Bollettino Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **6** (2013), n.3, p. 781–791.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2013_9_6_3_781_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2013.

Bollettino U. M. I.
(9) **VI** (2013), 781-791

Noether's Theorem on Gonality of Plane Curves for Hypersurfaces

FRANCESCO BASTIANELLI

Abstract. – *A well-known theorem of Max Noether asserts that the gonality of a smooth curve $C \subset \mathbb{P}^2$ of degree $d \geq 4$ is $d-1$, and any morphism $C \longrightarrow \mathbb{P}^1$ of minimal degree is obtained as the projection from one point of the curve. The most natural extension of gonality to n-dimensional varieties X is the degree of irrationality, that is the minimum degree of a dominant rational map $X \dashrightarrow \mathbb{P}^n$. This paper reports on the joint work* [4] *with Renza Cortini and Pietro De Poi, which aims at extending Noether's Theorem to smooth hypersurfaces $X \subset \mathbb{P}^{n+1}$ in terms of degree of irrationality. We show that both generic surfaces in $\mathbb{P}^3$ and generic threefolds in $\mathbb{P}^4$ of sufficiently large degree d have degree of irrationality $d-1$, and any dominant rational map of minimal degree is obtained as the projection from one point of the variety. Furthermore, we classify the exceptions admitting maps of minimal degree smaller than $d-1$, and we show that their degree of irrationality is $d-2$.*

1. – Introduction

Let C be a complex projective non-singular curve. The *gonality* of C is the minimum integer m such that the curve admits a non-constant morphism $f: C \longrightarrow \mathbb{P}^1$ of degree m.

The gonality of plane curves is governed by the following classical theorem, whose assertion goes back to Max Noether (see [17]). It is actually included in a wider statement describing the dimension and the geometry of linear systems of arbitrary degree on plane curves. We note further that Noether's original proof is incomplete, and the general assertion has been differently proved in the eighties by Ciliberto and Hartshorne (cf. [5, 11]).

NOETHER'S THEOREM *Let $C \subset \mathbb{P}^2$ be a smooth curve of degree $d \geq 4$. Then the gonality of C is* $\mathrm{gon}(C) = d-1$.

Moreover, any morphism $C \longrightarrow \mathbb{P}^1$ of degree $d-1$ is obtained projecting C from one of its points.

It is interesting to note that analogous results hold for certain classes of non-singular space curves. The paradigmatic statement is that—under additional hypothesis—the gonality of a smooth curve $C \subset \mathbb{P}^3$ is $\mathrm{gon}(C) = d - h$, where d is the degree of the curve and $h := \max\{\deg(\ell \cap C) | \ell \subset \mathbb{P}^3 \text{ is a line}\}$ is the maximal order of multisecant lines. For instance, this has been shown for complete intersections ([2]), curves lying on smooth quadric surfaces ([1, 15]), and arithmetically Cohen-Macaulay curves either being general ([10]) or lying on particular quartic surfaces ([7]). However, Noether's Theorem can not be extended in this terms to every space curve of large enough degree (cf. [9, Example 2.9] and [10]).

On the other hand, it would be interesting to investigate when the gonality of non-singular curves $C \subset \mathbb{P}^n$ can be computed by projecting C from suitable multisecant $(n-2)$–planes.

Throughout we deal instead with extensions of Noether's Theorem to non-singular hypersurfaces of projective spaces. In particular, we report on the joint work [4] with Renza Cortini and Pietro De Poi, where analogous statements have been proved for surfaces and threefolds.

In this setting, the most suitable notion extending gonality to a n-dimensional variety X is the *degree of irrationality*—usually denoted by $\mathrm{d}_r(X)$—that is the minimum integer m such that the variety admits a dominant rational map $f: X \dashrightarrow \mathbb{P}^n$ of degree m. In particular, the degree of irrationality of a curve C coincides with its gonality as any dominant rational map $C \dashrightarrow \mathbb{P}^1$ can be resolved to a morphism. Furthermore, the projection of a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d from one of its points does provide a dominant rational map $X \dashrightarrow \mathbb{P}^n$ of degree $d-1$.

