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Regularity for minimizers of non-autonomous non-quadratic functionals in the case $1 < p < 2$: an a priori estimate

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Abstract – We establish an a priori estimate for the second derivatives of local minimizers of integral functionals of the form

$$\mathscr{F}(v,\Omega) = \int_{\Omega} f(x, Dv(x))\, dx,$$

with convex integrand with respect to the gradient variable, assuming that the function that measures the oscillation of the integrand with respect to the x variable belongs to a suitable Sobolev space. The novelty here is that we deal with integrands satisfying subquadratic growth conditions with respect to gradient variable.

Riassunto – Ricaviamo una stima a priori per le derivate seconde di minimi locali di funzionali integrali del tipo

$$\mathscr{F}(v,\Omega) = \int_{\Omega} f(x, Dv(x))\, dx,$$

con integranda convessa rispetto alla variabile gradiente, assumendo che la funzione che misura l'oscillazione dell'integranda rispetto alla variabile x appartenga ad un opportuno spazio di Sobolev. La novità, qui, è che si tratta del caso in cui l'integranda soddisfi condizioni di crescita subquadratica rispetto alla variabile gradiente.

1 - INTRODUCTION

In this paper we consider integral functionals of the form

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$$\mathscr{F}(v,\Omega) = \int_\Omega f(x, Dv(x))dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is a Carathéodory map, such that $\xi \mapsto f(x,\xi)$ is of class $C^1(\mathbb{R}^{N\times n})$, and for an exponent $p \in (1,2)$ and some constants $L_1, L_2, \alpha > 0$ and $\mu \geq 0$ the following conditions are satisfied:

$$L_1(\mu^2 + |\xi|^2)^{\frac{p}{2}} \leq f(x,\xi) \leq L_2(\mu^2 + |\xi|^2)^{\frac{p}{2}}, \tag{1.2}$$

$$\langle D_\xi f(x,\xi) - D_\xi f(x,\eta), \xi - \eta \rangle \geq \alpha \left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi - \eta|^2, \tag{1.3}$$

for every $\xi, \eta \in \mathbb{R}^{N\times n}$ and for almost every $x \in \Omega$.

For what concerns the dependence of the energy density on the x-variable, we shall assume that the function $D_\xi f(x,\xi)$ is weakly differentiable with respect to x and that $D_x(D_\xi f) \in L^q(\Omega \times \mathbb{R}^{N\times n})$, for some $q > n$.
By the point-wise characterization of the Sobolev functions due to Hajlasz (Hajlasz P., 1996) this is equivalent to assume that there exists a nonnegative function $g \in L^q_{\text{loc}}(\Omega)$ such that

$$|D_\xi f(x,\xi) - D_\xi f(y,\xi)| \leq (g(x) + g(y))\,|x-y| \left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}} \tag{1.4}$$

for all $\xi \in \mathbb{R}^{N\times n}$ and for almost every $x, y \in \Omega$.

The regularity properties of minimizers of such integral functionals have been widely investigated in case the energy density $f(x,\xi)$ depends on the x-variable through a continuous function both in the superquadratic and in the subquadratic growth case. In fact, it is well known that the partial continuity of the vectorial minimizers can be obtained with a quantitative modulus of continuity that depends on the modulus of continuity of the coefficients (see for example (Acerbi E. and Fusco N., 1989), (Fusco N. and Hutchinson J. E., 1985), (Giaquinta M. and Modica G., 1986)) and the monographs (Giaquinta M., 1983), (Giusti E., 2003) for a more exhaustive treatment). For regularity results under general growth conditions, that of course include the superquadratic and the subquadratic ones, we refer to (Diening L. et al., 2009), and (Diening L. et al., 2011).

Recently, there has been an increasing interest in the study of the regularity under different assumptions on the function that measures the oscillation of the integrand $f(x,\xi)$ with respect to the x-variable.
This study has been successfully carried out when the oscillation of $f(x,\xi)$ with respect to the x-variable is controlled through a coefficient that belongs to a suitable Sobolev class of integer or fractional order and the assumptions (1.2)–(1.4) are satisfied with an exponent $p \geq 2$.
Let us remark that the regularity of the coefficients depends on the summability of their gradients and that also the case of possibly discontinous coefficients has been treated.

