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Abstract. – *The existence of multiple solutions to a Sturm-Liouville boundary value problem is presented. The approach adopted is based on multiple critical points theorems.*

1. – Introduction

In different fields of research, such as computer science, mechanical engineering, control systems, artificial or biological neural networks, economics and many others, the mathematical modeling of important questions leads naturally to consider nonlinear differential boundary value problems. For instance, a steady state temperature distribution in a rod (identified with a closed interval), or a semi-infinity porous medium initially filled with gas at a uniform pressure, are governed by nonlinear second order differential equations with suitable boundary conditions, as in particular, Neumann boundary conditions.

In this paper, we study the following nonlinear problem:

$$\begin{cases} -(\overline{p}u')' + \overline{r}u' + \overline{q}u = \lambda g(x,u) \\ u'(0) = u'(1) = 0, \end{cases} \tag{1}$$

where $\overline{p} \in C^1([0,1])$, $\overline{q}, \overline{r} \in C^0([0,1])$, with $\overline{p}$ and $\overline{q}$ positive functions, and λ is a positive real parameter.

The aim of this paper is to present some multiplicity results for the Neumann boundary value problem (1), as obtained in [3].

There is a wide literature that deals with multiplicity results for such a problem (see [6], [17], [18] and references therein) and, in the last few years, the existence of infinitely many solutions to Neumann problems has been widely investigated, for instance, in [7], [8], [9], [10], [11], [12], [13], [14], [16]. In [17] and [18], for the case $\overline{p} = 1$, $\overline{q} = 1$, $\overline{r} = 0$ and $\lambda = 1$, by using fixed point theorems, the existence of three solutions is established under a suitable behaviour of nonlinear term g which, in addition, must be sublinear at infinity. Our main tools to study the multiplicity of solutions for Neumann problems are critical point theorems.

First we use an important result due to B. Ricceri ([15]), as given in [5].

Other two results which insure three critical points will be used; in the first one the coercivity of the functional $\Phi - \lambda\Psi$ is required, in the second one a suitable sign hypothesis is assumed. The first result has been obtained in [4], the second one in [2].

2. – Main results

Consider the following problem

$$\begin{cases} -(pu')' + qu = \lambda f(x,u) \\ u'(0) = u'(1) = 0 \end{cases} \tag{2}$$

where $p \in C^1([0,1])$, $q \in C^0([0,1])$, $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ is a continuous function and λ is a positive real parameter.

Put $F(x,t) = \int_0^t f(x,\xi)d\xi$ for all $(x,t) \in [0,1] \times \mathbf{R}$,

$$p_0 = \min_{[0,1]} p(x) > 0, \; q_0 = \min_{[0,1]} q(x) > 0,$$

$$A := \liminf_{\xi \to +\infty} \frac{\int_0^1 \max_{|t| \le \xi} F(x,t)dx}{\xi^2}, \quad B := \limsup_{\xi \to +\infty} \frac{\int_0^1 F(x,\xi)dx}{\xi^2}$$

$$m = \min\{p_0, q_0\}, \quad k = \frac{m}{\|q\|_1} \tag{3}$$

where, as usual, $\|q\|_1 = \int_0^1 q(x)dx$,

$$\lambda_1 := \frac{\|q\|_1}{2B}, \quad \lambda_2 := \frac{m}{4A}. \tag{4}$$

We note that when $p = q = 1$ then $k = 1$.

Here, we give our first result on the existence of infinitely many solutions by requiring an oscillating behavior of the nonlinearity. (See [3] [Theorem 3.1])

THEOREM 1. – *Assume that*

$$\liminf_{\xi \to +\infty} \frac{\int_0^1 \max_{|t| \le \xi} F(x,t)dx}{\xi^2} < \frac{1}{2} k \limsup_{\xi \to +\infty} \frac{\int_0^1 F(x,\xi)dx}{\xi^2} \tag{5}$$

where k is given by (3).

