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Revisiting Aspects of Visualization in Mathematics Education

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La matematica richiede anzitutto immaginazione e interesse per vedere direttamente i problemi *e allora è istruttiva e anche divertente.*
(de Finetti L 1967, p. 1)

Words can "cite," but never "sight" their objects.
(Mitchell 1994, p. 152)

1. – Introduction

Bruno de Finetti's prescient contribution to mathematics and mathematics education regarding visualization is notable. This chapter is intended as an homage to his pioneering ideas. In his lovely book *Il "saper vedere" in Matematica* (de Finetti L 1967), among other things, de Finetti describes and exemplifies four main features of visualizing in mathematics:

- How to see easy things ("Saper vedere le cose facili", p. 8)
- How to see concrete things ("Saper vedere le cose concrete", p. 12)
- How to exploit dynamic vision ("Sfruttare una visione dinamica", p. 30)
- How to exploit global vision ("Sfruttare una visione globale", p. 35)

These are indeed basic premises which led the foundation of the subsequent work on what visualization may be in mathematics education and how can be integrated into its teaching and learning. The easiness and the concreteness enable us to envision abstract ideas and strategies, and the globality and the dynamism of images enable us to perceive and comprehend mathematical nuances. The visionary (double entendre intended) work of Bruno de Finetti re-emerged with full vigor more than two decades after he published his book.

In the last two decades, visualization in the teaching and learning of mathematics became a very central and active area of study. Only recently an entire issue of the *International Journal on Mathematics Education* (ZDM, Rivera et al. 2014) was devoted to the latest developments on visualization. A search of "visualization in mathematics education" in Google Scholar, restricted to 2014, yielded (in February 2015) almost 16,000 results (books, articles, reports). About the same number of entries can be found by searching overarching and important topics such as "socio cultural theories in mathematics education". In comparison, a similar search of "misconceptions in mathematics education" – a very heartfelt topic just a few decades ago – yielded only about half that number of entries.

The recent growing interest in visualization is possibly due to the inherent attractiveness of the topic to mathematicians, mathematics educators and textbook and software designers. It can be also attributed to the well-known truism that we live in a world dominated by the visual, to the point that "the potential for 'visual culture' to displace 'print culture' is an idea with implications as profound as the shift from oral culture to print culture." (Kirrane 1992, p. 58).

In the mathematics education community, the growing interest can also be attributed to the many still open research questions. Presmeg (2014, p. 151) lists thirteen "significant questions" and claims that "Many of the questions ... identified ... are still in need of investigation". She also stresses the need for addressing "newer questions that inevitably emerge, starting with – but not confined to – those I have suggested." (Presmeg 2014, p. 156).

This chapter is intended as a modest contribution to visualization, following up on previous work (Arcavi 2000) and addressing (at least partially) four of the questions from Presmeg's list:

– "What aspects of pedagogy are significant in promoting the strengths and obviating the difficulties of use of visualization in learning mathematics?" (Question 1)

- “What aspects of the use of different types of imagery and visualization are effective in mathematical problem solving at various levels?” (Question 3)
- “What conversion processes are involved in moving flexibly amongst various mathematical registers, including those of visual nature, thus combating the phenomenon of compartmentalization?” (Question 5)
- “How may be visualization be harnessed to promote mathematical abstraction and generalization?” (Question 10)

2. – A definition of visualization

On the basis of many studies, we have proposed in the past the following definition: “Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings.” (Arcavi 2000)

Given the complexity of this definition, some parsing of it may be in place. As in previous exercises that we undertook in order to make sense of multi-layered concepts such as variable (see, Arcavi & Schoenfeld, 1987), we can start by constraining ourselves to just one word to complete the following sentence: Visualization is ________. Such word should capture the essence of the term. According to the definition proposed above, we have three candidates for the one word: ‘ability’, ‘process’ and ‘product’. Still these three descriptors require, as shown above, to be completed by ‘of what’?, and they are: creation, interpretation, use and reflection upon pictures, images, diagrams. The definition also includes an answer to a “where?”: in our minds, on paper or with technological tools. Finally, the definition also includes purpose and goals: communication of information, thought, learning and understanding (whatever theoretical perspective is taken to define these loaded terms). At this point, it might be useful to apply some visual means to our proposed definition of visualization. Figure 1 is offered as a visually convenient arrangement of the verbal statements (a "diagram on paper") intended to help perceive and better communicate both the components of our proposed definition and the interconnections therein. It also reflects the parsing proposed above:

