Bollettino Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **6** (2013), n.3, p. 715–724.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2013_9_6_3_715_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2013.

Bollettino U. M. I.
(9) **VI** (2013), 715-724

A Unified Approach for Nonlinear Kantorovich-Type Operators

FLAVIA VENTRIGLIA - GIANLUCA VINTI

Abstract. – *In this paper we introduce some important results on the approximation by series and their generalizations for integral operators. In particular, we show some new results for nonlinear Kantorovich-type operators, contained in a recent publication, and several graphical examples*

1. – Introduction

In the last century, one of the most important results in signal processing is the famous Whittaker-Kotel'nikov-Shannon sampling Theorem which allows the reconstruction of a signal/function f, continuous with finite energy and band-limited in the whole real axis, by means of a cardinal series which takes into account the sample values of the signal/function uniformly distributed on $\mathbb{R}$. Precisely

THEOREM 1.1. – *Let $f \in L^2(\mathbb{R}) \cap C^0(\mathbb{R})$ be a function (being $C^0(\mathbb{R})$ the space of all continuous functions on $\mathbb{R}$) such that the support of its Fourier transform is contained in an interval $[-\pi w, \pi w]$, for $w > 0$. Then it is possible to reconstruct f on the whole real time-axis from the sequence $f\left(\frac{k}{w}\right)$ of its sample values, by means of the interpolation series*

$$f(t) = \sum_{k=-\infty}^{+\infty} f\left(\frac{k}{w}\right) sinc[\pi(wt-k)], \quad t \in \mathbb{R}.$$

For further information on the approximation by series see [1]. The next figure shows the approximation of a function by means of an interpolation series. It makes visible how the function envelops the graphs which are, for a fixed k, the terms of the interpolation series.

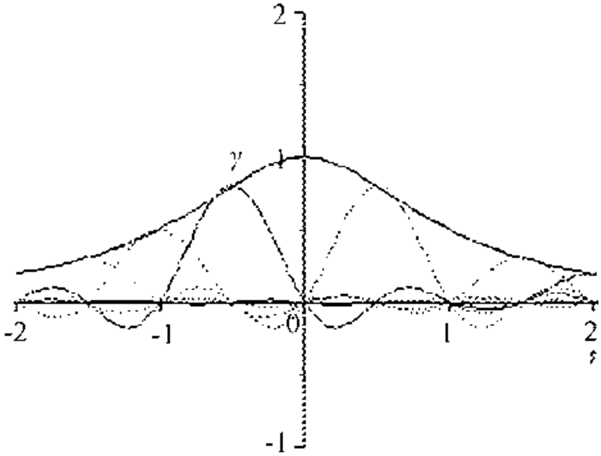

Fig. 1. – Sampling series.

However, the Theorem 1.1 has some disadvantages. First of all, in order to reconstruct the signal/function completely, the number of sample values should be infinite, which in practice does not occur. Furthermore, a signal $f \in L^2(\mathbb{R})$ cannot have limited duration, and in practice the signals have this property.

In order to avoid the above and other disadvantages, P. L. Butzer and his school (see [2]) replaced in the formulas of Theorem 1.1 the *sinc* function by a function φ, which is continuous with compact support. In this way, it is sufficient to know a finite number of sample values contained in the support of φ, obtaining a family of discrete operators, called generalized sampling operators, of the form

$$(S_w^{\varphi} f)(t) = \sum_{k=-\infty}^{+\infty} f\left(\frac{k}{w}\right)\varphi[(wt-k)], \quad t \in \mathbb{R},\ k \in \mathbb{Z},\ w > 0.$$

For generalized sampling operators the following convergence results hold.