Then for each $\lambda \in (\lambda_1, \lambda_2)$, where λ_1, λ_2 are given in (4), *the problem* (2) *possesses a sequence of pairwise distinct classical solutions.*

PROOF. – Our aim is to apply part (b) of [5] [Theorem 2.1]. Take as X the Sobolev space $W^{1,2}([0,1])$ endowed with the norm

$$\|u\| = \Big(\int_0^1 p(t)|u'(t)|^2dt + \int_0^1 q(t)|u(t)|^2dt\Big)^{\frac{1}{2}}.$$

For each $u \in X$, put

$$\Phi(u) := \frac{1}{2}\|u\|^2, \qquad \Psi(u) := \int_0^1 F(x, u(x))dx.$$

It is well known that the critical points in X of the functional $\Phi - \lambda\Psi$ are exactly the classical solutions of the problem (2).

Pick $\lambda \in]\lambda_1, \lambda_2[$. Let $\{c_n\}$ be a real sequence such that $\lim_{n\to\infty} c_n = +\infty$ and

$$\lim_{n\to\infty} \frac{\int_0^1 \max_{|t|\le c_n} F(x,t)dx}{c_n^2} = A.$$

Put $r_n = m\Big(\frac{c_n}{2}\Big)^2$ for all $n \in \mathbf{N}$. Taking into account $\|v\|^2 < 2r_n$ and $|v| \le \sqrt{\frac{2}{m}}\|v\|$ for every $t \in [0,1]$, $|v(t)| \le c_n$. One has

$$\varphi(r_n) = \inf_{\|u\|^2<2r_n} \frac{\sup_{\|v\|^2<2r_n} \int_0^1 F(x, v(x))dx - \int_0^1 F(x, u(x))dx}{r_n - \frac{\|u\|^2}{2}} \le \frac{\sup_{\|v\|^2<2r_n} \int_0^1 F(x, v(x))dx}{r_n}.$$

Hence,

$$\varphi(r_n) \le \frac{\int_0^1 \max_{|t|\le c_n} F(x,t)dx}{r_n} = 4\frac{\int_0^1 \max_{|t|\le c_n} F(x,t)dx}{mc_n^2} \qquad \forall n \in \mathbf{N}.$$

Then,

$$\gamma := \liminf_{n\to+\infty} \varphi(r_n) \le \frac{4A}{m} < +\infty.$$

Now, we claim that the functional $\Phi - \lambda\Psi$ is unbounded from below.

Let $\{d_n\}$be a real sequence such that $\lim_{n\to\infty} d_n = +\infty$ and

$$\lim_{n\to\infty} \int_0^1 \frac{F(x, d_n)dx}{d_n^2} = B. \tag{6}$$

For each $n \in \mathbf{N}$, we consider the functions $w_n(x) = d_n \in W^{1,2}([0,1])$.

Hence,

$$\|w_n\|^2 = d_n^2 \|q\|_1$$

and

$$\Phi(w_n) - \lambda \Psi(w_n) = \frac{\|w_n\|^2}{2} - \lambda \int_0^1 F(x, w_n(x))dx = \frac{d_n^2 \|q\|_1}{2} - \lambda \int_0^1 F(x, d_n)dx.$$

Now, if $B < +\infty$ let $\varepsilon \in \left]0, B - \frac{\|q\|_1}{2\lambda}\right[$. From (6) there exists ν_ε such that

$$\int_0^1 F(d_n) > (B - \varepsilon) d_n^2 \qquad \forall n > \nu_\varepsilon.$$

Therefore,

$$\Phi(w_n) - \lambda \Psi(w_n) = \frac{d_n^2 \|q\|_1}{2} - \lambda \int_0^1 F(x, d_n)dx < \frac{d_n^2 \|q\|_1}{2} - \lambda d_n^2 (B - \varepsilon) = d_n^2 \left(\frac{\|q\|_1}{2} - \lambda (B - \varepsilon) \right).$$