Thus it is natural to wonder whether Noether's theorem can be somehow extended in terms of degree of irrationality to hypersurfaces of $\mathbb{P}^{n+1}$.

The first result we present in this direction is a bound on the degree of irrationality of smooth hypersurfaces of arbitrary dimension (cf. [4, Theorem 1.2]).

THEOREM 1.1. – *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \geq n+3$. Then*

$$d - n \leq \mathrm{d}_r(X) \leq d - 1.$$

We note that this theorem overlaps the first part of Noether's one when $n = 1$. Moreover, we shall see that surfaces in $\mathbb{P}^3$ reach both the admissible values, whereas the lower bound fails to be sharp for threefolds in $\mathbb{P}^4$.

By focusing on small dimensional hypersurfaces, we can indeed provide a more detailed picture. Beside plane curves, the first case to describe is given by non-singular surfaces in $\mathbb{P}^3$ (cf. [4, Theorem 1.3]).

THEOREM 1.2. – *Let $X \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 5$. Then $\mathrm{d}_r(X) = d - 1$ unless one of the following occurs:*

(1) *X contains a twisted cubic;*
(2) *X contains a line ℓ and a rational curve C of degree c such that ℓ is a $(c-1)$-secant line of C;*

in these cases $\mathrm{d}_r(X) = d - 2$.

It is worth noticing that there exist surfaces as in (1) and (2) (see e.g. [8, p. 355] and [4, Example 4.8]). Moreover, the dominant rational maps $X \dashrightarrow \mathbb{P}^2$ of degree $d-2$ can be constructed explicitly by means of either the family of bisecant lines of the rational normal cubic, or the family of lines meeting both ℓ and C. Namely,

EXAMPLE 1.3. Let $X \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 6$ containing a twisted cubic Γ. For any $x \in X - \Gamma$, there exists a unique line ℓ_x through x intersecting twice the cubic (cf. Theorem 2.3), and each ℓ_x meets X at $d-2$ points outside Γ. We then consider the second symmetric product of the twisted cubic $\Gamma^{(2)} \cong \mathbb{P}^2$, and we define a dominant rational map $X \dashrightarrow \Gamma^{(2)}$ of degree $d-2$ by sending a general point $x \in X$ to the unordered pair $x_1 + x_2 \in \Gamma^{(2)}$ such that $\ell_x \cap \Gamma = \{x_1, x_2\}$.

On the other hand, provided that $X \subset \mathbb{P}^3$ is a smooth surface of degree $d \geq 6$ as in (2), we can construct analogously a dominant rational map of degree $d-2$ on the rational surface $\ell \times C$.

In the case of non-singular threefolds in $\mathbb{P}^4$, we prove a very similar result (see [4, Theorem 1.4]). In particular, we still provide a characterization—in terms of subvarieties—of those hypersurfaces having degree of irrationality $d-2$, and the dominant rational maps $X \dashrightarrow \mathbb{P}^3$ of minimal degree are obtained analogously. However, we can not decide whether these exceptions really occur.

THEOREM 1.4. – *Let $X \subset \mathbb{P}^4$ be a smooth threefold of degree $d \geq 7$. Then* $\mathrm{d}_r(X) = d - 1$ *unless one of the following occurs:*

(1) *X contains a non-degenerate rational scroll S of degree s and an $(s-1)$-secant line ℓ of S;*
(2) *X contains a non-degenerate rational surface S of degree s and a line $\ell \subset S$ such that the intersection of S outside ℓ with the general hyperplane H containing ℓ is a rational curve C and ℓ is an $(s-2)$-secant line of it;*

in these cases $\mathrm{d}_r(X) = d - 2$.