THEOREM 1.2. – *Let φ be an uniformly continuous and bounded function with compact support such that*

$$\sum_{k=-\infty}^{+\infty} \varphi(u-k) = 1, \quad u \in \mathbb{R}$$

then

- *if $f : \mathbb{R} \to \mathbb{R}$ is a bounded and continuous function at $s \in \mathbb{R}$, then $(S_w^{\varphi} f)(s) \to f(s), w \to +\infty$;*
- *if $f : \mathbb{R} \to \mathbb{R}$ is a bounded and uniformly continuous function, then $\|S_w^{\varphi} f - f\|_{\infty} \to 0, w \to +\infty$.*

In the 2007, C. Bardaro, P.L. Butzer, R.L. Stens and G. Vinti (see [4]) introduced the Kantorovich-type generalized sampling series. In this way, they extended the approximation results obtained by Butzer and his school to functions belonging to Orlicz spaces. Therefore, they enlarged the class of signals/functions which can be to reconstruct by approximation, also incorporating not necessarily continuous functions. They defined the Kantorovich sampling operators

$$(S_w^\varphi f)(t) = \sum_{k=-\infty}^{+\infty} \chi[(wx - t_k)] \left\{ \frac{w}{(t_{k+1} - t_k)} \int_{\frac{t_k}{w}}^{\frac{t_{k+1}}{w}} f(u)du \right\},$$

with $x \in \mathbb{R}, \{t_k\}_{k\in\mathbb{Z}}$ a suitable sequence of real numbers, $w > 0$ and where $\chi \in L^1(\mathbb{R})$ is a suitable kernel function.

We observe that, with respect to generalized sampling operators, the sample values $f\left(\frac{k}{w}\right)$ are replaced with the integral means $\frac{w}{(t_{k+1} - t_k)} \int_{\frac{t_k}{w}}^{\frac{t_{k+1}}{w}} f(u)du$, mimicking what was done by Kantorovich with the Bernestein polynomials. This approach, at the same time, reduces the "time jitter errors", that incur since, generally, it is not possible to evaluate the function precisely in the node $\frac{k}{w}$.

Vinti-Zampogni, in [6], extended the theory given in [4] to the more general case of nonlinear Kantorivich sampling series of the form

$$(S_w^\varphi f)(t) = \sum_{k=-\infty}^{+\infty} \chi \left(wx - t_k, \left\{ \frac{w}{(t_{k+1} - t_k)} \int_{\frac{t_k}{w}}^{\frac{t_{k+1}}{w}} f(u)du \right\} \right),$$

with $x \in \mathbb{R}, \{t_k\}_{k\in\mathbb{Z}}$ a suitable sequence of real numbers, $w > 0$ and where $\chi : \mathbb{R}^2 \to \mathbb{R}$ is a function satisfying suitable property.

The importance of this operators lies in their application to the image reconstruction. In fact, it is possible to prove that the Kantorovich sampling operators represent a good algorithm for the reconstruction and image enhancement. Their application to function belonging to Orlicz spaces allows to reconstruct the discontinuities, which are the points where the images have the greatest contrast (shades of gray), as in the edges. This is very useful, for example, in the medical field, where the images are now used not only for diagnostic purposes but also in the surgery, during which the detail of the edges is often crucial.

2. – Definitions

Following the line of generalization of C. Bardaro, J. Musielak and G. Vinti, see [3], considering functions defined in locally compact topological groups, we introduced, with G. Vinti, new operators, that give a unifying approach to the theory concerning the Kantorovich nonlinear operators.

Let $f: G \to \mathbb{R}$ be a measurable function, we defined the family of nonlinear operators

$$(T_w f)(s) = \int_H K_w \left(s - h_w(t), \frac{1}{\mu_G(B_w(t))} \int_{B_w(t)} f(u) d\mu_G(u) \right) d\mu_H(t),$$

with $w > 0, s \in G$, where

1. H and G are locally compact and σ-compact Hausdorff topological groups and, for the sake of simplicity, we assumed G to be abelian. Moreover, G and H are provided with their respective Haar measures on the class of Borel sets;
2. $\{h_w\}_{w>0}$, is a family of homeomorphisms of H into G;
3. $\{K_w\}_{w>0}$, with $K_w : G \times \mathbb{R} \to \mathbb{R}$, $w > 0$, is a family of measurable kernels satisfying suitable conditions.