From the choice of ε, one has

$$\lim_{n \to +\infty} [\Phi(w_n) - \lambda \Psi(w_n)] = -\infty.$$

If $B = +\infty$ we fix $M > \frac{\|q\|_1}{2\lambda}$. From 6 there exists ν_M such that

$$\int_0^1 F(x, d_n)dx > M d_n^2 \quad \forall n > \nu_M.$$

Moreover

$$\Phi(w_n) - \lambda \Psi(w_n) = \frac{d_n^2 \|q\|_1}{2} - \lambda \int_0^1 F(x, d_n)dx < \frac{d_n^2 \|q\|_1}{2} - \lambda M d_n^2 = d_n^2 \left(\frac{\|q\|_1}{2} - \lambda M \right).$$

Taking into account the choice of M, also in this case, one has

$$\lim_{n \to +\infty} [\Phi(w_n) - \lambda \Psi(w_n)] = -\infty.$$

By applying [5] [Theorem 2.1, part (b)], the functional $\Phi - \lambda \Psi$ admits a sequence u_n of critical points, and the conclusion is proven. □

Now, we point out our main results on the existence of three classical solutions. For more details, see [3] [Theorem 3.3 and Theorem 3.4].

THEOREM 2. – *Assume that:*

(*i*) *there exist two positive constants* c, d, *with* $c < d$, *such that:*

$$\frac{\int_0^1 \max_{t\in[-c,c]} F(x,t)dx}{c^2} < \frac{k}{2}\frac{\int_0^1 F(x,d)dx}{d^2}$$

when k *is given by* (4);

(*ii*) *there exist two positive constants* a, s, *with* $s < 2$, *such that:*

$$F(x,t) \leq a(1 + |t|^s)$$

for all $(x,t) \in [0,1] \times \mathbf{R}$.

Then, for each $\lambda \in \left]\frac{d^2\|q\|_1}{2\int_0^1 F(x,d)dx}, \frac{mc^2}{4\int_0^1 \max_{t\in[-c,c]} F(x,t)dx}\right[$, (2) *admits at least three classical solutions.*

THEOREM 3. – *Let* $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ *be a continuous function such that* $f(x,t) \geq 0$ *for all* $x \in [0,1]$ *and for all* $t \geq 0$. *Assume that there exist three positive constants* c_1, c_2, d, *with* $c_1 < d < \frac{c_2}{2}$ *such that*

$$(j) \quad \frac{\int_0^1 \max_{|t|\leq c_1} F(x,t)dx}{c_1^2} < \frac{m}{3\|q\|_1}\frac{\int_0^1 F(x,d)dx}{d^2};$$

$$(jj) \quad \frac{\int_0^1 \max_{|t|\leq c_2} F(x,t)dx}{c_2^2} < \frac{m}{6\|q\|_1}\frac{\int_0^1 F(x,d)dx}{d^2}.$$

Then, for all $\lambda \in \left(\frac{3}{4}\frac{d^2\|q\|_1}{\int_0^1 F(x,d)dx}, \min\left\{\frac{mc_1^2}{4\int_0^1 \max_{|t|<c_1} F(x,t)dx}, \frac{mc_2^2}{2\int_0^1 \max_{|t|<c_2} F(x,t)dx}\right\}\right)$, *the problem* (2) *admits three nonnegative classical solutions* u_1, u_2, u_3 *such that* $|u_i(t)| < c_2$ *for all* $t \in [0,1]$, $i = 1,2,3$.

3. – Multiple Solutions for the complete equation

Consider the problem

$$\begin{cases} -(\overline{p}u')' + \overline{r}u' + \overline{q}u = \lambda g(x,u), \\ u'(0) = u'(1) = 0. \end{cases} \tag{7}$$

Let $g : [0,1] \times \mathbf{R} \to \mathbf{R}$ be a continuous function, $\overline{p} \in C^1([0,1])$, $\overline{q}, \overline{r} \in C^0([0,1])$, and λ is a positive parameter. Moreover $\overline{p}$ and $\overline{q}$ are positive functions.