In the light of the previous results, both surfaces in $\mathbb{P}^3$ and threefolds in $\mathbb{P}^4$ having $\mathrm{d}_r(X) = d-2$ are characterized by the existence of some rational subvarieties. On the other hand, sufficiently general hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree $d \geq 2n+1$ do not contain rational curves (see [26, 6, 25]). Thus we deduce that when $n=2$ or $n=3$, the degree of irrationality of a generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq 2n+1$ is $\mathrm{d}_r(X) = d-1$.

Furthermore, we prove the following generalization of Noether's Theorem.

THEOREM 1.5. – *Let $1 \leq n \leq 3$, and let $X \subset \mathbb{P}^{n+1}$ be a generic hypersurface of degree $d \geq 2n+2$. Then the degree of irrationality of X is $\mathrm{d}_r(X) = d-1$.*

Moreover, any dominant rational map $f\colon X \dashrightarrow \mathbb{P}^n$ of degree $d-1$ is obtained projecting X from one of its points.

In the next section we outline the proofs of the above results, whereas the last section is devoted to some remarks and conjectures in view of further extensions of Noether's Theorem.

2. – Outline of the proofs

In this section we present the techniques our argument involves, and we outline the proofs of the results stated in the Introduction.

In the spirit of [13] and [3], we use Mumford's technique of *induced differentials* (see [16]) to provide restrictions of *Cayley-Bacharach* type on dominant rational maps $X \dashrightarrow \mathbb{P}^n$ from non-singular hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree $d \geq n+3$. On one hand, we estimate the degree of the maps, and we deduce Theorem 1.1. On the other, we show that the general fiber of a map $X \dashrightarrow \mathbb{P}^n$ of low degree consists of collinear points, so that any such a map specifies a families of lines in $\mathbb{P}^{n+1}$.

Then we turn to describing an important class of families known as *first order congruence of lines of* $\mathbb{P}^{n+1}$, and we establish a certain one-to-one correspondence connecting them with dominant rational maps $X \dashrightarrow \mathbb{P}^n$ of low degree.

In the light of this fact, we finally achieve our results by exploiting the classification of first order congruence of lines in low dimensional projective spaces, together with some classical results such as Lefschetz and Bertini's Theorems.

2.1 – *Mumford's trace map and Cayley-Bacharach condition*

Let $f\colon X \dashrightarrow Y$ be a dominant rational map of degree m between non-singular n-dimensional varieties. Let $U := \{y \in Y \mid \dim f^{-1}(y) = 0\}$ be the open set over which f is finite, and let $X^{(m)}$ be the m-fold symmetric product of X. So we can consider the morphism $\gamma\colon U \longrightarrow X^{(m)}$, which sends the general $y \in U$ to the unordered m-tuple $x_1 + \ldots + x_m$ such that $f^{-1}(y) = \{x_1, \ldots, x_m\}$.

By using Mumford's induced differentials (cf. [16, Section 2]), it is then possible to define the *trace map of* γ

$$\begin{array}{cccc} \mathrm{Tr}_\gamma\colon & H^{n,0}(X) & \longrightarrow & H^{n,0}(U) \\ & \omega & \longmapsto & \omega_\gamma \end{array}.$$

When the morphism γ has null trace map, the general 0-cycle $x_1 + \ldots + x_m \in \mathrm{Im}\gamma$ satisfies the *Cayley-Bacharach condition* with respect to the canonical linear series $|\omega_X|$, i.e. for any $i = 1, \ldots, m$ and for any effective divisor $K_X \in |\omega_X|$ passing through $x_1, \ldots, \widehat{x}_i, \ldots, x_m$, we have that $x_i \in K_X$ as well (see e.g. [13, Section 2]).

Given a dominant rational map $f\colon X \dashrightarrow \mathbb{P}^n$ from a non-singular hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d, it is immediate to check that $H^{n,0}(U) = \{0\}$, so that γ has null trace map. Thus the general point of $\mathrm{Im}\gamma$ satisfies the Cayley-Bacharach condition with respect to $|\omega_X| \cong \mathcal{O}_{\mathbb{P}^{n+1}}(d - n - 2)$, and we deduce the following (cf. [4, Theorem 2.5 and Example 2.7]).