Actually, it has been shown that the weak differentiability of the partial map $x \mapsto f(x,\xi)$ transfers to the gradient of the minimizers of the functional (1.1) (see (Carozza M. et al., 2011), (Eleuteri M. et al., 2016), (Eleuteri M. et al., 2016), (Giova R. and Passarelli di Napoli A., 2017), (Kristensen J. and Mingione G., 2010), (Passarelli di Napoli, 2014)) as well as to the gradient of the solutions of non linear elliptic systems (see (Baisón A. L. et al., 2017), (Clop A. et al., 2009), (Clop A. et al., 2017), (Cruz-Uribe D. et al., 2016), (Giova R., 2015), (Kuusi T. and Mingione G., 2012), (Passarelli di Napoli, 2014)) and of non linear systems with degenerate ellipticity in case $p \geq 2$. (see (Giova R., 2015)).
It is worth mentioning that the continuity of the coefficients is not sufficient to establish the higher differentiability of integer order of the minimizers, that has often revealed to be a crucial step in the investigation of other regularity properties.

As far as we know, no higher differentiability results are available for vectorial minimizers under the so-called subquadratic growth conditions, i.e. when the assumptions (1.2)–(1.4) hold true for an exponent $1 < p \leq 2$ in case of Sobolev coefficients.
The aim of this paper is to start the study of the higher differentiability properties of local minimizers of integral functional (1.1) under subquadratic growth condition. As a first step in this direction, here we shall establish the following a priori estimate for the second derivatives of the local minimizers.

Theorem 1.1. *Let $u \in W^{2,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ be a local minimizer of the functional $\mathscr{F}(v,\Omega)$ under the assumptions* (1.2)–(1.4). *If $q \geq \frac{2n}{p}$, than the following estimate*

$$\|D^2u\|_{L^p(B_r)} \leq C\left(\|Du\|_{L^p(B_R)} + 1\right) \tag{1.5}$$

holds true for every $0 < r < R$ such that $B_R \Subset \Omega$ with $C = C(\alpha, p, n, N, \|g\|_{L^q(B_R)})$.

The main tool in the proof is the use of the so called difference quotient method and a double iteration argument that allows us to reabsorb terms with critical summability. Respect to previous papers on this subject, new technical difficulties arise since in the subquadratic growth case some of the regularity properties of the integrand, valid in the superquadratic one, are lost.

2 - PRELIMINARY RESULTS

In this section we recall some standard definitions and collect several lemmas that we shall need to establish our results. We shall follow the usual convention and denote by C or c a general constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. All the norms we use on $\mathbb{R}^n$, $\mathbb{R}^N$ and $\mathbb{R}^{n\times N}$ will be the standard Euclidean ones and denoted by $|\cdot|$ in all cases. In particular, for matrices ξ, $\eta \in \mathbb{R}^{n\times N}$ we write $\langle \xi, \eta \rangle := \mathrm{trace}(\xi^T \eta)$ for the usual inner product of ξ and η, and $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$ for the corresponding Euclidean norm. When $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$ we write $a \otimes b \in$

$\mathbb{R}^{n\times N}$ for the tensor product defined as the matrix that has the element $a_r b_s$ in its r-th row and s-th column.

For a C^2 function $f\colon \Omega\times\mathbb{R}^{n\times N}\to\mathbb{R}$, we write

$$D_\xi f(x,\xi)[\eta] := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(x,\xi+t\eta)$$

and

$$D_{\xi\xi} f(x,\xi)[\eta,\eta] := \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} f(x,\xi+t\eta)$$

for $\xi,\,\eta\in\mathbb{R}^{n\times N}$ and for almost every $x\in\Omega$.
With the symbol $B(x,r)=B_r(x)=\{y\in\mathbb{R}^n:\ |y-x|<r\}$, we will denote the ball centered at x of radius r and

$$(u)_{x_0,r} = \fint_{B_r(x_0)} u(x)\,dx,$$

stands for the integral mean of u over the ball $B_r(x_0)$. We shall omit the dependence on the center when it is clear from the context.

2.1 - An auxiliary function

As usual, we shall use the following auxiliary function

$$V_p(\xi) := \left(\mu^2+|\xi|^2\right)^{\frac{p-2}{4}}\xi,\ \text{for all } \xi\in\mathbb{R}^{N\times n}. \tag{2.6}$$

2.2 - Some useful lemmas

The following results are proved in (Acerbi E. and Fusco N., 1989), and will be useful to estimate the L^p norm of D^2u, using the L^2 norm of the difference quotient of $V_p(Du)$.

Lemma 2.2. *For every $\gamma\in\left(-\frac{1}{2},0\right)$ and $\mu\geq 0$ we have*

$$\begin{aligned} c_0(\gamma)\left(\mu^2+|\xi|^2+|\eta|^2\right)^\gamma &\leq \int_0^1 \left(\mu^2+|t\xi+(1-t)\eta|^2\right)^\gamma dt \\ &\leq c_1(\gamma)\left(\mu^2+|\xi|^2+|\eta|^2\right)^\gamma, \end{aligned} \tag{2.7}$$

for every $\xi,\eta\in\mathbb{R}^k$.