Put $G(x,t) = \int_0^t g(x,\xi)d\xi$ for all $(x,t) \in [0,1] \times \mathbf{R}$ $k' = \dfrac{m'}{\|e^{-R}\overline{q}\|_1}$, $m' = \min\left\{ \min_{[0,1]} e^{-R}\overline{p}, \min_{[0,1]} e^{-R}\overline{q} \right\}$ where R is a primitive of $\dfrac{\overline{r}}{\overline{p}}$

$$\overline{\lambda}_1 = \frac{\|e^{-R}\overline{q}\|}{2 \limsup\limits_{\xi\to+\infty} \dfrac{\int_0^1 e^{-R}G(x,\xi)dx}{\xi^2}} \qquad \overline{\lambda}_2 = \frac{m'}{4 \liminf\limits_{\xi\to+\infty} \dfrac{\int_0^1 \max_{|t|<\xi} e^{-R}G(x,t)dx}{\xi^2}}.$$

Corollary 4. – *Assume that*

$$\liminf_{\xi\to+\infty} \frac{\int_0^1 \max_{|t|\le\xi} e^{-R}G(x,t)dx}{\xi^2} < \frac{1}{2}k' \limsup_{\xi\to+\infty} \frac{\int_0^1 e^{-R}G(x,\xi)dx}{\xi^2}. \tag{8}$$

Then, for each $\lambda \in \left(\overline{\lambda}_1, \overline{\lambda}_2\right)$, *the problem* (7) *possesses a sequence of pairwise distinct classical solutions.*

Remark 5. – *We observe that if g is autonomous the hypothesis* (8) *of Corollary* 4 *becomes*

$$\liminf_{\xi\to+\infty} \frac{\max_{|t|\le\xi} G(t)}{\xi^2} < \frac{1}{2}k' \limsup_{\xi\to+\infty} \frac{G(\xi)}{\xi^2}$$

and

$$\overline{\lambda}_1 = \frac{\|e^{-R}\overline{q}\|}{2\|e^{-R}\| \limsup\limits_{\xi\to+\infty} \dfrac{G(\xi)}{\xi^2}} \qquad \overline{\lambda}_2 = \frac{m'}{4\|e^{-R}\| \liminf\limits_{\xi\to+\infty} \dfrac{\max_{|t|<\xi} G(t)}{\xi^2}}$$

where $G(t) = \int_0^t g(\xi)dx$ *for all* $t \in \mathbf{R}$.

Moreover, we stress that $\overline{p}, \overline{q}$ *and* $\overline{r}$ *can depend on the variable* x.

In the sequel, we deal with the autonomous case.

COROLLARY 6. – *Let* $g : \mathbf{R} \to \mathbf{R}$ *be a continuous function and put*

$$G(t) = \int_0^t g(\xi)d\xi.$$

for all $t \in \mathbf{R}$

Assume that:

(i') *there exist two positive constants* c, d, *with* $c < d$, *such that:*

$$\frac{\max_{t\in[-c,c]} G(t)}{c^2} < \frac{k'}{3}\frac{G(d)}{d^2};$$

(ii') *there exist two positive constants* a, s, *with* $s < 2$, *such that:*

$$G(t) \le a(1 + |t|^s)$$

for all $t \in \mathbf{R}$.

Then, for each $\lambda \in \left(\frac{3}{4}\frac{d^2\|e^{-R}\overline{q}\|_1}{G(d)\|e^{-R}\|_1}, \frac{m'c^2}{4\|e^{-R}\|_1 \max_{t\in[-c,c]} G(t)}\right)$, *the problem* (7) *admits three classical solutions.*

COROLLARY 7. – *Let* $g : \mathbf{R} \to \mathbf{R}$ *be a continuous function such that* $g(x) \ge 0 \ \forall x > 0$. *Assume that there exist three positive constants* c_1, c_2, d, *with* $c_1 < d < \frac{c_2}{2}$ *such that*

(j') $$\frac{\max_{|t|\le c_1} G(t)}{c_1^2} < \frac{m'}{3\|e^{-R}\overline{q}\|_1}\frac{G(d)}{d^2};$$

(jj') $$\frac{\max_{|t|\le c_2} G(t)}{c_2^2} < \frac{m'}{6\|e^{-R}\overline{q}\|_1}\frac{G(d)}{d^2}.$$

