta.$$

We have

$$x_0 + \frac{1}{1+\lambda_0}\big(v_0 - F(x_0)\big) - \frac{\lambda_0}{1+\lambda_0}\big(w_0 - F(-x_0)\big) = \frac{1}{1+\lambda_0}\, v_0 - \frac{\lambda_0}{1+\lambda_0}\, w_0 \in H(x_0) \tag{7}$$

where the equality is due to (6), while the last relation is due to the convexity of $H(x_0)$ and $-w_0 \in -H(- x_0) = H(x_0)$.

Consequently

$$\begin{aligned}
\|x_0\| &\leqslant \gamma \cdot \left\| \frac{1}{1+\lambda_0}\big(v_0 - F(x_0)\big) - \frac{\lambda_0}{1+\lambda_0}\big(w_0 - F(- x_0)\big) \right\| \\
&\qquad [\text{by (7) and the definition of } \gamma] \\
&\leqslant \frac{\gamma}{1+\lambda_0} \cdot \Big(\operatorname{dist}\big(F(x_0), H(x_0)\big) + \delta + \lambda_0 \big\{ \operatorname{dist}\big(F(- x_0), H(- x_0)\big) + \delta \big\} \Big) \\
&< \varepsilon \\
&\qquad [\text{by (4)}]
\end{aligned}$$

Since this contradicts $\|x_0\| = \varepsilon$, we conclude that (5) holds and so we are done. $\square$

PROOF OF THEOREM 3. – We plan to apply the previous lemma. To this aim fix $\varepsilon_0 > 0$ such that $\tilde{\mu} := \mu + \varepsilon_0 < 1/\rho(T_1)$ and define $f(x) \in Z_1^m$ and $h(x) \subseteq Z_1^m$ by

$$f(x) := T_0\big(F(x)\big)\,,$$

$$h(x) := \big\{ y \in Z_1^m :\ \ y = T_0\, x_y \text{ with } x_y \in Z_1^m \text{ such that } \big| \, x_y \big| \leqslant \tilde{\mu} \, \big| \, x \, \big| \big\}.$$

In order to verify the hypotheses of the previous lemma, we start by noting that the monotonicity of $\|\cdot\|_{Z_1^m}$ guarantees that

$$\left| F(x) \right| \leqslant \mu \left| x \right| + \left| G(x) \right| \quad \Rightarrow \quad \| \left| F(x) \right| \|_{Z_1^m} \leqslant \mu \| \left| x \right| \|_{Z_1^m} + \| \left| G(x) \right| \|_{Z_1^m}$$

as well as that

$$\left| G(x) \right| \leqslant \varepsilon \left| x \right| \qquad \Rightarrow \qquad \| \left| G(x) \right| \|_{Z_1^m} \leqslant \varepsilon \| \left| x \right| \|_{Z_1^m}.$$

Together with the hypotheses of the theorem and with property (IV) of $\left| \cdot \right|$, we get

$$\|F(x)\|_{Z_1^m} = \| \left| F(x) \right| \|_{Z_1^m} \leqslant \mu \| \left| x \right| \|_{Z_1^m} + \| \left| G(x) \right| \|_{Z_1^m} = \mu \|x\|_{Z_1^m} + \|G(x)\|_{Z_1^m} \tag{8}$$

and

$$\lim_{\|x\|_{Z_1^m} \to \infty} \frac{\|G(x)\|_{Z_1^m}}{\|x\|_{Z_1^m}} = 0. \tag{9}$$

These inequalities imply that F is bounded on bounded sets, so that $T_0 \circ F$ is completely continuous. In view of properties (III) and (II) of $\left| \cdot \right|$ as well as of the monotonicity of $\|\cdot\|_{Z_1^m}$, every $h(x)$ is a bounded convex set. In the following steps, we demonstrate the remaining hypotheses of the lemma.

STEP 1. – $h(x) = -h(-x)$ *for all* $x \in Z_1^m$. Choose $y \in h(x)$. We have $y = T_0 x_y$ for some $x_y \in Z_1^m$ satisfying $\left| x_y \right| \leqslant \tilde{\mu} \left| x \right|$. Then $-y = T_0(-x_y)$ in view of the linearity of T_0. It follows (from property (II) of $\left| \cdot \right|$) that $-y \in h(-x)$ and so $h(x) \subseteq -h(-x)$. Exchanging the roles of x and $-x$ we get $-h(-x) \subseteq h(x)$ and we are done.

STEP 2. – *There exists* $\gamma > 0$ *such that* $\|x\|_{Z_1^m} \leqslant \gamma \|y\|_{Z_1^m}$ *whenever* $x + y \in h(x)$. If the contrary holds, then to each $n \geqslant 1$ there corresponds x_n and y_n such that

$$x_n + y_n \in h(x_n) \qquad \text{and} \qquad \|x_n\|_{Z_1^m} > n \|y_n\|_{Z_1^m}.$$

In view of the definition of $h(x_n)$, for each $n \geqslant 1$ there is $u_n \in Z_1^m$ such that

$$\left| u_n \right| \leqslant \tilde{\mu} \left| x_n \right| \qquad \text{and} \qquad x_n + y_n = T_0 u_n.$$

Dividing these relations by $\|x_n\|_{Z_1^m}$ and setting $z_n := x_n / \|x_n\|_{Z_1^m}$ and $v_n := u_n / \|x_n\|_{Z_1^m}$, we get

$$\left| v_n \right| \leqslant \tilde{\mu} \left| z_n \right| \tag{10}$$

from property (II) of $|\!|\cdot|\!|$, while from the linearity of T_0 we get

$$z_n + \frac{y_n}{\|x_n\|_{Z_1^m}} = T_0\, v_n. \tag{11}$$

Applying property (IV) of $|\!|\cdot|\!|$ and the monotonicity of $\|\cdot\|_{Z_1^m}$ to (10) we see that the sequence $(v_n)_n$ is bounded. Therefore $(v_n)_n$ is weakly sequentially compact by the reflexivity of Z_1^m and by the Eberlein-Šmulian's theorem. Moreover, from $\|x_n\|_{Z_1^m} > n\,\|y_n\|_{Z_1^m}$ we see that

$$\frac{y_n}{\|x_n\|_{Z_1^m}} \to 0.$$

These remarks and the compactness of T_0 imply the existence of $n_k \uparrow \infty$ such that $z_{n_k} \to z_0$ and $v_{n_k} \rightharpoonup v_0$ for suitable z_0 and v_0. Then, taking limits in (11) using the compactness of T_0, we get $z_0 = T_0\, v_0$. We claim that

$$|\!|\, v_0 \,|\!| \leqslant \tilde{\mu}\; |\!|\, z_0 \,|\!| \,. \tag{12}$$

To state it, noticing that $\tilde{\mu}\; |\!|\, z_0 \,|\!| = |\!|\, \tilde{\mu}\, z_0 \,|\!|$ by virtue of property (II) of $|\!|\cdot|\!|$, it suffices to show that every positive and continuous linear functional h on Z_1^m satisfies

$$h(|\!|\, v_0 \,|\!|) \leqslant h(|\!|\, \tilde{\mu}\, z_0 \,|\!|),$$

as granted by Theorem 2.4 of KRASNOSELSKI-LIFSHITS-SOBOLEV [16]. Fix such an h. We have

$$h\big(|\!|\, v_{n_k} \,|\!|\big) \leqslant h\big(|\!|\, \tilde{\mu}\, z_{n_k} \,|\!|\big) \tag{13}$$

due to (10), to property (II) of $|\!|\cdot|\!|$ and to the positivity of h. Properties (II) and (III) of $|\!|\cdot|\!|$ imply that $h \circ |\!|\cdot|\!|$ is a convex function. As it is also continuous, we take limits in (13) using Corollary III.8 of BREZIS [5] on the function at the left-hand side and the continuity on the function at the right-hand side, obtaining

$$h(|\!|\, v_0 \,|\!|) \leqslant \liminf_k\, h\big(|\!|\, v_{n_k} \,|\!|\big) \leqslant h(|\!|\, \tilde{\mu}\, z_0 \,|\!|)$$

which is what we required to conclude that (12) holds. Then from property (I) of $|\!|\cdot|\!|$ and from (12) we get

$$|\!|\, z_0 \,|\!| = |\!|\, T_0\, v_0 \,|\!| \leqslant T_0\, \tilde{\mu}\; |\!|\, z_0 \,|\!| \,. \tag{14}$$

Applying Theorem 2 to the map $z \rightsquigarrow \tilde{\mu}\, z$ in Z_1^m, we see that the successive approximations

$$w_0 := |\!|\, z_0 \,|\!| \qquad , \qquad w_{n+1} := T_0\, \tilde{\mu}\, w_n \;\; (n \geqslant 1)$$

converge to a point w_∞ satisfying $w_\infty = \tilde{\mu}\, T_0\, w_\infty$. By (14) and the positivity of T_0, the sequence $(w_n)_n$ is increasing, hence $w_\infty \geqslant \lvert z_0 \rvert$. But $\|z_0\|_{Z_1^m} = 1$, hence $\lvert z_0 \rvert > 0$ (by property (IV) of $\lvert \cdot \rvert$) and so $w_\infty \neq 0$. Then $1/\tilde{\mu}$ is an eigenvalue of T_0, contradicting the hypothesis $\tilde{\mu} < 1/\rho(T_1) = 1/\rho(T_0)$. Conclusion: γ does exist.