Lemma 2.3. *For every $\gamma\in\left(-\frac{1}{2},0\right)$ and $\mu\geq 0$ we have*

$$(2\gamma+1)|\xi-\eta| \leq \frac{\left|\left(\mu^2+|\xi|^2\right)^\gamma\xi-\left(\mu^2+|\eta|^2\right)^\gamma\eta\right|}{\left(\mu^2+|\xi|^2+|\eta|^2\right)^\gamma} \leq \frac{c(k)}{2\gamma+1}|\xi-\eta|, \tag{2.8}$$

for every $\xi, \eta \in \mathbb{R}^k$.

The next lemma can be proved using an iteration technique, and will be needed in the following, where we will refer to this as Iteration Lemma.

Lemma 2.4 (Iteration Lemma)**.** *Let* $h : [\rho, R] \to \mathbb{R}$ *be a nonnegative bounded function,* $0 < \theta < 1$*,* $A, B \geq 0$ *and* $\gamma > 0$*. Assume that*

$$h(r) \leq \theta h(d) + \frac{A}{(d-r)^\gamma} + B$$

for all $\rho \leq r < d \leq R_0 < R$*. Then*

$$h(\rho) \leq \frac{cA}{(R_0 - \rho)^\gamma} + cB,$$

where $c = c(\theta, \gamma) > 0$*.*

For the proof we refer to (Giusti E., 2003), [Lemma 6.1].

2.3 - Finite difference and difference quotient

In what follows, we denote, for every function f, for $h \in \mathbb{R}$, and being e_s the unit vector in the x_s direction, the finite difference

$$\tau_{s,h} f(x) := f(x + he_s) - f(x).$$

Here we recall some properties of the finite difference.

Proposition 2.5. *Let* f *and* g *be two functions such that* $f, g \in W^{1,p}(\Omega, \mathbb{R}^N)$ *with* $p \geq 1$*, and let us consider the set*

$$\Omega_{|h|} := \{x \in \Omega : dist(x, \partial\Omega) > |h|\}.$$

Then the following properties hold:

1. $\tau_{s,h} f \in W^{1,p}(\Omega_{|h|}, \mathbb{R}^N)$ *and*

$$D_i(\tau_{s,h} f) = \tau_{s,h}(D_i f);$$

2. *if at least one of the functions* f *or* g *has support contained in* $\Omega_{|h|}$*, then*

$$\int_\Omega f \tau_{s,h} g dx = \int_\Omega g \tau_{s,-h} f dx;$$

3. *we have*

$$\tau_{s,h}(fg)(x) = f(x + he_s)\tau_{s,h} g(x) + g(x)\tau_{s,h} f(x).$$

The following lemmas describe fundamental properties of finite differences and difference quotients of Sobolev functions.

Lemma 2.6. *If* $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$, $1 < p < +\infty$, $s \in \{1,...,n\}$ *and* $f, D_s f \in L^p(B_R, \mathbb{R}^N)$, *then*

$$\int_{B_\rho} |\tau_{s,h} f(x)|^p dx \le |h|^p \int_{B_R} |D_s f(x)|^p dx.$$

Moreover, for $\rho < R$, $|h| < \frac{R-\rho}{2}$,

$$\int_{B_\rho} |f(x+he_s)|^p dx \le c(n,p) \int_{B_R} |f(x)|^p dx.$$

Lemma 2.7. *Let* $f : \mathbb{R}^n \to \mathbb{R}^N$, $f \in L^p(B_R, \mathbb{R}^N)$ *with* $1 < p < +\infty$. *Suppose that there exist* $\rho \in (0,R)$ *and* $M > 0$ *such that*

$$\sum_{s=1}^{n} \int_{B_\rho} |\tau_{s,h} f(x)|^p dx \le M^p |h|^p$$

for every $h < \frac{R-\rho}{s}$. *Then* $f \in W^{1,p}(B_R, \mathbb{R}^N)$. *Moreover*

$$\|Df\|_{L^p(B_\rho)} \le M.$$

3 - PROOF OF THEOREM 1.1

It is well known that every local minimizer of the functional (1.1) is a weak solution $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ of the corresponding Euler-Lagrange system, i.e.

$$\mathrm{div} A(x, Du(x)) = 0, \tag{3.9}$$

where we set

$$A_i^\alpha(x,\xi) := D_{\xi_i^\alpha} f(x,\xi), \text{ for all } \alpha = 1,...,N \text{ and } i = 1,\dots,n. \tag{3.10}$$

Assumptions (1.2) and (1.3) can be written as

$$|A(x,\xi)| \le c(\mu^2 + |\xi|^2)^{\frac{p-1}{2}}, \text{ for some constant } c \ge 0, \tag{3.11}$$

$$\langle A(x,\xi) - A(x,\eta), \xi - \eta \rangle \ge \alpha |\xi - \eta|^2 \left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}}, \tag{3.12}$$

for every ξ, $\eta \in \mathbb{R}^{n \times N}$ and for almost every $x \in \Omega$.