LEMMA 2.1. – *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \geq n + 3$, and let $f\colon X \dashrightarrow \mathbb{P}^n$ be a dominant rational map of degree m. Then*

i) the degree of the map is $m \geq d - n$;
ii) if in addiction $m \leq 2d - 2n - 3$ and $y \in \mathbb{P}^n$ is general, the points $x_1, \ldots, x_m \in f^{-1}(y)$ are collinear.

We point out that the first assertion provides the lower bound on $\mathrm{d}_r(X)$ of Theorem 1.1, whereas the upper bound is the obvious one obtained projecting $X \subset \mathbb{P}^{n+1}$ from one of its points.

On the other hand, any dominant rational map $f\colon X \dashrightarrow \mathbb{P}^n$—as in the second assertion—specifies a n-dimensional subvariety $B' \subset \mathbb{G}(1, n+1)$ which parameterizes the lines in $\mathbb{P}^{n+1}$ containing the sets of collinear points $f^{-1}(y)$.

2.2 – *First order congruences of lines in* $\mathbb{P}^{\mathbf{n+1}}$

Let $\mathbb{G}(1, n+1)$ be the Grassmannian of lines in $\mathbb{P}^{n+1}$, and let us denote by $\ell_b \subset \mathbb{P}^{n+1}$ the line parameterized by the point $b \in \mathbb{G}(1, n+1)$. A *congruence of lines in* $\mathbb{P}^{n+1}$ is a flat family of lines

$$\begin{array}{l} \Lambda := \left\{(b, P) \in B \times \mathbb{P}^{n+1} | P \in \ell_b\right\} \\ \big\downarrow \\ B \end{array}$$

obtained as the pullback of the universal family under a desingularization map $B \longrightarrow B'$ of an irreducible n-dimensional subvariety $B' \subset \mathbb{G}(1, n+1)$.

The *order* of the congruence is defined as the degree of the projection $\Lambda \longrightarrow \mathbb{P}^{n+1}$, that is the number of lines of the family passing through the general point of $\mathbb{P}^{n+1}$. In particular, a *first order congruence of lines in* $\mathbb{P}^{n+1}$ is a n-dimensional family of lines such that the general point of $\mathbb{P}^{n+1}$ lies on a unique line of the family.

Moreover, a point $P \in \mathbb{P}^{n+1}$ is said to be a *fundamental point* of the congruence if it lies on infinitely many lines of the family, and the set F of fundamental points is called *fundamental locus*.

Congruences of lines are very classical geometric objects, which have been studied long since (see e.g. [12, 24, 14]), aiming for a classification in dependence on the number and the dimension of the components of their fundamental locus.

First order congruences of lines in $\mathbb{P}^2$ have a very simple description.

EXAMPLE 2.2. A congruence of lines in $\mathbb{P}^2$ is a family parameterized over a curve B lying on the dual plane $\mathbb{G}(1,2) \cong \mathbb{P}^2$. Moreover, the order of the congruence equals the degree of such a curve. Thus $\Lambda \longrightarrow B$ is a first order congruence of lines in $\mathbb{P}^2$ if and only if $B \subset \mathbb{G}(1,2)$ parameterizes the star of lines through a fixed point $F \in \mathbb{P}^2$, which is the fundamental locus.

On the other hand, the more the dimension grows, the more the combinatorics are involved. In particular, the following describes the case of $\mathbb{P}^3$ (cf. e.g. [23, 19]).