Moreover, $\forall w > 0$, we defined a cover of G, $\mathcal{B}_w = (B_w(t))_{t \in H}$, such that:

(i) $0 < \mu_G(B_w(t)) < +\infty$ for every $t \in H$ e $w > 0$;
(ii) for every $w > 0$ and $t \in H, h_w(t) \in B_w(t)$;
(iii) if $U \in \mathcal{U}$, there exists $\overline{w} > 0$ such that for every $w > \overline{w}$ we have $h_w(t) - B_w(t) \subset U$, for $t \in H$.

We require, as is usually used in approximation theory (see [3]), that the following properties are satisfied:

K_w.1) $K_w(s - h_w(\,\cdot\,), u) \in L^1_{\mu_H}(H)$ for every $u \in \mathbb{R}, s \in G$ and $K_w(s, 0) = 0$, for every $s \in G$;

K_w.2) let K_w be an (L_w, ψ)-Lipschitz kernel, i.e. there exists a family of measurable functions $L_w : G \to \mathbb{R}^+_0$, such that

$$|K_w(s, u) - K_w(s, v)| \leq L_w(s)\psi(|u - v|),$$

for every $s \in G, u, v \in \mathbb{R}$ and some φ-function $\psi : \mathbb{R}^+_0 \to \mathbb{R}^+_0$, i.e. a continuous, non decreasing function, such that $\psi(0) = 0$, $\psi(u) > 0$ for every $u > 0$ and $\psi(u) \to +\infty$ as $u \to +\infty$;

K_w.3) for every $n \in \mathbb{N}$ end $w > 0$, putting

$$r^w_n(s) = \sup_{\frac{1}{n} \leq |u| \leq n} \left| \frac{1}{u} \int_H K_w(s - h_w(t), u) d\mu_H(t) - 1 \right|, \quad s \in G,$$

we have $\lim_{w \to +\infty} r^w_n(s) = 0$ uniformly with respect to $s \in G$.

Moreover, we assume that the family $\{L_w\}_{w>0}$ satisfies the following conditions:

L_w.1) $L_w(s - h_w(\,\cdot\,)) \in L^1_{\mu_H}(H)$ for every $s \in G$;

L_w.2) there exists $M > 0$ such that

$$\int_H L_w(s - h_w(t)) d\mu_H \leq M,$$

for every $s \in G$ and $w > 0$;

L_w.3) for every $U \in \mathcal{U}$, if we put $U_{s,w} = \{t \in H : s - h_w(t) \in U\}$, we have

$$\lim_{w \to +\infty} \int_{H \setminus U_{s,w}} L_w(s - h_w(t)) d\mu_H = 0,$$

uniformly with respect to $s \in G$.

Now we show that, choosing suitable topological groups and functions, we find, as a particular case of our operators, the nonlinear sampling operators of Zampogni-Vinti. Let $G = (\mathbb{R}, +)$, provided with its Lebesgue measure, $H = (\mathbb{Z}, +)$ provided with its counting measure, $h_w(k) = \dfrac{t_k}{w}$, for every $k \in \mathbb{Z}$, where $(t_k)_{k \in \mathbb{Z}}$ is a sequence of real numbers, such that there exist $\delta, \Delta > 0$ with $\delta < t_{k+1} - t_k < \Delta < +\infty$, and $B_w(k) = \left[\dfrac{t_k}{w}, \dfrac{t_{k+1}}{w}\right]$. Then for every measurable function $f : \mathbb{R} \to \mathbb{R}$, our operators become

$$(T_w f)(s) = \sum_{k=-\infty}^{+\infty} K_w\left(s - \frac{t_k}{w}, \frac{w}{\Delta_k} \int_{\frac{t_k}{w}}^{\frac{t_{k+1}}{w}} f(u) du\right),$$

which, for the kernels $K_w(s, \cdot) = K(ws, \cdot)$, are the nonlinear Kantorovich sampling operators studied in [6]. Moreover, as we will see in what follows, further suitable choices of the groups G and H, of their Haar measures and of the family $\{h_w\}_{w>0}$, allow to obtain, as particular case of our operators, the nonlinear Kantorovich convolution operators and nonlinear Kantorovich-Mellin type operators, widely used in approximation theory.