Concerning the dependence on the x-variable, assumption (1.4) translates into the following

$$|A(x,\xi) - A(y,\xi)| \le (g(x) + g(y)) |x - y| \left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}} \tag{3.13}$$

for every $\xi \in \mathbb{R}^{N \times n}$ and for almost every $x, y \in \Omega$.

Proof of Theorem 1.1. Let us fix a ball $B_R(x_0) = B_R$ of radius $R \in (0, \operatorname{dist}(x_0, \partial\Omega))$, and consider $\frac{R}{2} < r < \tilde{s} < t < \tilde{t} < \lambda r < R < 1$, with $1 < \lambda < 2$. Let's test the equation (3.9) with the function $\varphi = \tau_{s,-h}(\eta^2 \tau_{s,h} u)$, where $\eta \in C_0^\infty(B_t)$ is a cut off function such that $\eta = 1$ on $B_{\tilde{s}}$, $|D\eta| \le \frac{c}{t-\tilde{s}}$.
With this choice of φ, and by 2 of Proposition 2.5, we get

$$\int_{B_R} \left\langle \tau_{s,h} A(x, Du(x)), D(\eta^2(x)(\tau_{s,h} u(x))) \right\rangle dx = 0.$$

After some manipulations, and dropping the vector e_s to simplify the notations, we can write the last equality as follows

$$\begin{aligned}
I_0 &:= \int_{B_R} \left\langle A(x+h, Du(x+h)) - A(x+h, Du(x)), \eta^2(x) D(\tau_{s,h} u(x)) \right\rangle dx \\
&= -\int_{B_R} \left\langle A(x+h, Du(x)) - A(x, Du(x)), \eta^2(x) D(\tau_{s,h} u(x)) \right\rangle dx \\
&\quad - \int_{B_R} \left\langle \tau_{s,h} A(x, Du(x)), 2\eta(x) D\eta(x) \otimes \tau_{s,h} u(x) \right\rangle dx \\
&= -\int_{B_R} \left\langle A(x+h, Du(x)) - A(x, Du(x)), \eta^2(x) D(\tau_{s,h} u(x)) \right\rangle dx \\
&\quad - \int_{B_R} \left\langle A(x, Du(x)), \tau_{s,-h}\Big(2\eta(x) D\eta(x) \otimes \tau_{s,h} u(x)\Big) \right\rangle dx \\
&= -\int_{B_R} \left\langle A(x+h, Du(x)) - A(x, Du(x)), \eta^2(x) D(\tau_{s,h} u(x)) \right\rangle dx \\
&\quad - \int_{B_R} \left\langle A(x, Du(x)), \tau_{s,-h}\Big(2\eta(x) D\eta(x)\Big) \otimes \tau_{s,h} u(x) \right\rangle dx \\
&\quad - \int_{B_R} \left\langle A(x, Du(x)), 2\eta(x) D\eta(x) \otimes \tau_{s,-h}\Big(\tau_{s,h} u(x)\Big) \right\rangle dx \\
&=: I + II + III.
\end{aligned} \tag{3.14}$$

Previous equality implies that

$$I_0 \le |I| + |II| + |III|. \tag{3.15}$$

In order to estimate the integral $|I|$, we use the hypothesis (3.13) and Young's inequality, as follows

$$\begin{aligned}
|I| &\le c|h| \int_{B_R} \eta^2(x) \left(g(x) + g(x+h)\right) \left(\mu^2 + |Du(x)|^2\right)^{\frac{p-1}{2}} |D\tau_{s,h} u(x)| dx \\
&\le c|h| \int_{B_R} \eta^2(x) \left(g(x) + g(x+h)\right) \left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p-1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\cdot |D(\tau_{s,h}u(x))|dx \\
&\le \varepsilon \int_{B_R} \eta^2(x)|D(\tau_{s,h}u(x))|^2 \left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}} dx \\
&+c_\varepsilon |h|^2 \int_{B_R} \eta^2(x)\left(g^2(x)+g^2(x+h)\right) \\
&\cdot \left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p}{2}} dx. \qquad (3.16)
\end{aligned}
$$