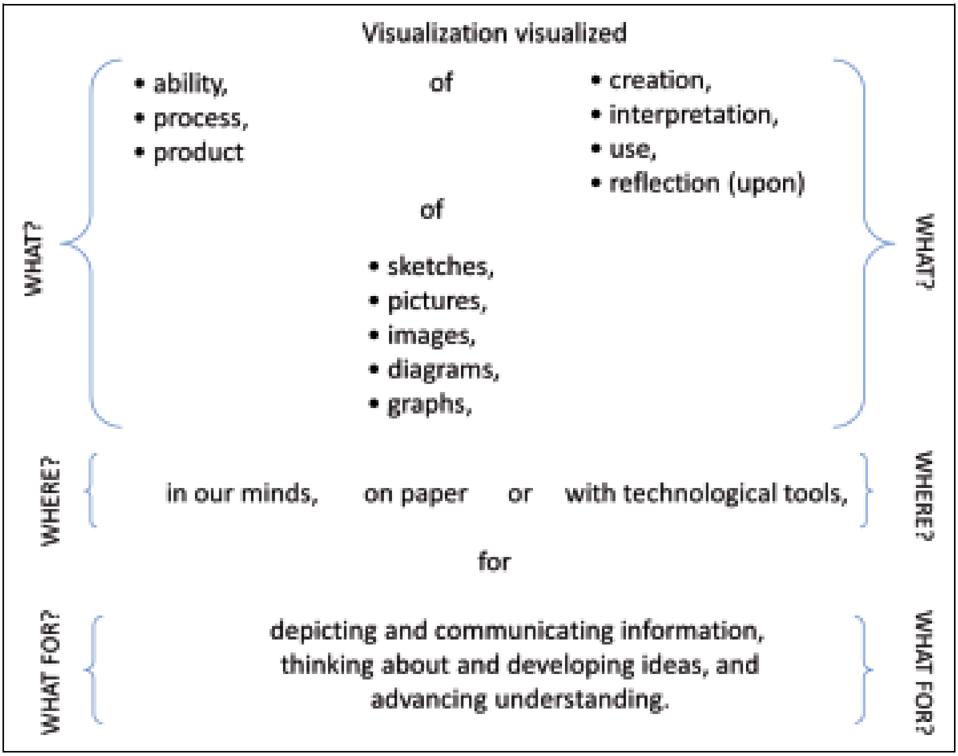

Figure 1. A visual display of the proposed definition of visualization

Following on this definition, we concentrate on some aspects of pedagogy, use and conversion process (to use Presmeg's formulations). Specifically, we address three themes:

– visualization and sense making,
– visualization and systematicity, and
– visualization and proof.

These issues will be discussed on the shoulders of some illustrative examples, and are proposed as the basis for further developments and examination.

3. – Visualization and sense making

As human beings, we sometimes have the feeling or the intellectual experience that leads us to say: ‘this makes sense to me’ or alternatively ‘this does not make sense to me’ (or paraphrasing de Finetti’s: this is easy to me, this is concrete to me). What does this saying entail? It may refer to what some describe as an ‘aha! moment’, an insight, in which we *feel* how pieces fall into place, how ideas suddenly cohere and connect to each other. It may consist of the impression that something resonates with what we think, and is aligned with previous experiences and understandings. It may be reflected by our ability to explain something to ourselves or to others in a way that satisfies our inner intellectual demands, dispelling ‘haziness’ and uncertainty, and producing intellectual satisfaction. Making sense may also imply the perception or the recognition of something through the senses or through the intellect, regarding it as reasonable, plausible, akin to what can be expected, and producing a sense of meaningfulness for oneself. One of the main challenges of mathematics education is to provide repeated opportunities for students to participate in experiences which support the development of this inner feeling, and not only to engage in the reproduction (for themselves or for others) of ideas and strategies that are expected of them.

Sense making may involve some of our five senses, especially vision. It is perhaps in that respect that the English expression “I see” is sometimes also a synonym of “I understand”, rather than the mere act of sensorial vision, but possibly intricately connected to it.

Consider the following example from elementary arithmetic, to calculate:

$$\frac{1}{9}+\frac{1}{8}\cdot\frac{1}{9}\,.$$

This is a simple exercise which can be solved in several ways, for example:

$$\frac{1}{9}+\frac{1}{8}\cdot\frac{1}{9}=\frac{1}{9}+\frac{1}{72}=\frac{8+1}{72}=\frac{1}{8}\,.$$

Or,

$$\frac{1}{9}+\frac{1}{8}\cdot\frac{1}{9}=\frac{1}{9}\left(1+\frac{1}{8}\right)=\frac{1}{9}\cdot\frac{9}{8}=\frac{1}{8}.$$

Some sophisticated savant suggested replacing the multiplication by a subtraction and that yields the result directly as follows:

$$\frac{1}{9}+\frac{1}{8}\cdot\frac{1}{9}=\frac{1}{9}+\frac{1}{8}-\frac{1}{9}=\frac{1}{8}.$$

This substitution is warranted once we know that the product of two unit fractions with consecutive denominators is equal to their positive difference.

So far we have shown different arithmetic solutions. There is a very interesting visual solution which appears in Smudge (1999, p. 8) without any further explanation, as shown in Figure 2.