STEP 3. – $\lim\limits_{\|x\|_{Z_1^m}\to\infty} \operatorname{dist}\big(f(x), h(x)\big)/\|x\|_{Z_1^m} = 0$. Given $x \in Z_1^m$, define $y_x \in Z_1^m$ by the formula

$$y_x := \frac{\mu\, \|x\|_{Z_1^m}\, F(x)}{1 + \mu\, \|x\|_{Z_1^m} + \|G(x)\|_{Z_1^m}}.$$

When $\|x\|_{Z_1^m} > \delta_{\varepsilon_0}$, we have

$$\begin{aligned}
\lvert y_x \rvert &= \frac{\mu\, \|x\|_{Z_1^m}}{1 + \mu\, \|x\|_{Z_1^m} + \|G(x)\|_{Z_1^m}}\, \lvert F(x) \rvert \\
&\qquad \text{[by property (II) of } \lvert \cdot \rvert \text{]} \\
&\leqslant \lvert F(x) \rvert \\
&\leqslant \mu\, \lvert x \rvert + \lvert G(x) \rvert \leqslant \tilde{\mu}\, \lvert x \rvert \\
&\qquad \text{[by the definition of } \varepsilon_0\text{]},
\end{aligned}$$

hence $T_0\, y_x \in h(x)$. For $\|x\|_{Z_1^m} > \delta_{\varepsilon_0}$ we have

$$\begin{aligned}
\|f(x) - T_0\, y_x\|_{Z_1^m} &\leqslant \|T_0\|\, \|F(x) - y_x\|_{Z_1^m} \\
&= \|T_0\| \left| 1 - \frac{\mu\, \|x\|_{Z_1^m}}{1 + \mu\, \|x\|_{Z_1^m} + \|G(x)\|_{Z_1^m}} \right| \|F(x)\|_{Z_1^m} \\
&\leqslant \|T_0\|\, \frac{1 + \|G(x)\|_{Z_1^m}}{1 + \mu\, \|x\|_{Z_1^m} + \|G(x)\|_{Z_1^m}} \left\{ \mu \|x\|_{Z_1^m} + \|G(x)\|_{Z_1^m} \right\} \\
&\qquad \text{[by (8)]} \\
&\leqslant \|T_0\| \left\{ 1 + \|G(x)\|_{Z_1^m} \right\}
\end{aligned}$$

which implies $\lim\limits_{\|x\|_{Z_1^m}\to\infty} \operatorname{dist}\big(f(x), h(x)\big)/\|x\|_{Z_1^m} = 0$ in view of (9).

Thus all the hypotheses of the previous lemma are fulfilled, hence f has a fixed point and we are done. ☐

3. – Applications to systems of Dirichlet problems

In this section we apply the previous results to systems of Dirichlet problems

$$\begin{cases} L_i u_i = f_i(x,u) & \text{in } \Omega \\ & \qquad (i=1,\dots,m) \\ u_i = 0 & \text{on } \partial\Omega \end{cases}$$

where L_i is a uniformly elliptic differential operator of second-order in divergence form

$$L_i u := -\sum_{h,k=1}^{N} \left(a^i_{hk}(x)\cdot u_{x_h}\right)_{x_k} + c_i(x)\cdot u$$

with $a^i_{hk} \in C^{1,\alpha}(\overline{\Omega})$, $c_i \in C^{0,\alpha}(\overline{\Omega})$ and $c_i \geqslant 0$ on $\overline{\Omega}$.

The standing notations and assumptions of this section are the following:

- α is a fixed member of $]0,1[$;
- $\Omega \subseteq \mathbb{R}^N$ is a bounded domain of class $C^{2,\alpha}$;
- $\nu(x)$ denotes the outer normal at $x \in \partial\Omega$;
- $\mathbb{R}^{m\times m}$ is the set of $m\times m$ square matrices;
- if the symbol μ denotes a matrix or a matrix function, then

$$\mu_{hk}$$

denotes the element of μ with indices h, k;
- $\mathbb{R}^m$ is endowed with the standard order:

$$x \leqslant y \qquad \Leftrightarrow \qquad x_i \leqslant y_i \ \ (i=1,\dots,m)$$

for x, $y \in \mathbb{R}^m$.

The smoothness of $\partial\Omega$ and of the coefficients of the L_i's guarantees the validity of the Schauder's theorem as well as of the following

STRONG MAXIMUM PRINCIPLE. – *For every non-constant $u \in C^2(\Omega)\cap C^0(\overline{\Omega})$) we have*

$$\begin{cases} L_i u \geqslant 0 & \text{in } \Omega \\ u\big|_{\partial\Omega} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} u > 0 & \text{in } \Omega \\ \dfrac{\partial}{\partial\nu} u(x) < 0 & \text{in } \partial\Omega \end{cases}.$$

which we need for the order structures involved with our arguments.

To implement the abstract scheme of the previous section, there is a flexibility in the choice of the spaces X_i, Y_i, Z_i (in connection with the smoothness requested on the solutions, as well as with the use of Theorem 1 or 2), that we shall use only to change Z_i according to the needs. Moreover, two lemmas are necessary; the first surely known to experts (but I do not know of any written reference), while the other is suggested by the work of AHMAD-LAZER [1].

LEMMA 3. – *For every* $x_0 \in \partial\Omega$ *there exist* $\delta_{x_0}, \varepsilon_{x_0} > 0$ *such that the set*

$$W_{x_0} := \{x + t\,\nu(x) : \quad -\delta_{x_0} < t \leqslant 0,\, x \in B(x_0, \varepsilon_{x_0}) \cap \partial\Omega\}$$

is a neighborhood of x_0 *in* $\overline{\Omega}$.

PROOF. – Fix $x_0 \in \partial\Omega$. According to one of the equivalent definitions of domain of class $C^{2,\alpha}$ explained in §§ 4.7 and 7.5 of DUISTERMAAT-KOLK [9], there exist a neighborhood U of x_0 and a function $\Phi : U \to \mathbb{R}$ of class $C^{2,\alpha}$ such that:

- $\operatorname{grad} \Phi(x) \neq 0$ for all $x \in U$,
- $U \cap \Omega = \{x \in U : \quad \Phi(x) < 0\}$,
- $U \cap \partial\Omega = \{x \in U : \quad \Phi(x) = 0\}$,
- $\nu(x) := \operatorname{grad} \Phi(x)/\|\operatorname{grad} \Phi(x)\|$ for every $x \in U \cap \partial\Omega$.

Since Φ is of class C^2, ν is continuously differentiable. Therefore the map ϕ defined by

$$\phi(x, t) := \big(x + t\,\nu(x), t\big)$$

is a continuously differentiable map $U \times \mathbb{R} \to \mathbb{R}^N \times \mathbb{R}$. Its derivative at the point $(x_0, 0)$ has the form

$$\left(I_{\mathbb{R}^N} + t\frac{d}{dx}\nu(x_0), 1\right).$$

Thus the sign of the determinant of the associated matrix is given by the sign of the "minor" $I_{\mathbb{R}^N} + t\dfrac{d}{dx}\nu(x_0)$. As

$$\lim_{t\to 0} I_{\mathbb{R}^N} + t\,\frac{d}{dx}\nu(x_0) = I_{\mathbb{R}^N},$$

for small t's the determinant of the matrix associated to $I_{\mathbb{R}^N} + t\,\dfrac{d}{dx}\nu(x_0)$ will be approximately equal to 1. Consequently ϕ fulfils the condition of the Local Inversion Theorem in a neighborhood of $(x_0, 0)$ and so the range of ϕ is a neighborhood of $(x_0, 0)$. Its image by the canonical projection $\mathbb{R}^{N+1} \to \mathbb{R}^N$ is a neighborhood of x_0 in $\mathbb{R}^N$, whence we find the desired conclusion. □