Now, we estimate $|II|$ by (3.11) and the properties of η thus obtaining

$$
\begin{aligned}
|II| &\le \frac{c|h|}{(t-\tilde{s})^2}\int_{B_t}\left(\mu^2+|Du(x)|^2\right)^{\frac{p-1}{2}}|\tau_{s,h}u(x)|dx \\
&\le \frac{c|h|}{(t-\tilde{s})^2}\left(\int_{B_t}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{p-1}{p}}\left(\int_{B_t}|\tau_{s,h}u(x)|^p dx\right)^{\frac{1}{p}}, \qquad (3.17)
\end{aligned}
$$

where, in the last inequality, we used Hölder's inequality. By virtue of the first inequality of Lemma 2.6, we obtain

$$
|II| \le \frac{c|h|^2}{(t-\tilde{s})^2}\int_{B_{\tilde{t}}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx. \qquad (3.18)
$$

The term $|III|$ is estimated using the hypothesis (3.11) again, the properties of η, Hölder's inequality and Lemma 2.6, as follows

$$
\begin{aligned}
|III| &\le \frac{c}{t-\tilde{s}}\int_{B_t}\left(\mu^2+|Du(x)|^2\right)^{\frac{p-1}{2}}|\tau_{s,-h}(\tau_{s,h}u(x))|dx \\
&\le \frac{c}{t-\tilde{s}}\left(\int_{B_t}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{p-1}{p}}\left(\int_{B_t}|\tau_{s,-h}(\tau_{s,h}u(x))|^p dx\right)^{\frac{1}{p}} \\
&\le \frac{c|h|}{t-\tilde{s}}\left(\int_{B_t}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{p-1}{p}}\left(\int_{B_{\tilde{t}}}|\tau_{s,h}Du(x))|^p dx\right)^{\frac{1}{p}}, \qquad (3.19)
\end{aligned}
$$

where in the last inequality we used Lemma 2.6 and (1) of Proposition 2.5.

By the assumption (3.12), we get

$$
|I_0| \ge \alpha \int_{B_R}\eta^2(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\tau_{s,h}Du(x)|^2 dx. \qquad (3.20)
$$

Inserting estimates (3.16), (3.18), (3.19) and (3.20) in (3.15), we obtain

$$
\begin{aligned}
&\alpha \int_{B_R} \eta^2(x) \left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}} |\tau_{s,h} Du(x)|^2 dx \\
&\le \varepsilon \int_{B_R} \eta^2(x) |D(\tau_{s,h} u(x))|^2 \left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}} dx \\
&+ c_\varepsilon |h|^2 \int_{B_R} \eta^2(x) \left(g^2(x)+g^2(x+h)\right) \left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p}{2}} dx \\
&+ \frac{c|h|^2}{(t-\tilde{s})^2} \int_{B_{\tilde{t}}} \left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}} dx \\
&+ \frac{c|h|}{t-\tilde{s}} \left(\int_{B_t} \left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}} dx \right)^{\frac{p-1}{p}} \left(\int_{B_{\tilde{t}}} |\tau_{s,h} Du(x))|^p dx \right)^{\frac{1}{p}}. \qquad (3.21)
\end{aligned}
$$

Choosing $\varepsilon = \frac{\alpha}{2}$ in the previous estimate, we can reabsorb the first integral in the right hand side by the left hand side thus getting

$$
\begin{aligned}
&\int_{B_R} \eta^2(x) \left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}} |\tau_{s,h} Du(x)|^2 dx \\
&\le c|h|^2 \int_{B_R} \eta^2(x) \left(g^2(x)+g^2(x+h)\right) \left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p}{2}} dx \\
&+ \frac{c|h|^2}{(t-\tilde{s})^2} \int_{B_{\tilde{t}}} \left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}} dx \\
&+ \frac{c|h|}{t-\tilde{s}} \left(\int_{B_t} \left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}} dx \right)^{\frac{p-1}{p}} \left(\int_{B_{\tilde{t}}} |\tau_{s,h} Du(x))|^p dx \right)^{\frac{1}{p}}, \qquad (3.22)
\end{aligned}
$$

with $c = c(\alpha, p, n, N)$.