3. – Results

Before introducing our results, we give some definitions useful in what follows. It is well known that, if $f : G \to \mathbb{R}$ is a measurable function, then

$$I_\varphi^G(f) = \int_G \varphi(|f(u)|) d\mu_G(u)$$

denote, for every φ-function φ, a modular functional, which generates the Orlicz space $L^\varphi(G)$ defined by

$$L^\varphi(G) = \{f : G \to \mathbb{R} \quad \text{measurable} : I_\varphi^G(\lambda f) < +\infty, \quad \text{for some} \quad \lambda > 0\}.$$

Finally, we introduce on $L^\varphi(G)$ the norm of Luxemburg

$$\|f\|_\varphi := \inf\left\{\lambda > 0 : I_\varphi^G\left(\frac{f}{\lambda}\right) \leq 1\right\},$$

and given $f, (f_n)_{n\in\mathbb{N}} \subseteq L^\varphi(G)$ we define two types of convergence:

1. the modular convergence, i.e. $\exists\lambda > 0$ such that
$$\lim_{n\to+\infty} I_\varphi^G(\lambda(f_n - f)) = 0, \quad \text{and}$$
2. the Luxemburg's norm convergence, i.e. $\forall\lambda > 0$
$$\lim_{n\to+\infty} \|f_n - f\|_\varphi := \lim_{w\to+\infty} I_\varphi^G(\lambda(f_n - f)) = 0.$$

In [5] we proved the following results.

THEOREM 3.1. – *Let $f : G \to \mathbb{R}$ be bounded and uniformly continuous. Then*
$$\lim_{w\to+\infty} \|(T_w f) - f\|_\infty = 0.$$

THEOREM 3.2. – *Let $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a convex φ-function and $f \in C_C(G)$, where $C_C(G)$ is the set of continuous functions with compact support on G. Then*
$$\lim_{w\to+\infty} \|(T_w f) - f\|_\varphi = 0.$$

THEOREM 3.3. – *For every $f \in L^\varphi(G) \cap \mathcal{Y}$, there exists $\mu > 0$ such that*
$$\lim_{w\to+\infty} I_\varphi^G[\mu(T_w f - f)] = 0,$$
where φ and η are two convex φ-functions such that for every $\lambda \in]0,1[$ there exists $C_\lambda \in]0,1[$ such that
$$\varphi(C_\lambda \psi(u)) \leq \eta(\lambda u) \tag{1}$$
for every $u \in \mathbb{R}_0^+$, and we suppose that

L_w.4) *$L_w \in L^1_{\mu_G}(G)$, for every $w > 0$, and*

L_w.5) *given $N > 0$, there exists a subspace $\mathcal{Y} \subset L^\eta(G)$ with $\mathcal{Y} \supset C_C^\infty(G)$ such that for every $f \in \mathcal{Y}$,*
$$\limsup_{w\to+\infty} \alpha_w I_\eta^H \left[\frac{\lambda}{\mu(B_w(\,\cdot\,))} \int_{B_w(\cdot)} f(z)\, d\mu_G(z) \right] \leq N I_\eta^G[\lambda f],$$
for every $\lambda > 0$.

In particular, in the proof of previous theorem we use the property of the Orlicz spaces, of the functions K_w and L_w, the result of norm convergence for functions with compact support and the density of the set $C_C^\infty(G)$ in $L^\varphi(G)$ with respect to the modular convergence. Moreover, we remark that condition L_w.5) is not assured in general: however, when H and G are subgroups of $\mathbb{R}$, as in the applications shown in the next section, the “Kantorovich” nature of the operators considered here allows us to discharge $L_w.5)$ on the kernels $L_w, w > 0$.