LEMMA 4. – *In the subspace*

$$C_0^1 := \{u \in C^1(\overline{\Omega}, \mathbb{R}^m) : \quad u\big|_{\partial\Omega} = 0\}$$

of $C^1(\overline{\Omega}, \mathbb{R}^k)$ *the set*

$$P := \{u \in C_0^1 : \quad u_i \geqslant 0 \textit{ for all } i\}$$

is a closed cone. Its interior $\mathring{P}$ *in* C_0^1 *is non-empty and is characterized by*

$$u \in \mathring{P} \qquad \Leftrightarrow \qquad \begin{cases} u_i > 0 & \text{on } \Omega \\ & \qquad (i = 1, \dots, m) \\ \dfrac{\partial}{\partial \nu} u_i < 0 & \text{on } \partial\Omega \end{cases}.$$

PROOF. – Obviously P is a closed cone in C_0^1. First suppose that $u \in \mathring{P}$. Let e_1 be the normalized positive eigenfunction related to the first eigenvalue of $-\Delta$ in $W_0^{1,2}(\Omega)$. Set $e := (e_1, \dots, e_1)$ with e_1 repeated m times and set $u_n := u - e/n$. As $u \in \mathring{P}$, $u_n \in \mathring{P}$ for n large, say for $n \geqslant n_0$. For $n \geqslant n_0$ we have $u_i \geqslant e_1/n > 0$ on Ω for every i. It remains to show that $\dfrac{\partial}{\partial \nu} u_i < 0$ on $\partial\Omega$. Assume $\dfrac{\partial}{\partial \nu} u_i(x_0) \geqslant 0$ with $x_0 \in \partial\Omega$ and argue by contradiction. For $n \geqslant n_0$ we have

$$\begin{aligned} 0 &< \frac{\partial}{\partial \nu} u_i(x_0) - \frac{\partial}{\partial \nu} \frac{e_1(x_0)}{n} = \frac{\partial}{\partial \nu} u_{ni}(x_0) \\ &= \lim_{t \to 0} \frac{u_{ni}\big(x_0 + t\, \nu(x_0)\big)}{t} \\ &\qquad \text{[in view of the definition of exterior derivative]} \\ &= \frac{d}{dt}\Big|_{t=0} u_{ni}\big(x_0 + t\, \nu(x_0)\big). \end{aligned}$$

Then by continuity there is $\varepsilon > 0$ such that the derivative of

$$v(t) := u_{ni}\big(x_0 + t\, \nu(x_0)\big)$$

is positive whenever $|t| \leqslant \varepsilon$. Moreover, the definition of exterior normal guarantees that $x_0 + t\, \nu(x_0) \in \Omega$ for negative t's sufficiently close to 0. So for $t_0 < 0$ sufficiently small in absolute value, the derivative of v on $[t_0, 0]$ will be positive and $x_0 + t_0\, \nu(x_0) \in \Omega$. Now we apply the mean value theorem to v on $[t_0, 0]$ and we deduce that $u_{ni}\big(x_0 + t_0\, \nu(x_0)\big) < 0$ because $v' > 0$ and $v(0) = 0$. This contradicts $u_{ni} \geqslant 0$ on Ω, hence the implication "$\Rightarrow$" holds.

To state the reverse implication, suppose that $u \in C_0^1$ fulfils the following conditions

$$\begin{cases} u_i > 0 & \text{on } \Omega \\ & \qquad (i = 1, \dots, m) \\ \dfrac{\partial}{\partial \nu} u_i < 0 & \text{on } \partial\Omega \end{cases}$$

and let us show by contradiction that $u \in \mathring{P}$. If $u \notin \mathring{P}$, then there are $u_n \in C_0^1 \setminus P$ such that $u_n \to u$ in C^1. Thus to every n there corresponds x_n and i_n satisfying $x_n \in \Omega$ and $u_{ni_n}(x_n) < 0$. Passing to a subsequence if necessary, we assume that

$i_n \equiv \text{const} =: i_0$ for every n and that $x_n \to x_0$. By Theorem 7.5 on p. 268 of DUGUNDJI [8]

$$u_n(x_n) \to u(x_0) \qquad \text{and} \qquad \operatorname{grad} u_n(x_n) \to \operatorname{grad} u(x_0). \tag{15}$$

Consequently $u_{i_0}(x_0) \leqslant 0$, hence $x_0 \in \partial\Omega$. By virtue of Lemma 3, there exist δ_{x_0}, $\varepsilon_{x_0} > 0$ such that the set

$$W_{x_0} := \{x + t\,\nu(x) : \ -\delta_{x_0} < t \leqslant 0,\, x \in B(x_0, \varepsilon_{x_0}) \cap \partial\Omega\}$$

is a neighborhood of x_0 in $\overline{\Omega}$. After possibly shrinking δ_{x_0} and ε_{x_0}, we may assume that $\frac{\partial}{\partial\nu} u_{i_0} < 0$ on W_{x_0}. In view of

$$\frac{\partial}{\partial\nu}\, u_i(x) = \big(\operatorname{grad} u_i(x) \,\big|\, \nu(x)\big)$$

and of (15), there is n_0 such that

$$\frac{\partial}{\partial\nu}\, u_{n\,i_0}(x) = \big(\operatorname{grad} u_{ni_0}(x) \,\big|\, \nu(x)\big) < 0 \qquad\qquad (n \geqslant n_0;\ x \in W_{x_0}).$$

As W_{x_0} is a neighborhood of x_0 in $\overline{\Omega}$, by taking n_0 larger if necessary, we assume that $x_n \in W_{x_0}$ for $n \geqslant n_0$. Fix $n \geqslant n_0$, so that $x_n = y_n + t_n\,\nu(y_n)$ with $y_n \in B(x_0, \varepsilon_{x_0}) \cap \partial\Omega$ and $-\delta_{x_0} < t_n < 0$ [note that $t_n < 0$ because $x_n \notin \partial\Omega$ since $u_n(x_n) \neq 0$]. Applying the mean value theorem to the function

$$v(s) := u_{ni_0}\big(y_n + s\,\nu(y_n)\big)$$

on $[t_n, 0]$, we see that $u_{ni_0}(x_n) > 0$ because $v' < 0$, $v(0) = 0$ and $v'(s)$ is the derivative at $s = 0$ of $u_{ni_0}\big(y + s\,\nu(y_n)\big)$. This is a contradiction, hence we are done. □

Now we are ready to outline the abstract scheme for Dirichlet problems. For every $i \in \{1, \ldots, m\}$ we define:

- $X_i := \{u \in C^1(\overline{\Omega}) : \ u\big|_{\partial\Omega} = 0\}$ as a subspace of $C^1(\overline{\Omega})$;
- $Y_i := C^{0,\alpha}(\overline{\Omega})$;
- $Z_i := L^2(\Omega)$ or $Z_i := C^0(\overline{\Omega})$, always specifying the choice;
- $P_i \subseteq X_i$, $Q_i \subseteq Y_i$ and $R_i \subseteq Z_i$ are the subsets of non-negative functions;
- G_i is the Green's function of L_i subjected to the Dirichlet boundary condition;
- T_i is the solution operator defined by the Green function:

$$(T_i\, u)(x) = \int_\Omega G_i(x, y)\, u(y)\, dy;$$

- $\big|\cdot\big|_i$ is the map $Z_i \to R_i$ which to every $u \in Z_i$ associates the function $x \rightsquigarrow |u(x)|$;
- $\mathcal{M}^+$ is the set of all matrix functions in $C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^{m\times m})$ whose entries are all positive a.e.

In view of our assumptions on L_i and $\partial\Omega$, well-known results ensure that $G_i \geqslant 0$.

From a theorem of Schauder it follows that $T_i(Y_i) \subseteq C^{2,\alpha}(\overline{\Omega})$, so that $T_i\big|_{Y_i}$ is a compact operator $Y_i \to X_i$. By the Strong Maximum Principle and Lemma 4, $T_i\big|_{Y_i}$ is strongly positive.

Let Z_i be either $L^2(\Omega)$ or $C^0(\overline{\Omega})$. Obviously the map $\boldsymbol{|}\cdot\boldsymbol{|}_i$ satisfies properties (i) and (iii) in § 2, while (ii) is true because

$$\boldsymbol{|}\, T_i\, v\, \boldsymbol{|}_i(x) = \left|\int\limits_{\Omega} G_i(x,y)\, v(y)\, dy\right| \leqslant \int\limits_{\Omega} G_i(x,y)\, |v(y)|\, dy = \big(T_i\, \boldsymbol{|}\, v\, \boldsymbol{|}_i\big)(x)$$

in view of the positivity of the Green function and the definition of $\boldsymbol{|}\cdot\boldsymbol{|}_i$.