Dividing previous estimate by $|h|^2$ and using Lemma 2.3, we have

$$
\begin{aligned}
&\int_{B_R} \eta^2(x) \frac{|\tau_{s,h}(V_p(Du(x)))|^2}{|h|^2} dx \\
&\le c \int_{B_R} \eta^2(x) \left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}} \frac{|\tau_{s,h} Du(x)|^2}{|h|^2} dx \\
&\le c \int_{B_R} \eta^2(x) \left(g^2(x)+g^2(x+h)\right) \left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p}{2}} dx \\
&+ \frac{c}{(t-\tilde{s})^2} \int_{B_{\tilde{t}}} \left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}} dx
\end{aligned}
$$

$$+\frac{c}{t-\tilde{s}}\left(\int_{B_t}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{p-1}{p}}\left(\int_{B_{\tilde{t}}}\frac{|\tau_{s,h}Du(x))|^p}{|h|^p}dx\right)^{\frac{1}{p}}. \tag{3.23}$$

Now, by Hölder's inequality and Lemma 2.3, we get

$$\begin{aligned}
&\int_{B_R}\eta^2(x)\frac{\left|\tau_{s,h}Du(x)\right|^p}{|h|^p}dx\\
&\le\int_{B_R}\eta^2(x)\frac{\left|\tau_{s,h}\left(V_p\left(Du(x)\right)\right)\right|^p}{|h|^p}\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p(2-p)}{4}}dx\\
&\le\left(\int_{B_R}\eta^2(x)\frac{\left|\tau_{s,h}\left(V_p\left(Du(x)\right)\right)\right|^2}{|h|^2}dx\right)^{\frac{p}{2}}\\
&\cdot\left(\int_{B_R}\eta^2(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{2-p}{2}},
\end{aligned} \tag{3.24}$$

and therefore, combining (3.23) and (3.24), we have

$$\begin{aligned}
&\int_{B_R}\eta^2(x)\frac{\left|\tau_{s,h}Du(x)\right|^p}{|h|^p}dx\\
&\le c\left\{\int_{B_R}\eta^2(x)\left(g^2(x)+g^2(x+h)\right)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p}{2}}dx\right.\\
&+\frac{1}{(t-\tilde{s})^2}\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\\
&\left.+\frac{1}{t-\tilde{s}}\left(\int_{B_t}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{p-1}{p}}\left(\int_{B_{\tilde{t}}}\frac{|\tau_{s,h}Du(x))|^p}{|h|^p}dx\right)^{\frac{1}{p}}\right\}^{\frac{p}{2}}\\
&\cdot\left\{\int_{B_R}\eta^2(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p}{2}}dx\right\}^{\frac{2-p}{2}}.
\end{aligned} \tag{3.25}$$

Using Young's inequality with exponents $\frac{2}{p}$ and $\frac{2}{2-p}$, which is legitimate since $1<p<2$, and the properties of η, we have

$$\begin{aligned}
&\int_{B_R}\eta^2(x)\frac{\left|\tau_{s,h}Du(x)\right|^p}{|h|^p}dx\\
&\le c\int_{B_R}\eta^2(x)\left(g^2(x)+g^2(x+h)\right)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p}{2}}dx
\end{aligned}$$

$$+c\left(1+\frac{1}{(t-\tilde{s})^2}\right)\int_{B_{\tilde{t}}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx$$

$$+\frac{c}{t-\tilde{s}}\left(\int_{B_t}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{p-1}{p}}\left(\int_{B_{\tilde{t}}}\frac{|\tau_{s,h}Du(x))|^p}{|h|^p}dx\right)^{\frac{1}{p}}. \qquad (3.26)$$

Using Young's inequality with exponents p and $\frac{p}{p-1}$ to estimate the last integral in the right hand side, we obtain

$$\int_{B_R}\eta^2(x)\frac{\left|\tau_{s,h}Du(x)\right|^p}{|h|^p}dx \le c\int_{B_{\tilde{t}}}g^2(x)dx+c\int_{B_{\tilde{t}}}g^2(x)\left|Du(x)\right|^p dx$$

$$+c\left(1+\frac{1}{(t-\tilde{s})^2}+\frac{1}{(t-\tilde{s})^{\frac{p}{p-1}}}\right)\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx$$

$$+\frac{1}{2}\int_{B_{\lambda r}}\frac{\left|\tau_{s,h}Du(x)\right|^p}{|h|^p}dx. \qquad (3.27)$$

Recalling that $\eta=1$ on $B_{\tilde{s}}$, we obtain

$$\int_{B_{\tilde{s}}}\frac{\left|\tau_{s,h}Du(x)\right|^p}{|h|^p}dx \le \frac{1}{2}\int_{B_{\tilde{t}}}\frac{\left|\tau_{s,h}Du(x)\right|^p}{|h|^p}dx$$

$$+c\int_{B_{\tilde{t}}}g^2(x)dx+c\int_{B_{\tilde{t}}}g^2(x)\left|Du(x)\right|^p dx$$

$$+c\left(1+\frac{1}{(t-\tilde{s})^2}+\frac{1}{(t-\tilde{s})^{\frac{p}{p-1}}}\right)\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx. \qquad (3.28)$$