4. – Applications

As application of this theory we now consider the Orlicz space defined by the function $\varphi(u) = u^p, u \geq 0$, $p \geq 1$; in this case the space $L^\varphi(G)$ coincides with the space $L^p(G)$. If we consider, for example, $\psi(u) = u$, we can take $\eta(u) = \varphi(u)$, and $C_\lambda = \lambda$, in this way the condition (1) is satisfied and condition $L_w.5)$ becomes:

$$\limsup_{w\to+\infty} \alpha_w \left\| \frac{1}{\mu_G(B_w(\,\cdot\,))} \int_{B_w(\cdot)} f(u)d\mu_G(u) \right\|^p_{L^p(H)} \leq N\|f\|^p_{L^p(G)},$$

In this context our result is stated as follows.

THEOREM 4.1. – *For every $f \in \mathcal{Y}$, we have*

$$\lim_{w\to+\infty} \|T_w f - f\|_{L^p(G)} = 0.$$

We observe that it is possible to make analogue considerations for other Orlicz spaces as $L^\alpha \log^\beta L$-spaces (where $\varphi_{\alpha,\beta}(u) = u^\alpha \log^\beta(u + e)$, $u \geq 0$, $\alpha \geq 1$, $\beta > 0$) or exponential spaces (where $\varphi_\alpha(u) = e^{u^\alpha} - 1, u \geq 0, \alpha > 0$).

Finally, we consider some graphical representations.

In order to obtain graphical examples of nonlinear Kantorovich-Mellin operators for a function $f \in L^p(\mathbb{R}), 1 \leq p < +\infty$, we consider $G = H = (\mathbb{R}^+, \cdot)$ provided with logarithmic measure, $h_w(t) = t$ and $B_w(t) = \left[\frac{tw}{w+1}, \frac{t(w+1)}{w}\right]$. So, our operators take now the form:

$$(M_w f)(s) = \int_0^{+\infty} K_w\left(\frac{s}{t}, \frac{1}{2\log\left(\frac{w+1}{w}\right)} \int_{\frac{(tw)}{w+1}}^{\frac{t(w+1)}{w}} f(u)\frac{du}{u}\right)\frac{dt}{t}.$$

As kernels we use the functions

$$K_w(s, u) = L_w(s)g_w(u), \quad w > 0, s \in \mathbb{R}^+, u \in \mathbb{R},$$

where our assumption $K_w.i), i = 1, \ldots, 3$ and $L_w.i), i = 1, \ldots, 4$ are satisfied if we consider

$$\mathbf{L} := L_w(x) = wx^w\chi_{]0,1[}(x), \quad w > 0, x \in \mathbb{R}^+,$$

and

$$g_w(x) = \begin{cases} x^{1-\frac{1}{w}} & \text{if } 0 < \ \ x < \ \ 1 \\ x & \text{otherwise.} \end{cases}$$

We consider, in order to obtain its approximation with previous operators, the function

$$f(x) = \begin{cases} x & 0 \leq x < 2 \\ -\dfrac{3}{x^3} & x \geq 2 \end{cases}$$

The next graph show as, for different w, we obtain a more precise approximation of the function f.