The norms $\|\cdot\|_{L^2}$ and $\|\cdot\|_{C^2}$ are clearly monotone.

Thus we are perfectly poised to apply the general scheme of the previous section.

To write systems in vector form, we introduce the notation

$$L := \operatorname{diag}(L_1, \ldots, L_m).$$

The following lemma is the counterpart of Lemma 1 for elliptic systems.

LEMMA 5. – *For each $\mu \in \mathcal{M}^+$ there is a unique positive eigenvalue $\lambda_0(\mu)$ of the Dirichlet problem*

$$\begin{cases} Lu = \lambda\, \mu(x) \cdot u \quad \text{in } \Omega \\ u\big|_{\partial\Omega} = 0 \end{cases} \tag{16}$$

which has an eigenvector with positive components in Ω and turns out to be smaller than the absolute values of the other eigenvalues of (16).

Moreover: if μ, $\nu \in \mathcal{M}^+$ satisfy the two conditions

- *$\mu_{hk} \leqslant \nu_{hk}$ for all h and k,*
- *for every h there exist k_h and $x_h \in \Omega$ such that $\mu_{hk_h}(x_h) < \nu_{hk_h}(x_h)$,*

then $\lambda_0(\mu) > \lambda_0(\nu)$.

PROOF. – We use the above notations together with those introduced in the previous section.

For each $\mu \in \mathcal{M}^+$ we define an operator $S_\mu \in \mathcal{L}(Y)$ by the formula

$$(S_\mu\, u)(x) := \left(\sum_k \mu_{1k}(x)\, u_k(x), \ldots, \sum_k \mu_{mk}(x)\, u_k(x)\right) \qquad (u \in Y,\ x \in \overline{\Omega}).$$

Since μ and u are C^α, $S_\mu\, u \in Y$. Clearly the norm $\|S_\mu\|$ is bounded above by the maximum of the sup norms of the entries of μ, while

$$u \in Q \setminus \{0\} \quad \Rightarrow \quad \begin{cases} (S_\mu u)_i \in Q_i \setminus \{0\} \\ \big| (S_\mu u)_i \big|_i \leqslant \big(S_\mu(\, |u_1|_1, \dots, |u_m|_m)\big)_i \end{cases} \qquad (i = 1, \dots, m)$$

the first relation on the right-hand side being due to the fact that $u \in Q \setminus \{0\}$ has all components non-negative and at least one which is positive on an open subset (by continuity). Therefore $S_\mu \in \mathcal{S}^+$.

Now suppose that μ, $\nu \in \mathcal{M}^+$ satisfy the conditions of the second part of the lemma, i.e. that $\mu_{hk} \leqslant \nu_{hk}$ for all h and k, and that for every h there exist k_h and $x_h \in \Omega$ such that $\mu_{hk_h}(x_h) < \nu_{hk_h}(x_h)$. By virtue of Lemma 4, $u \in \mathring{P}$ has positive all components in Ω, so that for every h we have

$$\sum_k \mu_{hk}(x_h)\, u_k(x_h) < \sum_k \nu_{hk}(x_h)\, u_k(x_h)$$

which implies

$$u \in \mathring{P} \qquad \Rightarrow \qquad S_\mu u < S_\nu u$$

in the order of Y.

Thus we simply have to apply Lemma 1 to the operators $T \circ S_\mu$ taking into account that their eigenvalues are the inverses of the eigenvalues of the corresponding Dirichlet problems. □

In the remaining part of the section we shall freely use the notations introduced above, including those of the statement of Lemma 5.

THEOREM 4. – *Let* $A : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}^{m \times m}$ *and* $g : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}^m$ *be locally Lipschitz. If there exits* $\mu \in \mathcal{M}^+$ *and* $\rho > 0$ *such that*

- $\lambda_0(\mu) > 1$, $\lambda_0(\mu)$ *being the first eigenvalue of the Dirichlet problem* (16),
- $|A_{hk}(x, u)| \leqslant \mu_{hk}(x)$ *for all* x, h, k *and all* u *with* $\|u\| \geqslant \rho$,

and if

$$\lim_{\|u\| \to \infty} \frac{g(x, u)}{\|u\|} = 0$$

uniformly on x, *then the Dirichlet problem*

$$\begin{cases} Lu = A(x, u) \cdot u + g(x, u) \quad \text{in } \Omega \\ u\big|_{\partial\Omega} = 0 \end{cases}$$

has at least one classical solution.

In view of the second part of Lemma 5, $\lambda_0(\mu) > 1$ when both there is $\nu \in \mathcal{M}^+$ such that $\mu_{hk} \leqslant \lambda_0(\nu)\, \nu_{hk}$ for all h and k and for every h there exist k_h and $x_h \in \Omega$ such that $\mu_{hk_h}(x_h) < \lambda_0(\nu)\, \nu_{hk_h}(x_h)$. Therefore in the scalar case, i.e. when $m = 1$, the requirement "μ is less than the first eigenvalue" implies $\lambda_0(\mu) > 1$, hence a famous Hammerstein theorem in [12] is included in Theorem 4.

PROOF. – We apply Theorem 1 choosing $Z_i = L^2(\Omega)$ for all i, so that we may identitify Z with $L^2(\Omega, \mathbb{R}^m)$. Set

$$\mathcal{M} := \{\gamma \in L^2(\Omega, \mathbb{R}^{m\times m}) : \; |\gamma_{hk}| \leqslant \mu_{hk} \text{ a.e.}\}.$$

Being a bounded subset of $L^2(\Omega, \mathbb{R}^{m\times m})$, $\mathcal{M}$ is weakly compact, hence weakly sequentially compact (by Eberlein-Šmulian's theorem) and contains the weak limit of every convergent sequence in it. For each $\gamma \in \mathcal{M}$ let $S_\gamma \in \mathcal{L}(Z)$ be defined by the formula

$$(S_\gamma u)(x) := \left(\sum_k \gamma_{1k}(x)\, u_k(x), \ldots, \sum_k \gamma_{mk}(x)\, u_k(x)\right) \qquad (u \in Z,\; x \in \overline{\Omega}).$$

Clearly $\gamma \rightsquigarrow S_\gamma$ is a continuous map $L^2(\Omega, \mathbb{R}^{m\times m}) \to \mathcal{L}(Z)$. Consequently

$$\mathcal{S} := \{S_\gamma : \; \gamma \in \mathcal{M}\}$$

is a weakly sequentially compact subset of $\mathcal{L}(Z)$ containing the weak limit of every convergent sequence in it.

If $u = (T \circ S_\gamma)u$ with $u \in Z$, then for each i the component u_i is a solution of the scalar Dirichlet problem

$$\begin{cases} L_i v = \sum_{k\neq i} \gamma_{ik}\, u_k(x) + \gamma_{ii}(x)\, v & \text{in } \Omega \\ v\big|_{\partial\Omega} = 0 \end{cases}$$

whose right-hand is bounded in absolute value by a quantity $h + \text{const}\,|v|$ with $h \in L^2$, so an application of the bootstrap procedure ensures that $u_i \in Y_i$.

Setting $S_0 := S_\mu\big|_Y$, the hypotheses of Theorem 1 related to $\mathcal{S}$ are either just verified or obvious.

Now we fix $u_0 \in \mathbb{R}^m$ such that $\|u_0\| = \rho$ and set

$$B(x,u) \;:=\; \begin{cases} A(x,u) & \text{if } \|u\| \geqslant \rho \\ A\left(x, \dfrac{\rho}{\|u\|}\, u\right) & \text{if } 0 < \|u\| < \rho\,, \\ A(x,u_0) & \text{if } \|u\| = 0 \end{cases}$$

$$G(x,u) \;:=\; A(x,u)\cdot u - B(x,u)\cdot u + g(x,u).$$

Clearly the given Dirichlet problem can be rewritten in the equivalent form

$$\begin{cases} Lu = B(x,u)\cdot u + G(x,u) & \text{in } \Omega \\ u\big|_{\partial\Omega} = 0 \end{cases}.$$

Consequently u is a solution if and only if

$$u = T\Big(S_{B\left(\cdot,u(\cdot)\right)}\, u + G\big(\cdot, u(\,\cdot\,)\big)\Big).$$

From the hypotheses of the theorem we have $B\big(\cdot, u(\,\cdot\,)\big) \in \mathcal{M}$ for every $u \in Z$ and

$$\lim_{\|u\|_Y\to\infty} \frac{\|G\big(\cdot,u(\,\cdot\,)\big)\|_Y}{\|u\|_Y} = 0,$$

while $u \rightsquigarrow S_{B\left(\cdot,u(\cdot)\right)}\, u$ is a continuous map $Y \to Y$ by virtue of the definition of B. Then Theorem 1 provides the conclusion. □

Now we derive three corollaries, the first being a trivial consequence of Thorem 4, while the other two have been inspired by LASOTA [18],[19].