Since the previous estimate holds for every $r<\tilde{s}<t<\tilde{t}<\lambda r$, and the constant appearing in (3.28) are indipendent of t, we can pass to the limit as $t\to\tilde{t}$, thus getting

$$\int_{B_{\tilde{s}}}\frac{\left|\tau_{s,h}Du(x)\right|^p}{|h|^p}dx \le \frac{1}{2}\int_{B_{\tilde{t}}}\frac{\left|\tau_{s,h}Du(x)\right|^p}{|h|^p}dx$$

$$+c\int_{B_{\tilde{t}}}g^2(x)dx+c\int_{B_{\tilde{t}}}g^2(x)\left|Du(x)\right|^p dx$$

$$+c\left(1+\frac{1}{(\tilde{t}-\tilde{s})^2}+\frac{1}{(\tilde{t}-\tilde{s})^{\frac{p}{p-1}}}\right)\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx. \qquad (3.29)$$

By virtue of Lemma 2.4, we have

$$\int_{B_r} \frac{\left|\tau_{s,h} Du(x)\right|^p}{|h|^p} dx \le c \int_{B_{\lambda r}} g^2(x) dx + c \int_{B_{\lambda r}} g^2(x) \, |Du(x)|^p \, dx$$
$$+ c \left(1 + \frac{1}{r^2(\lambda - 1)^2} + \frac{1}{r^{\frac{p}{p-1}} (\lambda - 1)^{\frac{p}{p-1}}} \right) \int_{B_{\lambda r}} \left(\mu^2 + |Du(x)|^2 \right)^{\frac{p}{2}} dx \quad (3.30)$$

and so, by Lemma 2.6,

$$\int_{B_r} |D^2 u(x)|^p dx \le c \int_{B_{\lambda r}} g^2(x) dx + c \int_{B_{\lambda r}} g^2(x) \, |Du(x)|^p \, dx$$
$$+ c \left(1 + \frac{1}{r^2(\lambda - 1)^2} + \frac{1}{r^{\frac{p}{p-1}} (\lambda - 1)^{\frac{p}{p-1}}} \right) \int_{B_{\lambda r}} \left(\mu^2 + |Du(x)|^2 \right)^{\frac{p}{2}} dx. \quad (3.31)$$

To go further in the estimate, we have to study the term

$$\int_{B_{\lambda r}} g^2(x) \, |Du(x)|^p \, dx. \quad (3.32)$$

To do this, our first step is to apply Hölder's inequality with exponents $\frac{q}{2}$ and $\frac{q}{q-2}$, thus obtaining

$$\int_{B_{\lambda r}} g^2(x) \, |Du(x)|^p \, dx \le \left(\int_{B_{\lambda r}} g^q(x) dx \right)^{\frac{2}{q}} \left(\int_{B_{\lambda r}} |Du(x)|^{\frac{pq}{q-2}} \, dx \right)^{\frac{q-2}{q}}, \quad (3.33)$$

and the second integral in the right hand side term of (3.33) converges for $\frac{pq}{q-2} \le \frac{np}{n-p}$, that is $q \ge \frac{2n}{p}$.

Now we distinguish between two cases.

Case I $\left(q = \frac{2n}{p} \right)$. In case $q = \frac{2n}{p}$, estimate (3.33) becomes

$$\int_{B_{\lambda r}} g^2(x) |Du(x)|^p dx \le \left(\int_{B_{\lambda r}} g^{\frac{2n}{p}}(x) dx \right)^{\frac{p}{n}} \cdot \left(\int_{B_{\lambda r}} |Du(x)|^{\frac{np}{n-p}} dx \right)^{\frac{n-p}{n}}. \quad (3.34)$$

Now we observe that, if $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$, then $Du \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and, by Sobolev's embedding Theorem, $W^{1,p}_{\mathrm{loc}}(\Omega) \hookrightarrow L^{p^*}_{\mathrm{loc}}(\Omega)$, where $p^* = \frac{np}{n-p}$.
So, for a positive constant $c = c(n,p)$, we have

$$\int_{B_{\lambda r}} g^2(x)|Du(x)|^p dx \leq c \left(\int_{B_{\lambda r}} g^q(x) dx \right)^{\frac{2}{q}} \int_{B_{\lambda r}} \left(|D^2u(x)|^p + |Du(x)|^p \right) dx. \tag{3.35}$$