THEOREM 2.3. – *Let $B \subset \mathbb{G}(1,3)$ be a surface. Then $\Lambda \longrightarrow B$ is a first order congruence of lines in $\mathbb{P}^3$ if and only if the fundamental locus F is one of the following:*

(a) *a point, where B is the star of lines through F;*
(b) *a twisted cubic, where B is the family of bisecant lines of F;*
(c) *a non-degenerate reducible curve consisting of a rational curve C of degree c and a $(c-1)$-secant line ℓ, where B is the family of lines meeting both C and ℓ;*
(d) *a non-reduced line ℓ and B is the family of lines $\bigcup_{\pi \in B_\ell} B_{\rho(\pi)}$, where $B_\ell \subset \mathbb{G}(2,3)$ is the pencil of planes containing ℓ, $\rho\colon B_\ell \longrightarrow \ell$ is a non-constant map, and for any plane $\pi \in B_\ell$, $B_{\rho(\pi)} \subset \mathbb{G}(1,3)$ is the star of lines on π passing through the point $\rho(\pi) \in \ell$.*

Although the classification of first order congruences of lines in $\mathbb{P}^4$ is not complete, there are series of papers on this topic (see e.g. [18, 20, 21]) providing a quite detailed picture (cf. [22, Table 1]), which have been crucial to prove Theorem 1.4.

In order to relate congruences of lines and dominant rational maps from hypersurfaces in $\mathbb{P}^{n+1}$, we introduce the following notation. Let $X \subset \mathbb{P}^{n+1}$ be a non-singular hypersurface of degree $d \geq 2n+1$, and let $\Lambda \longrightarrow B$ be a first order congruence of lines in $\mathbb{P}^{n+1}$ having fundamental locus F. We denote by $F_{B|X}$ the union of the irreducible components of F entirely contained in X, and by $\delta_{B|X}$ we mean the number of intersection points—counted with multiplicity—of a general line of the congruence with X at $F_{B|X}$.

Since the n-dimensional variety $B \subset \mathbb{G}(1, n+1)$ is rational (cf. [18, Theorem 7]), the congruence $\Lambda \longrightarrow B$ induces a dominant rational map $f_B\colon X \dashrightarrow \mathbb{P}^n$ of degree $d - \delta_{B|X}$, which is obtained by sending the general $x \in X - F_{B|X}$ to the unique point $b \in B$ such that $x \in \ell_b$. On the other hand, we already observed by Lemma 2.1 that any dominant rational map $f\colon X \dashrightarrow \mathbb{P}^n$ of low degree specifies a family $\Lambda_f \longrightarrow B_f$ of lines in $\mathbb{P}^{n+1}$, which turns out to be a first order congruence.

Indeed [4, Theorem 4.3] assures that the above constructions are somehow each other's inverses. Namely,

THEOREM 2.4. – *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \geq 2n+3-k$, with $0 \leq k \leq 2$. Then*

i) any dominant rational map $f\colon X \dashrightarrow \mathbb{P}^n$ of degree $m \leq d-k$ induces a first order congruence $\Lambda_f \longrightarrow B_f$ of lines in $\mathbb{P}^{n+1}$;

ii) any first order congruence $\Lambda \longrightarrow B$ of lines in $\mathbb{P}^{n+1}$ induces a dominant rational map $f_B\colon X \dashrightarrow \mathbb{P}^n$ of degree $m = d - \delta_{B|X}$.

In particular, the congruence induced by the map f_B coincides with B, and—up to birational isomorphisms of $\mathbb{P}^n$—the dominant rational map induced by the congruence $\Lambda_f \longrightarrow B_f$ is f.

2.3 – *Proofs*

By virtue of the previous subsections, proving Theorems 1.2, 1.4 and 1.5 is now almost straightforward.

Moreover, we can provide a different proof of Noether's Theorem as follows.

PROOF OF NOETHER'S THEOREM. Let $X \subset \mathbb{P}^2$ be a smooth curve of degree $d \geq 4$. Therefore $\mathrm{gon}(X) = \mathrm{d}_r(X) = d-1$ by Theorem 1.1.