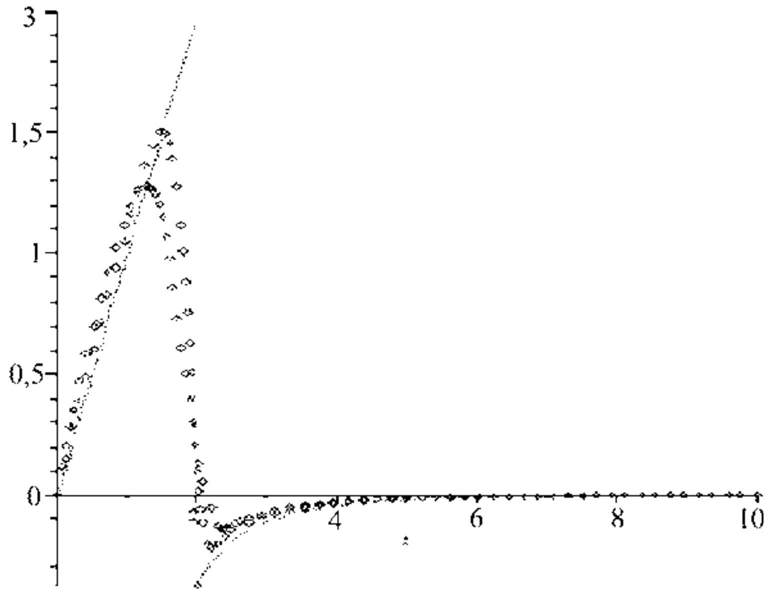

Fig. 2. – The nonlinear Kantorovich Mellin type operators $(M_w^{\mathbf{L}} f)(x)$ for $w = 5, 10$.

Analogously, in the case of Kantorovich convolution sampling operators, our operators take the form

$$(S_w^G f)(s) = \sum_{k \in \mathbb{Z}} G(ws - k) g_w \left(\frac{w}{\Delta_k} \int_{\frac{k}{w}}^{\frac{k+1}{w}} f(u) du \right), \quad w > 0,$$

or

$$(C_w^G f)(s) = \int_{\mathbb{R}} G(ws - k) g_w \left(\frac{w}{2} \int_{\frac{(k-1)}{w}}^{\frac{(k+1)}{w}} f(u) du \right) dk, \quad w > 0,$$

where we use as kernel function G, for example, Fejer's kernel

$$F(x) = \frac{1}{2} sinc^2 \left(\frac{x}{2} \right),$$

or combination of positive B-splines

$$M_n(x) = \frac{1}{(n-1)!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left(\frac{n}{2} + x - j \right)_+^{n+1}.$$

In particular, if we consider the function

$$f(x) = \begin{cases} \dfrac{1}{x^2} & x < -1 \\ -1 & -1 \leq x < 0 \\ 2 & 0 \leq x < 2 \\ -\dfrac{3}{x^3} & x \geq 2, \end{cases}$$

and as kernel G a combination of positive B-splines we obtain the following graph.

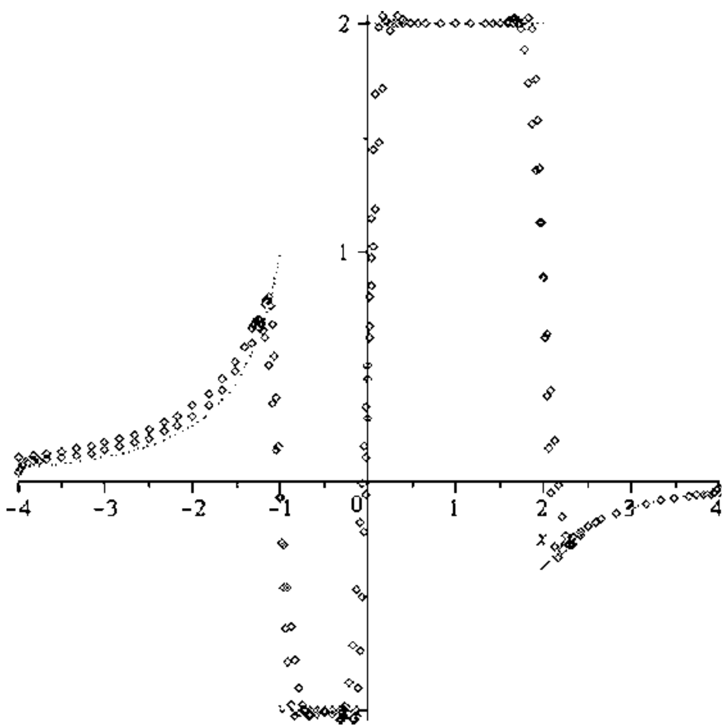

Fig. 3. – The nonlinear Kantorovich convolution operators are $(C_w^G f)(x)$ with $G(x) = 3M_4(x) - 4M_3(x)$ per $w = 5, 10, 15$.

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Received November 28, 2012 and in revised form November 30, 2013