By the assumption $g \in L^q_{\text{loc}}(\Omega)$, and by the absolute continuity if the integral, there exists $R_0 > 0$ such that, for every $R < R_0$, we have

$$c \left(\int_{B_R} g^q(x) dx \right)^{\frac{2}{q}} < \frac{1}{2}. \tag{3.36}$$

For this choice of R, joining (3.31), (3.33), (3.35), (3.36), we get:

$$\int_{B_r} \left|D^2u(x)\right|^p dx \leq c \int_{B_{\lambda r}} g^2(x) dx + \frac{1}{2} \int_{B_{\lambda r}} \left|D^2u(x)\right|^p dx$$
$$+ c \left(1 + \frac{1}{r^2(\lambda-1)^2} + \frac{1}{r^{\frac{p}{p-1}} (\lambda-1)^{\frac{p}{p-1}}} \right) \int_{B_{\lambda r}} \left(\mu^2 + |Du(x)|^2 \right)^{\frac{p}{2}} dx. \tag{3.37}$$

Since (3.37) holds for all r and for all $\lambda \in (1,2)$, setting $\rho = r$, $R_0 = \lambda r$, $\gamma = \frac{p}{p-1}$ and

$$h(\rho) = \int_{B_\rho} \left|D^2u(x)\right|^p dx,$$

by Lemma 2.4, we have

$$\|D^2u\|_{L^p(B_r)} \leq c(\alpha, p, n, N, \|g\|_{L^q(B_R)}) \left(\|Du\|_{L^p(B_R)} + 1 \right). \tag{3.38}$$

A standard covering argument yields the conclusion.

Case II $\left(q > \frac{2n}{p} \right)$. For $q > \frac{2n}{p}$ we have $p < \frac{pq}{q-2} < \frac{np}{n-p}$, and setting $\theta := \frac{2n}{pq} < 1$, we have

$$\frac{q-2}{pq} = \frac{1-\theta}{p} + \frac{\theta(n-p)}{np},$$

and we can use the interpolation inequality to estimate the last integral in (3.33) as follows

$$\left(\int_{B_{\lambda r}} |Du(x)|^{\frac{pq}{q-2}} dx \right)^{\frac{q-2}{q}} \leq \left[\left(\int_{B_{\lambda r}} |Du(x)|^p dx \right)^{\frac{1-\theta}{p}} \right.$$

$$\cdot \left(\int_{B_{\lambda r}} |Du(x)|^{\frac{np}{n-p}} dx \right)^{\frac{\theta(n-p)}{np}} \Bigg]^p, \tag{3.39}$$

and by (3.33), recalling the definition of θ, we get

$$\int_{B_{\lambda r}} g^2(x)|Du(x)|^p dx \le \left(\int_{B_{\lambda r}} g^q(x) dx \right)^{\frac{2}{q}} \left(\int_{B_{\lambda r}} |Du(x)|^p dx \right)^{\frac{pq-2n}{pq}}$$
$$\cdot \left(\int_{B_{\lambda r}} |Du(x)|^{\frac{np}{n-p}} dx \right)^{\frac{2(n-p)}{pq}}. \tag{3.40}$$

Then, by Sobolev's embedding Theorem, we have

$$\int_{B_{\lambda r}} g^2(x)|Du(x)|^p dx \le \left(\int_{B_{\lambda r}} g^q(x) dx \right)^{\frac{2}{q}} \left(\int_{B_{\lambda r}} |Du(x)|^p dx \right)^{\frac{pq-2n}{pq}}$$
$$\cdot \left(\int_{B_{\lambda r}} \left(|Du^2(x)|^p + |Du(x)|^p \right) dx \right)^{\frac{2n}{pq}}. \tag{3.41}$$

Now, since $q > \frac{2n}{p}$, we can use Young's inequality with exponents $\left(\frac{pq}{pq-2n}, \frac{pq}{2n} \right)$, thus getting, for every $\varepsilon > 0$,

$$\int_{B_{\lambda r}} g^2(x)|Du(x)|^p dx \le c \left(\int_{B_{\lambda r}} g^q(x) dx \right)^{\frac{2}{q}} \left[c_\varepsilon \int_{B_{\lambda r}} |Du(x)|^p dx \right.$$
$$\left. + \varepsilon \int_{B_{\lambda r}} \left(|D^2u(x)|^p + |Du(x)|^p \right) dx \right]. \tag{3.42}$$

Now we choose ε such that

$$\varepsilon \cdot \left[c \left(\int_{B_R} g^q(x) dx \right)^{\frac{2}{q}} \right] < \frac{1}{2}, \tag{3.43}$$

so that we can obtain the estimate (3.37) again, and apply Lemma 2.4 in the same way, thus getting (3.38) in this case too.

We remark that, differently from the previous case, when $q > \frac{2n}{p}$, we don't need to use a covering argument to conclude. In fact, in (3.43), we just choose a suitable value of ε, which depends on the norm of g in $L^q(B_R)$, while the radius of the ball on which the integral in the left hand side is taken does not depend on the L^q-norm of g: here, differently from (3.36), we don't use the absolute continuity of the integral. □

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