COROLLARY 1. – *Let $f : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}^m$ be continuously differentiable with $\partial f/\partial u$ locally Lipschitz. Let $g : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}^m$ be locally Lipschitz. If there exits $\mu \in \mathcal{M}^+$ and $\rho > 0$ such that*

- $\lambda_0(\mu) > 1$, *$\lambda_0(\mu)$ being the first eigenvalue of the Dirichlet problem* (16),
- $\left|\dfrac{\partial f_i(x,u)}{\partial u_k}\right| \leqslant \mu_{ik}(x)$ *for all x, i, k and all u with $\|u\| \geqslant \rho$,*

and if

$$\lim_{\|u\|\to\infty} \frac{g(x,u)}{\|u\|} = 0$$

uniformly on x, then the Dirichlet problem

$$\begin{cases} Lu = f(x,u) + g(x,u) & \text{in } \Omega \\ u\big|_{\partial\Omega} = 0 \end{cases}$$

has at least one classical solution. The solution is unique when $\rho = 0$ and $g \equiv 0$.

PROOF. – The existence is a direct consequence of Theorem 4: by setting

$$A(x,u) := \int_0^1 \frac{\partial}{\partial u} f(x, \xi u)\, d\xi$$

we have the representation $f(x,u) = f(x,0) + A(x,u)\,u$.

To state the uniqueness when $\rho = 0$ and $g \equiv 0$, we use the definitions as in the proof of Theorem 4 for $\gamma \in \mathcal{M}$ and $S_\gamma \in \mathcal{L}(Z)$. Setting

$$\begin{aligned} F(u) &:= f\big(\cdot, u(\,\cdot\,)\big) \qquad (u \in Y) \\ A_{uv}(x) &:= \int_0^1 \frac{\partial}{\partial u} f\Big(x, v(x) + \xi\,\big(u(x) - v(x)\big)\Big)\,d\xi \qquad (u,\, v \in Y) \end{aligned}$$

we obtain the representation

$$F(u) - F(v) = S_{A_{uv}}\,(u - v).$$

Now we apply the corollary to Theorem 1 and get the desired conclusion about uniqueness. □

COROLLARY 2. – *Let $f : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}^m$ be locally Lipschitz. If there exist $\mu \in \mathcal{M}^+$ and a locally Lipschitz function $\beta : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}^+$ such that*

- *the only $u \in W^{1,2}(\Omega, \mathbb{R}^m)$ such that*

$$\begin{cases} |L_i u_i(x)| \leqslant \displaystyle\sum_{j=1}^m \mu_{ij}(x)\,|u_j(x)| & \text{in } \Omega \\ & \qquad (i = 1, \ldots, m) \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

  is $u \equiv 0$,
- $\lim\limits_{y \uparrow \infty} \beta(x,y)/y = 0$ *uniformly in x,*
- $\big|f_i(x,u)\big| \leqslant \sum_{j=1}^m \mu_{ij}(x)\,|u_j| + \beta(x, \|u\|)$ *for all i, x, u,*

then there exists a classical solution to

$$\begin{cases} Lu = f(x,u) & \text{in } \Omega \\ u\big|_{\partial\Omega} = 0 \end{cases}.$$

PROOF. – Setting

$$\gamma_i(x,u) := \sum_{j=1}^m \mu_{ij}(x)\,|u_j| + \beta(x, \|u\|) + 1 \qquad (i = 1, \ldots, m)$$

and

$$\sigma(\tau) := \begin{cases} \operatorname{sgn}(\tau) & \text{if } \tau \in \mathbb{R} \setminus \{0\} \\ 0 & \text{if } \tau = 0 \end{cases},$$

we can write

$$f_i(x,u) = \frac{f_i(x,u)}{\gamma_i(x,u)}\left\{\sum_{j=1}^{m} \mu_{ij}(x)\,\sigma(u_j)\,u_j + \beta(x,\|u\|)+1\right\}$$
$$= \sum_{j=1}^{m} \frac{f_i(x,u)\,\mu_{ij}(x)\,\sigma(u_j)}{\gamma_i(x,u)}\,u_j + \frac{f_i(x,u)\,\{\beta(x,\|u\|)+1\}}{\gamma_i(x,u)}$$

to obtain the representation

$$f(x,u) = A(x,u)\cdot u + g(x,u)$$

where $A(x,u)$ is the matrix functions with entries

$$A_{ij}(x,u) := \frac{f_i(x,u)\,\mu_{ij}(x)\,\sigma(u_j)}{\gamma_i(x,u)}$$

and

$$g_i(x,u) := \frac{f_i(x,u)\,\{\beta(x,\|u\|)+1\}}{\gamma_i(x,u)}.$$

Clearly A is locally Lipschitz and

$$|A_{ij}(x,u)| \leqslant \mu_{ij}(x) \qquad \text{(all } i,j,x,\ u)$$

as well as

$$\lim_{\|u\|\to\infty} \frac{\|g(x,u)\|}{\|u\|} = 0$$

uniformly in x. Thus to apply Theorem 4, we simply have to show that $\lambda_0(\mu) > 1$. Proceeding by contradiction, if $\lambda_0(\mu) \leqslant 1$ then (as μ is $C^{0,\alpha}$) there is a non-null classical solution u^0 of

$$\begin{cases} Lu^0 = \lambda_0(\mu)\,\mu(x)\cdot u^0 & \text{in } \Omega \\ u^0\big|_{\partial\Omega} = 0 \end{cases}.$$

Thus

$$\begin{cases} |L_i u_i^0(x)| \leqslant \displaystyle\sum_{j=1}^{m} \mu_{ij}(x)\,|u_j^0(x)| & \text{in } \Omega \\ & \qquad (i=1,\dots,m) \\ u_i^0 = 0 & \text{on } \partial\Omega \end{cases},$$

which contradicts the hypotheses, hence we are done. □

Note that the spectral assumption of the next corollary involves only the eigenvalues of a real matrix, not the eigenvalues from a PDE.

COROLLARY 3. – *Let* $f : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}^m$ *be locally Lipschitz. If there exist* $\mu \in \mathcal{M}^+$ *and a locally Lipschitz function* $\beta : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

- *the spectral radius of the matrix* A *with entries*

$$A_{ij} := \|T_i\| \, \|\mu_{ij}\|_\infty$$

is less than 1, $\|T_i\|$ *being the norm of* $T_i : L^2(\Omega) \to L^2(\Omega)$,

- $\lim\limits_{y\uparrow\infty} \beta(x,y)/y = 0$ *uniformly on* x,
- $\left|f_i(x,u)\right| \leqslant \sum\limits_{j=1}^{m} \mu_{ij}(x)\,|u_j| + \beta(x,\|u\|)$ *for all* i, x, u,

then there exists a classical solution to

$$\begin{cases} Lu = f(x,u) & \text{in } \Omega \\ u\big|_{\partial\Omega} = 0 \end{cases}.$$

PROOF. – In view of the previous corollary, it suffices to show that the $u \equiv 0$ is the only map in $W^{1,2}(\Omega, \mathbb{R}^m)$ satisfying

$$\begin{cases} |L_i u_i(x)| \leqslant \sum\limits_{j=1}^{m} \mu_{ij}(x)\,|u_j(x)| & \text{in } \Omega \\ & \qquad (i = 1, \dots, m) \\ u = 0 & \text{on } \partial\Omega \end{cases}.$$

By contradiction, suppose that u is a non-trivial solution of this inequality. Setting

$$v_i := \|u_i\|_{L^2} \qquad (i = 1, \dots, m),$$

and substituting $u_i = T_i(L_i u_i)$ we have

$$\begin{aligned} v_i &\leqslant \|T_i\| \, \|L_i u_i\|_{L^2} \\ &\leqslant \|T_i\| \left\| \sum_{j=1}^{m} \mu_{ij}(x)\,|u_j(x)| \right\|_{L^2} \\ &\leqslant \|T_i\| \sum_{j=1}^{m} \left\| \mu_{ij}(x)\,|u_j(x)| \right\|_{L^2} \\ &\leqslant \|T_i\| \sum_{j=1}^{m} \|\mu_{ij}\|_\infty \, \|u_i\|_{L^2} \\ &= \sum_{j=1}^{m} A_{ij}\, v_j. \end{aligned}$$