On the other hand, setting $n = k = 1$ in Theorem 2.4, any morphism $f\colon X \longrightarrow \mathbb{P}^1$ of degree $d-1$ induces a first order congruence $\Lambda_f \longrightarrow B_f$ of lines in $\mathbb{P}^2$. Therefore B_f parameterizes a star of lines through a fixed point $F \in \mathbb{P}^2$, which is the fundamental locus of the congruence (cf. Example 2.2). Since $\deg f = d-1 = d - \delta_{B|X}$, we conclude that $F \in X$ and $f\colon X \longrightarrow \mathbb{P}^1$ is the projection from F. □

Then we turn to the degree of irrationality of non-singular surfaces in $\mathbb{P}^3$ and we prove Theorem 1.2.

Proof of Theorem. 1.2. Let $X \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 5$. Hence its degree of irrationality $\mathrm{d}_r(X)$ equals either $d-2$ or $d-1$ by Theorem 1.1.

In order to characterize the former case, we suppose that X admits a dominant rational map $f: X \dashrightarrow \mathbb{P}^2$ of degree $d-2$. By Theorem 2.4, the map f is induced by the associated first order congruence $\Lambda_f \longrightarrow B_f$ of lines in $\mathbb{P}^3$. Moreover, $\deg f = d - 2 = d - \delta_{B_f|X}$ and hence the general line $\ell_b \subset \Lambda_f$ intersects twice X at $F_{B_f|X}$. Clearly, the intersection multiplicity between ℓ_b and X at any reduced component of $F_{B_f|X}$ is one. We refer throughout to the classification of first order congruence of lines in $\mathbb{P}^3$ provided by Theorem 2.3.

If the general line $\ell_b \subset \Lambda_f$ meets X at two distinct points of $F_{B_f|X}$, then the fundamental locus F is either a twisted cubic as in (b), or a non-degenerate reducible curve as in (c). In particular, the whole fundamental locus F is contained in the surface X, and we obtain conditions (1) and (2) in the statement of Theorem 1.2.

On the other hand, if the general line $\ell_b \subset \Lambda_f$ met X at a double point of the fundamental locus, the surface X would be singular. This is obvious if $F \in X$ were a point as in (a), whereas [4, Lemma 4.6] uses Bertini's Theorem recursively to show that, if F were a non-reduced line as in (d), any plane containing F would be tangent to X at the same point. □

The proof of Theorem 1.4 is very similar to the previous one, and we refer the reader to [4, Section 4] for details. Initially, we note that the degree of irrationality of a smooth threefold $X \subset \mathbb{P}^4$ of degree $d \geq 7$ satisfies $d - 3 \leq \mathrm{d}_r(X) \leq d - 1$. Then we consider a dominant rational map $f: X \dashrightarrow \mathbb{P}^3$ of degree $\deg f \leq d - 2$, together with its associated congruence $\Lambda_f \longrightarrow B_f$. Finally, we analyze $F_{B_f|X}$ in the light of Lefschetz Theorem and the description of first order congruences of lines in $\mathbb{P}^4$. In particular, there are only two admissible configurations, which lead to $\mathrm{d}_r(X) = d - 2$ and are described by conditions (1) and (2) of Theorem 1.4.

Then we conclude this section by proving Noether's Theorem for generic hypersurfaces $X \subset \mathbb{P}^{n+1}$, having dimension $1 \leq n \leq 3$ and degree $d \geq 2n + 2$.

Proof of Theorem. 1.5. The case of plane curves is covered by Noether's result. Therefore we set $n = 2, 3$ and we consider a dominant rational map $f: X \dashrightarrow \mathbb{P}^n$ of minimal degree. Since $\deg f \leq d - 1$ and $d \geq 2n + 2$, Theorem 2.4 guarantees that there exists a first order congruence $\Lambda_f \longrightarrow B_f$ of lines in $\mathbb{P}^{n+1}$ induced by f.