Then, for all $\lambda \in \left(\frac{3}{4}\frac{d^2\|e^{-R}\overline{q}\|_1}{G(d)\|e^{-R}\|_1}, \min\left\{\frac{m'c_1^2}{4\|e^{-R}\|_1 \max_{|t|<c_1} G(t)}, \frac{m'c_2^2}{2\|e^{-R}\|_1 \max_{|t|<c_2} G(t)}\right\}\right)$,
the problem (7) *admits three nonnegative classical solutions.*

EXAMPLE 8. *Put*

$$a_n := \frac{2n!(n+2)! - 1}{4(n+1)!}, \quad b_n := \frac{2n!(n+2)! + 1}{4(n+1)!}$$

for every $n \in \mathbf{N}$ *and define the positive continuous function* $f : \mathbf{R} \to \mathbf{R}$

$$f(\xi) = \begin{cases} \dfrac{32(n+1)!^2[(n+1)!^2 - n!^2]}{\pi}\sqrt{\dfrac{1}{16(n+1)!^2} - \left(\xi - \dfrac{n!(n+2)}{2}\right)^2} + 1 & \text{if } \xi \in \cup_{n\in\mathbf{N}}[a_n, b_n], \\ 1 & \text{otherwise.} \end{cases}$$

One has $\int_{n!}^{(n+1)!} f(t)dt = \int_{a_n}^{b_n} f(t)dt = (n+1)!^2 - n!^2$, *for every* $n \in \mathbf{N}$. *Then, one has* $\lim_{n\to+\infty} \dfrac{F(b_n)}{b_n^2} = 4$ *and* $\lim_{n\to+\infty} \dfrac{F(a_n)}{a_n^2} = 0$. *Therefore, by a simple computation, we obtain* $\liminf_{\xi\to+\infty} \dfrac{F(\xi)}{\xi^2} = 0$ *and* $\limsup_{\xi\to+\infty} \dfrac{F(\xi)}{\xi^2} = 4$. *So,*

$$0 = \liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} < \frac{2}{e-1} = \frac{1}{2(e-1)} \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2}.$$

Hence, from Corollary 4, *for each* $\lambda > \dfrac{1}{8}$, *the problem*

$$\text{(9)} \qquad \begin{cases} -u'' + u' + u = \lambda f(u) \\ u'(0) = u'(1) = 0 \end{cases}$$

has a sequence of pairwise distinct positive classical solutions.

EXAMPLE 9. *The problem*

$$\text{(10)} \qquad \begin{cases} -\left(e^{-x}u'\right)' + e^x u = 2\lambda x u^{10}(11 - u) \\ u'(0) = u'(1) = 0 \end{cases}$$

admits at least three solutions for each $\lambda \in \left(\dfrac{3(e-1)e^2}{2^{11}}, \dfrac{e}{4}\right)$.

In fact, if we choose, for instance, $c = 1$ *and* $d = 2$, *hypotheses of Theorem* 2 *are satisfied.*

EXAMPLE 10. *Consider the following problem*

$$\text{(11)} \qquad \begin{cases} -u'' + u = \lambda f(x, u) \\ u'(0) = u'(1) = 0. \end{cases}$$

Let $f(x, t) = g(x)h(t)$, *where* $g(x) : [0, 1] \to \mathbf{R}$, $g(x) = x$, *and*

$$h(t) = \begin{cases} 1, & if\ \ t \in (-\infty, 1]; \\ t^{10}, & if\ \ t \in (1, 2]; \\ 2^{10}, & if\ \ t \in (2, 400]; \\ \widetilde{h}(t), & if\ \ t \in (400, +\infty[, \end{cases}$$

where $\widetilde{h}(t)$ *is an arbitrary function.*

Choosing $c_1 = 1$, $d = 2$ *and* $c_2 = 400$, *the function* $f(x, t)$ *satisfies the hypotheses of Theorem* 3, *and, for each* $\lambda \in \left(\frac{33}{2^{11} - 1}, \frac{2^7 5^3}{\left(\frac{1 - 2^{11}}{11} + 2^{13} 5 \right)} \right)$, *the given problem admits at least three positive classical solutions* u_i *such that* $|u_i(t)| < 400$ *for all* $t \in [0, 1]$, $i = 1, 2, 3$.

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