Consequently, for each i there exists $\theta_i \in [0,1]$ such that

$$v_i = \sum_{j=1}^{m} \theta_i A_{ij} v_j.$$

Moreover, $v_i \neq 0$ for at least one i because u is non-trivial. Thus the above identity means that 1 is an eigenvalue of the matrix $B := (\theta_i A_{ij})_{ij}$ with eigenvector $(v_1, \ldots, v_m)$. Now we apply a theorem of Frobenius saying that when two non-negative matrices satisfy the inequalities $\beta_{ij} \leqslant \gamma_{ij}$ among their entries, then the same inequality holds among their spectral radius (this is corollary 8.1.19 on p. 491 of HORN-JOHNSON [13]). Therefore

$$\theta_i A_{ij} \leqslant A_{ij} \text{ for all } i, j \qquad \Rightarrow \qquad 1 \leqslant \rho(B) \leqslant \rho(A)$$

contradicting the assumption $\rho(A) < 1$, $\rho(\,\cdot\,)$ denoting the spectral radius. □

The following result shows "non-resonance below the first eigenvalue" for non-symmetric elliptic systems. For the scalar Dirichlet problem, the same conclusion has been obtained by a different method (based on the traditional elliptic estimates) in § 2 of HAI-SCHMITT [11] and earlier for two-point BVPs by ALBRECHT [2], TIPPETT [21] and MAWHIN [20]. The existence result for elliptic systems goes back to Theorem 5.5 of KAZDAN-WARNER [14]. Our theorem improves that of Hai-Schmitt also because it shows convergence of the successive approximations in C^0 rather than in L^2.

Since the proof of the next theorem is based on Theorem 2, it is related to an elliptic system where all the differential operators are equal. We have selected L_1 for this role simply to fix our notation. Thus $L := \operatorname{diag}(L_1, \ldots, L_1)$.

The following norms on $\mathbb{R}^m$ fulfil the assumptions required of $\|\cdot\|_{\mathbb{R}^m}$ in the next theorems:

$$\|x\| := (x_1^2 + \cdots + x_m^2)^{1/2}, \qquad \|x\|_\infty := \max_i |x_i|, \qquad \|x\|_1 := |x_1| + \cdots + |x_m|.$$

THEOREM 5. – *Let $\|\cdot\|_{\mathbb{R}^m}$ be a norm on $\mathbb{R}^m$ with the following properties:*

- *$\|\cdot\|_{\mathbb{R}^m}$ is monotone with respect to the standard order of $\mathbb{R}^m$,*
- *$\|(1,0,\ldots,0)\|_{\mathbb{R}^m} = 1$.*

Let $f : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}^m$ be locally Lipschitz and let λ_0 be the first eigenvalue of the scalar Dirichlet problem

$$\begin{cases} L_1 v = \lambda v & \text{in } \Omega \\ v|_{\partial\Omega} = 0 \end{cases}.$$

If there exists a positive constant $\mu < \lambda_0$ such that

$$\|f(x,u) - f(x,v)\|_{\mathbb{R}^m} \leqslant \mu \|u - v\|_{\mathbb{R}^m} \qquad (\text{all } x, u, v),$$

then the Dirichlet problem

$$\begin{cases} L_1 u_i = f_i(x,u) & \text{in } \Omega \\ & \qquad (i=1,\dots,m) \\ u_i = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique classical solution and it is the uniform limit of the following sequence of successive approximations:

$$\begin{cases} L_1(u_{n+1})_i = f_i(x,u_n) & \text{in } \Omega \\ & \qquad (i=1,\dots,m) \\ (u_{n+1})_i = 0 & \text{on } \partial\Omega \end{cases}$$

starting with any $u_0 \in C^0(\overline{\Omega}, \mathbb{R}^m)$.

PROOF. – We apply Theorem 2 with the choice $Z_1 := C^0(\overline{\Omega})$, where the symbols R_1, T_1 and $\lvert\!\lvert \cdot \rvert\!\rvert_1$ are defined above. We identify Z_1^m with $C^0(\overline{\Omega}, \mathbb{R}^m)$, so that $\|\cdot\|_{Z_1^m}$ is the sup norm. The norm $\|\cdot\|_{Z_1^m}$ is monotone in Z_1^m because $\|\cdot\|_{\mathbb{R}^m}$ is monotone in $\mathbb{R}^m$. We set $T_0 := (T_1, \dots, T_1)$ and define $\lvert\!\lvert \cdot \rvert\!\rvert : Z_1^m \to R_1^m$ by

$$\lvert\!\lvert u \rvert\!\rvert (x) := \big(\|u(x)\|_{\mathbb{R}^m}, 0, \dots, 0\big) \qquad (x \in \overline{\Omega}).$$

Using the notations of Theorem 2, we note that properties (II) and (III) of $\lvert\!\lvert \cdot \rvert\!\rvert$ are clear. Property (IV) holds because

$$\begin{aligned} \|\lvert\!\lvert u \rvert\!\rvert\|_{Z_1^m} &= \sup_x \|\lvert\!\lvert u \rvert\!\rvert(x)\|_{\mathbb{R}^m} = \sup_x \big\|\big(\|u(x)\|_{\mathbb{R}^m}, 0, \dots, 0\big)\big\|_{\mathbb{R}^m} \\ &= \sup_x \big(\|u(x)\|_{\mathbb{R}^m} \ \|(1,0,\dots,0))\|_{\mathbb{R}^m}\big) \\ &= \sup_x \|u(x)\|_{\mathbb{R}^m} \\ &\quad \text{[by the hypotheses about } \|\cdot\|_{\mathbb{R}^m}] \\ &= \|u\|_{Z_1^m}. \end{aligned}$$

To state (I), we observe that for every $x \in \overline{\Omega}$ and $u \in Z_1^m$ we have in the standard order of $\mathbb{R}^m$:

$$\begin{aligned} \lvert\!\lvert T_0 u \rvert\!\rvert (x) &= \big(\|(T_0 u)(x)\|_{\mathbb{R}^m}, 0, \dots, 0\big) \\ &= \left(\left\| \int_\Omega G_1(x,y)\, u(y)\, dy \right\|_{\mathbb{R}^m}, 0, \dots, 0\right) \\ &\leqslant \left(\int_\Omega G_1(x,y)\, \|u(y)\|_{\mathbb{R}^m}\, dy, 0, \dots, 0\right) \\ &\quad \text{[by properties of integrals and the definition of the standard order of } \mathbb{R}^m] \\ &= (T_0 \lvert\!\lvert u \rvert\!\rvert)(x) \end{aligned}$$

which means that $\lvert\!\lvert T_0 u \rvert\!\rvert \leqslant T_0 \lvert\!\lvert u \rvert\!\rvert$, i.e. (I).

Setting $F(u) := f\big(\cdot, u(\,\cdot\,)\big)$, from the hypotheses we have

$$\big| F(u) - F(v) \big| \leqslant \mu \big| u - v \big| \qquad (u,\, v \in Z_1^m).$$

Moreover, $\lambda_0 = 1/\rho(T_1)$. Thus the conclusion follows from Theorem 2. □

In view of the following statements, recall that $f : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}^k$ satisfies

- the *Carathéodory conditions* when $f(\cdot, u)$ is measurable for every u and $f(x, \cdot)$ is continuous for a.e. x;
- the *generalized Carathéodory conditions* when f satisfies the Carathéodory conditions and moreover to every bounded subset $B \subseteq \mathbb{R}^m$ there corresponds $h_B \in L^1(\Omega)$ such that

$$\|f(x, u)\| \leqslant h_B(x) \qquad (\text{a.e. } x \in \overline{\Omega},\, u \in B).$$

THEOREM 6. – *Let* $\|\cdot\|_{\mathbb{R}^m}$ *and* λ_0 *be as in the previous theorem. Let* $f : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}^m$ *satisfy the Carathéodory conditions. If there exist both a positive constant* $\mu < \lambda_0$ *and a function* $\beta : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}^+$ *with the following properties:*

- β *satisfies the generalized Carathéodory conditions and there exist* $h \in L^2(\Omega)$ *and* $\gamma > 0$ *such that* $\beta(x, y) \leqslant h(x) + \gamma \cdot |y|$ *for all* x *and* y*;*
- $\|f(x, u)\|_{\mathbb{R}^m} \leqslant \mu\, \|u\|_{\mathbb{R}^m} + \beta(x, \|u\|_{\mathbb{R}^m})$ *for all* x *and* u*;*
- $\lim_{\|u\|_{\mathbb{R}^m} \to \infty} \beta(x, \|u\|_{\mathbb{R}^m}) / \|u\|_{\mathbb{R}^m} = 0$ *uniformly on* x*;*

then the Dirichlet problem

$$\begin{cases} L_1 u_i = f_i(x, u) & \text{in } \Omega \\ & \qquad (i = 1, \ldots, m) \\ u_i = 0 & \text{on } \partial\Omega \end{cases}$$

has a weak solution.