Let F be the fundamental locus of the congruence, and let $F_{B_f|X}$ the union of the irreducible components of F contained in X. From the description of first order congruences of lines in $\mathbb{P}^3$ (cf. Theorem 2.3) and in $\mathbb{P}^4$ (see [4, Table 1]), we deduce that either F is a point, or it consists of rational components. Furthermore, $F_{B_f|X}$ is non-empty as $\deg f = d - \delta_{B_f|X} \leq d - 1$.

On the other hand, X does not contain rational subvarieties (cf. [26, 6]). Hence $F = F_{B_f|X}$ is a point of X, and $\Lambda_f \longrightarrow B_f$ is the family of lines of $\mathbb{P}^{n+1}$ passing through F. Thus $f: X \dashrightarrow \mathbb{P}^n$ is the dominant rational map of degree $\mathrm{d}_r(X) = d - 1$ projecting X from the point $F \in X$.

3. – Final remarks

In this section we briefly deal with the degree of irrationality of non-singular hypersurfaces of arbitrary dimension. In particular, we discuss whether the techniques we used may lead to further extensions of Noether's Theorem. On one hand, we look for a characterization of hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree d having $\mathrm{d}_r(X) \leq d - 2$, as in Theorems 1.2 and 1.4. On the other, we focus on generic hypersurfaces along the lines of Theorem 1.5.

To this aim, we note that both Theorem 1.1—bounding $\mathrm{d}_r(X)$—and Theorem 2.4—connecting dominant rational maps of minimal degree with first order congruences of lines in $\mathbb{P}^{n+1}$—hold for any dimension n.

In low dimensional cases, our argument uses the descriptions of first order congruences of lines in $\mathbb{P}^3$ and $\mathbb{P}^4$ to decide whenever the irreducible components of the fundamental locus may lie on the hypersurface X. However, a detailed picture of first order congruences in arbitrary dimension is still missing. Moreover, it turns out that even to construct examples of hypersurfaces $X \subset \mathbb{P}^{n+1}$ having $\mathrm{d}_r(X) \leq d - 2$ is a difficult task. Indeed, we can not determine whether these varieties really occur when n is odd, whereas something more is known otherwise.

EXAMPLE 3.1. When $n = 2k$ is even, the family of lines in $\mathbb{P}^{n+1}$ meeting two given k-planes is a first order congruence. Thus any hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq 3$ containing both these linear spaces has $\mathrm{d}_r(X) = d - 2$. For example, the smooth hypersurface defined by the equation $x_0^d - x_0 x_1^{d-1} + \ldots + x_{2k}^d - x_{2k} x_{2k+1}^{d-1} = 0$ contains the k-planes $x_0 = \ldots = x_{2k} = 0$ and $x_0 - x_1 = \ldots = x_{2k} - x_{2k+1} = 0$.

If instead $n = 2k - 1$, analogous constructions fail as no smooth hypersurface in $\mathbb{P}^{2k}$ contains a linear space of dimension at least k.

Turning to generic hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree $d \geq 2n + 1$, we already mentioned that they do not contain rational curves (cf. [6]).

On the other hand, rational curves cover the whole fundamental locus of every known first order congruences of lines in $\mathbb{P}^{n+1}$, which has been classified and differs from the star of lines through a fixed point. Furthermore, the same fact holds for many other non-classified cases (see e.g. [19, Theorem 7]). Therefore it seems natural to conjecture that for any first order congruence of lines in $\mathbb{P}^{n+1}$, each positive dimensional irreducible component of the fundamental locus contains rational curves.

If this is the case, then none of these component shall be contained on a generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq 2n+1$, and we shall achieve the following.

CONJECTURE 3.2. *Let $X \subset \mathbb{P}^{n+1}$ be a generic hypersurface of degree $d \geq 2n+1$. Then the degree of irrationality of X is $\mathrm{d}_r(X) = d-1$.*

If in addiction $d \geq 2n+2$, any dominant rational map $X \dashrightarrow \mathbb{P}^n$ of degree $d-1$ is obtained projecting X from one of its points.

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Received December 6, 2012 and in revised form December 8, 2013