PROOF. – Now we set $Z_1 := L^2(\Omega)$ and we identify Z_1^m with $L^2(\Omega, \mathbb{R}^m)$. The norm $\|\cdot\|_{Z_1^m}$ is monotone because $\|\cdot\|_{\mathbb{R}^m}$ is so. Following the patterns of the proof of the previous theorem, it is easily seen that the map $\big| \cdot \big| : Z_1^m \to R_1^m$ defined by

$$\big| u \big| (x) := \big(\|u(x)\|_{\mathbb{R}^m}, 0, \ldots, 0\big) \qquad (x \in \overline{\Omega})$$

satisfies properties (I)-(IV) mentioned in the statetment of Theorem 2. It is a continuous map $L^2(\Omega, \mathbb{R}^m) \to L^2(\Omega, \mathbb{R}^m)$ by virtue of the following inequalities:

$$\begin{aligned}
\| \,|u_n| - |u_0|\, \|_{L^2}^2 &= \int_\Omega \big\| \big(\|u_n(x)\|_{\mathbb{R}^m}, 0, \dots, 0\big) - \big(\|u_0(x)\|_{\mathbb{R}^m}, 0, \dots, 0\big) \big\|_{\mathbb{R}^m}^2 \, dx \\
&= \int_\Omega \big\| \big(\|u_n(x)\|_{\mathbb{R}^m} - \|u_0(x)\|_{\mathbb{R}^m}, 0, \dots, 0\big) \big\|_{\mathbb{R}^m}^2 \, dx \\
&= \int_\Omega \big| \|u_n(x)\|_{\mathbb{R}^m} - \|u_0(x)\|_{\mathbb{R}^m} \big|^2 \, \big\|(1, 0, \dots, 0)\big\|_{\mathbb{R}^m}^2 \, dx \\
&\leqslant \int_\Omega \big\| u_n(x) - u_0(x) \big\|_{\mathbb{R}^m}^2 \, dx \\
&= \big\| u_n - u_0 \big\|_{L^2}^2 .
\end{aligned}$$

Setting

$$F(u) := f\big(\cdot, u(\,\cdot\,)\big) \quad \text{and} \quad G(u) := \big(\beta(\cdot, \|u(\,\cdot\,)\|_{\mathbb{R}^m}), 0, \dots, 0\big) \qquad (u \in Z_1^m),$$

in the standard order of $\mathbb{R}^m$ we have

$$|\,F(u)\,| \leqslant \mu \,|\,u\,| + |\,G(u)\,| \qquad \text{(for all } u)$$

and to each $\varepsilon > 0$ there corresponds $\delta_\varepsilon > 0$ such that

$$|\,G(u)\,| \leqslant \varepsilon\,|\,u\,| \qquad (\|u\|_{Z_1^m} > \delta_\varepsilon).$$

In view of a well-known theorem of Krasnoselski (cf. § I.2 of KRASNOSELSKI [17]), F and G are continuous and bounded on bounded sets.

Thus the conclusion follows from Theorem 3. □

4. – Applications to systems of conjugate BVPs

In this section we consider systems of conjugate BVPs of the type

$$\text{(17)} \qquad \begin{cases} L_i u_i = f_i(t, u) \\ u_i^{(j)}(t_{ih}) = 0 \end{cases} \qquad (1 \leqslant i \leqslant m;\; 1 \leqslant h \leqslant k_i;\; 0 \leqslant j \leqslant n_{ih} - 1)$$

where the order and the boundary conditions may change from equation to equation:

- $k_i \geqslant 2$;
- $2 \leqslant n_{i1} + \dots + n_{ik_i} =: n_i$;
- $a = t_{i1} < \dots < t_{ik_i} = b$;
- L_i is an ordinary differential operator acting on the members of $C^{n_i}([a, b])$ which has the form

$$L_i u_i := u^{(n_i)} + a_{i1}(t)\, u^{(n_i - 1)} + \dots + a_{in_i}(t)\, u$$

and fulfils the following assumption: L_i is disconjugate, its coefficients a_{il} are continuous functions and the Green's function corresponding to it and the given boundary condition, is non-negative.

In addition to these, our standing notations and assumptions for the section include:

- p_i is the Levin polynomial corresponding to the BVP for the i^{th} equation, i.e.
$$p_i(t) := (t - t_{i1})^{n_{i1}} \cdots (t - t_{ik_i})^{n_{ik_i}};$$
- G_i is the Green's function corresponding to the BVP for the i^{th} scalar equation, i.e.
$$\begin{cases} L_i v = h(t) \\ v^{(j)}(t_{ih}) = 0 \end{cases} \quad (h; j) \qquad \Leftrightarrow \qquad v(t) = \int_a^b G_i(t,s)\, h(s)\, ds.$$

Our terminology is based on COPPEL [6] and ELIAS [10].

According to what is proved on pp. 108-109 of COPPEL [6], the sign of G_i is characterized by

$$0 < \frac{G_i(t,s)}{p_i(t)} \leqslant \text{const} =: \gamma_i \qquad (a \leqslant t \leqslant b,\ a < s < b). \tag{18}$$

Consequently, conjugate multipoint BVPs have non-negative Green's functions when n_{ih} is even for every $h \geqslant 2$, even if n_{i1} is odd.

To fit problem (17) into the abstract scheme of § 2 under the above assumptions, for each i we set

$$X_i = Y_i := \Big\{ v \in C([a,b]) : \sup_{t \neq t_{i1}, \dots, t_{ik_i}} \frac{|v(t)|}{p_i(t)} < \infty \Big\}$$

$$P_i = Q_i := \big\{ v \in C([a,b]) : \ v(t) \geqslant 0 \text{ for all } t \big\}$$

$$\|v\|_{X_i} = \|v\|_{Y_i} := \|v\|_\infty + \sup_{t \neq t_{i1}, \dots, t_{ik_i}} \frac{|v(t)|}{p_i(t)} \qquad (v \in X_i)$$

$$(T_i v)(t) := \int_a^b G_i(t,s)\, v(s)\, ds \qquad (v \in Z_i;\ a \leqslant t \leqslant b)$$

$$Z_i := C^0([a,b]) \text{ or } L^1([a,b]) \text{ or } L^2([a,b]), \text{ depending on the context,}$$

$$\big| v \big|_i (t) := |v(t)| \qquad (v \in Z_i;\ a \leqslant t \leqslant b)$$

$$\big| u \big| (t) := \big(\|u(t)\|_{\mathbb{R}^m}, 0, \dots, 0 \big) \qquad (u \in Z_1^m;\ a \leqslant t \leqslant b)$$

$$\mathcal{M}^+ := \big\{ \mu \in C^0([a,b], \mathbb{R}^{m \times m}) : \ \mu_{ij} \geqslant 0 \text{ a.e.} \big\}$$

$$L := \operatorname{diag}(L_1, \dots, L_m).$$

As is proven in § 2 of DEGLA [7], $(X_i, \|\cdot\|_{X_i}, P_i)$ is an ordered Banach space with $\mathring{P}_i \neq \emptyset$ and T_i is a compact linear operator such that

$$v \in P_i \setminus \{0\}) \qquad \Rightarrow \qquad T_i v \in \mathring{P}_i$$

i.e. $T_i\big|_{X_i}$ is strongly positive in $(X_i, \|\cdot\|_{X_i}, P_i)$.

When $v \in L^1([a,b])$, then $T_i v$ is continuous because G_i is so. Moreover, from (18) we deduce

$$\big|(T_i v)(t)\big| = \left|\int_a^b G_i(t,s)v(s)\,ds\right| = \left|\int_a^b \frac{G_i(t,s)}{p_i(t)}\,p_i(t)\,v(s)\,ds\right| \leqslant \gamma_i\,\|v\|_{L^1}\,p_i(t).$$

Therefore $T_i v \in X_i$ whenever $v \in Z_i$.

With this material in mind, the proofs of the following results are quite similar (*mutatis mutandi*) to those of the similar results in the previous section. Thus they are omitted.

LEMMA 6. – *For each $\mu \in \mathcal{M}^+$ there is a unique positive eigenvalue $\lambda_0(\mu)$ of the conjugate BVP*

$$\text{(19)} \qquad \begin{cases} Lu = \lambda\,\mu(t)\cdot u \\ u_i^{(j)}(t_{ih}) = 0 \qquad\qquad (\textit{all } i,\, h,\, j) \end{cases}$$

which has an eigenvector u satisfying

$$\inf_{t \neq t_{i1},\ldots,t_{ik_i}} \frac{u_i(t)}{p_i(t)} > 0 \qquad\qquad (i = 1,\ldots,m)$$

and happens to be smaller than the absolute value of any other eigenvalue of (19).

Moreover, if μ, $\nu \in \mathcal{M}^+$ fulfil the two conditions

- *$\mu_{ij} \leqslant \nu_{ij}$ for all i and j,*
- *there are i_0, j_0, t_0 such that $\mu_{i_0 j_0}(t_0) < \nu_{i_0 j_0}(t_0)$,*

then $\lambda_0(\mu) > \lambda_0(\nu)$.

The next theorem generalizes Theorem 4 of VIDOSSICH [22]) and Theorem 5.2 of DEGLA [7].

THEOREM 7. – *Let $A : [a,b] \times \mathbb{R}^m \to \mathbb{R}^{m\times m}$ and $g : [a,b] \times \mathbb{R}^m \to \mathbb{R}^m$ satisfy the generalized Carathéodory conditions. If there exist $\mu \in \mathcal{M}^+$ and $\rho > 0$ such that*

- *$\lambda_0(\mu) > 1$, $\lambda_0(\mu)$ being the first eigenvalue of* (19),
- *$|A_{hk}(t,u)| \leqslant \mu_{hk}(t)$ for all t, h, k and all u with $\|u\| \geqslant \rho$,*

and if

$$\lim_{\|u\|\to\infty} \frac{g(t,u)}{\|u\|} = 0$$

uniformly on t, then the conjugate BVP

$$\begin{cases} Lu = A(t,u)\cdot u + g(t,u) \\ u_i^{(j)}(t_{ih}) = 0 \qquad\qquad (all\ \ i,\ h,\ j) \end{cases}$$

has at least one solution.

COROLLARY 1. – *Let* $f : [a,b]\times\mathbb{R}^m \to \mathbb{R}^m$ *be continuously differentiable and let* $g : [a,b]\times\mathbb{R}^m \to \mathbb{R}^m$ *satisfy the generalized Carathéodory conditions. If there exist* $\mu \in \mathcal{M}^+$ *and* $\rho > 0$ *such that*

- $\lambda_0(\mu) > 1$*,* $\lambda_0(\mu)$ *being the first eigenvalue of* (19)*,*
- $\left|\dfrac{\partial f_i(t,u)}{\partial u_k}\right| \leqslant \mu_{ik}(t)$ *for all* t*,* i*,* k *and all* u *with* $\|u\| \geqslant \rho$*,*

and if

$$\lim_{\|u\|\to\infty} \frac{g(t,u)}{\|u\|} = 0$$

uniformly on t, then there is a solution to the conjugate BVP

$$\begin{cases} Lu = f(t,u) + g(t,u) \\ u_i^{(j)}(t_{ih}) = 0 \qquad\qquad (all\ \ i,\ h,\ j) \end{cases}.$$

The solution is unique when $\rho = 0$ *and* $g \equiv 0$*.*

COROLLARY 2. – *Let* $f : [a,b]\times\mathbb{R}^m \to \mathbb{R}^m$ *satisfy the Carathéodory conditions. Suppose there exist* $\mu \in \mathcal{M}^+$ *and a function* $\beta : [a,b]\times\mathbb{R}^+ \to \mathbb{R}^+$ *satisfying the generalized Carathéodory conditions such that*

- *the only* $u \in C^0([a,b],\mathbb{R}^m)$ *satisfying*

$$\begin{cases} |L_i u_i(t)| \leqslant \sum\limits_{j=1}^{m} \mu_{ij}(t)\,|u_j(t)| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (all\ \ i,\ h,\ j) \\ u_i^{(j)}(t_{ih}) = 0 \end{cases}$$

  is $u \equiv 0$*,*
- $\lim\limits_{y\uparrow\infty} \beta(t,y)/y = 0$ *uniformly in t,*
- $\left|f_i(t,u)\right| \leqslant \sum\limits_{j=1}^{m} \mu_{ij}(t)\,|u_j| + \beta(t,\|u\|)$ *for all* i*,* t*,* u*.*

Then there exists a solution to the conjugate BVP

$$\begin{cases} Lu = f(t,u) \\ u_i^{(j)}(t_{ih}) = 0 \qquad\qquad (all\;\; i,\, h,\, j) \end{cases}.$$

COROLLARY 3. – *Let $f : [a,b] \times \mathbb{R}^m \to \mathbb{R}^m$ satisfy the Carathéodory conditions. If there exist $\mu \in \mathcal{M}^+$, a function $\beta : [a,b] \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the generalized Carathéodory conditions and*

- *the spectral radius of the matrix A with entries*

$$A_{ij} := \|T_i\| \, \|\mu_{ij}\|_\infty$$

is less than 1, *$\|T_i\|$ being the norm of $T_i : L^2([a,b]) \to L^2([a,b])$,*

- $\lim\limits_{y\uparrow\infty} \beta(t,y)/y = 0$ *uniformly on t,*
- $\big|f_i(t,u)\big| \leqslant \sum\limits_{j=1}^{m} \mu_{ij}(t)\,|u_j| + \beta(t,\|u\|)$ *for all i, t, u,*

then there exists a solution to the conjugate BVP

$$\begin{cases} Lu = f(t,u) + g(t,u) \\ u_i^{(j)}(t_{ih}) = 0 \qquad\qquad (all\;\; i,\, h,\, j) \end{cases}.$$

THEOREM 8. – *Let $\|\cdot\|_{\mathbb{R}^m}$ be a norm on $\mathbb{R}^m$ with the following properties:*

- *$\|\cdot\|_{\mathbb{R}^m}$ is monotone with respect to the standard order of $\mathbb{R}^m$,*
- $\|(1,0,\ldots,0)\|_{\mathbb{R}^m} = 1$.

Let $f : [a,b] \times \mathbb{R}^m \to \mathbb{R}^m$ satisfy the generalized Carathéodory conditions and let λ_0 be the first eigenvalue of the scalar conjugate BVP

$$\begin{cases} L_1 v_i = \lambda\, v_i \\ \\ v_i^{(j)}(t_{ih}) = 0 \end{cases} \qquad (all\;\; i,\, h,\, j)\,.$$

If there exists a positive constant $\mu < \lambda_0$ such that

$$\|f(t,u) - f(t,v)\|_{\mathbb{R}^m} \leqslant \mu\, \|u - v\|_{\mathbb{R}^m} \qquad\qquad (all\;\; t,u,v),$$

then the conjugate BVP

$$\begin{cases} L_1 u_i = f_i(t,u) \\ \\ u_i^{(j)}(t_{ih}) = 0 \end{cases} \qquad (all\;\; i,\, h,\, j)$$

has a unique solution and it is the uniform limit of the sequence of successive *approximations*

$$\begin{cases} L_1(u_{n+1})_i = f_i(t,u_n) \\ (u_{n+1})_i^{(j)}(t_{ih}) = 0 \end{cases} \qquad (all\ \ i,\, h,\, j)$$

with u_0 *any continuous map.*

THEOREM 9. – *Let* $\|\cdot\|_{\mathbb{R}^m}$ *and* λ_0 *be as in the previous theorem. Let* $f : [a,b] \times \mathbb{R}^m \to \mathbb{R}^m$ *satisfy the Carathéodory conditions. If there exist both a positive constant* $\mu < \lambda_0$ *and a function* $\beta : [a,b] \times \mathbb{R}^+ \to \mathbb{R}^+$ *with the following properties:*

- β *satisfies the generalized Carathéodory conditions and there exist* $h \in L^2([a,b])$ *and* $\gamma > 0$ *such that* $\beta(t,y) \leqslant h(t) + \gamma \cdot |y|$ *for all t and y;*
- $\|f(t,u)\|_{\mathbb{R}^m} \leqslant \mu\,\|u\|_{\mathbb{R}^m} + \beta(t, \|u\|_{\mathbb{R}^m})$ *for all t and u;*
- $\lim\limits_{\|u\|_{\mathbb{R}^m}\to\infty} \beta(t,\|u\|_{\mathbb{R}^m})/\|u\|_{\mathbb{R}^m} = 0$ *uniformly on t;*

then conjugate BVP

$$\begin{cases} L_1 u_i = f_i(t,u) \\ u_i^{(j)}(t_{ih}) = 0 \end{cases} \qquad (all\ \ i,\, h,\, j)$$

has a solution.

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