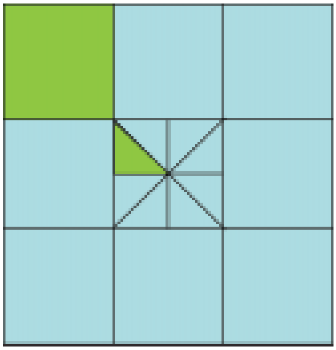

Figure 2. A visualization of $\frac{1}{9}+\frac{1}{8}\cdot\frac{1}{9}=\frac{1}{8}$

As is the case with other visual solutions, this diagram may require some consideration before it 'makes sense'. The large square is divided into nine equal squares, thus each of them constitutes a ninth of the whole. The central small square is subdivided into eight congruent triangles, thus each of them constitutes an eighth of it. If we take the larger square as the unit, then the triangular subdivision of the central small square constitutes an eighth of a ninth of the large square.

Therefore, the painted area (see Figure 3) is a geometrical representation of the expression $\frac{1}{9}+\frac{1}{8}\cdot\frac{1}{9}$.

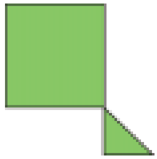

Figure 3. A visualization of $\frac{1}{9}+\frac{1}{8}\cdot\frac{1}{9}$

If one observes carefully in order to make sense of the representation, one realizes that the whole square is made up of eight combinations of one small square and a small triangle attached to it. Thus, $\frac{1}{9}+\frac{1}{8}\cdot\frac{1}{9}=\frac{1}{8}$.

It can be claimed that the graphical display 'makes more sense' than a concatenation of operations since it visually connects the operation to the meaning of a fraction as part of a whole and provides a global insight of what is being calculated and how. Moreover, depending on how we look, the visual diagram offers a representation of either the left hand side of the equality or its right hand side (or both simultaneously): if one looks at the small square and the attached triangle as two separate figures, we see the addition of $\frac{1}{9}+\frac{1}{8}\cdot\frac{1}{9}$ ($\frac{1}{9}$ being the small square and $\frac{1}{8}\cdot\frac{1}{9}$ the small triangle). However, if we look (globally) at the two figures as a single 'unit', one sees the eighth of the large square, and thus the figure highlights the result of the calculation, namely, $\frac{1}{8}$. In other words, the figure is reasonable and resonates with our inner feeling of understanding because it represents both the calculation (a process), and its result (a product) and thus is also constitutes an explanation of it. In this case Visualization is a mediator, a catalyzer, and a facilitator for sense making.

At this point, it might be fair to point out a limitation of visualization as it emerges from this particular example. Our aim in mathematics is generalization, can we make the case that $\frac{1}{n+1}+\frac{1}{n}\cdot\frac{1}{n+1}=\frac{1}{n}$ for any integer n that is the square of an odd number (recall that the square in the visualization had a central small square)? As in the previous case, the sense making intended can be further nurtured by comparing and contrasting with other representations. Such comparisons may highlight not only how algebra lends itself well to generalizations, but also may inspire us to look for visualization of this property for numbers that are not necessarily squares of odd numbers. For example, it may inspire to create a representation for $\frac{1}{3}+\frac{1}{2}\cdot\frac{1}{3}=\frac{1}{2}$, as follows (see Figure 4) as well as for other cases.

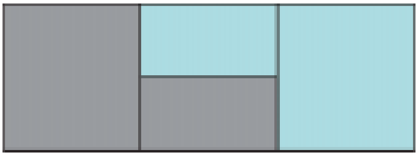

Figure 4. A visualization of $\frac{1}{3}+\frac{1}{2}\cdot\frac{1}{3}=\frac{1}{2}$

4. – Visualization and systematicity

Much has been said about the limitations of visualization. For example, consider the rough sketches of the graphs of the two functions $y = log_{\frac{1}{16}}x$ and $y = (\frac{1}{16})^x$ (Eisenberg 2000) as shown in Figure 5.

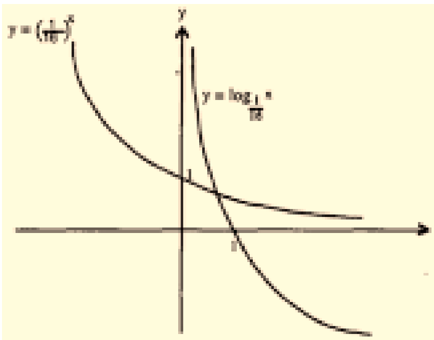

Figure 5. Sketch of the graphs of $y = log_{\frac{1}{16}}x$ and $y = (\frac{1}{16})^x$

According to the sketched graphs, there is one solution for the equation $log_{\frac{1}{16}}x = (\frac{1}{16})^x$. However, this equation has three solutions!

Some of us may find difficult to visualize two concave curves of this type intersecting three times. Graphing these particular functions may not be helpful due to scaling problems. However, changing the basis of the logarithms to 0.01, one can produce a satisfactorily visually convincing image (Figure 6), showing the relative positions of the two graphs, and thus illustrating how two concave curves can intersect in three points.

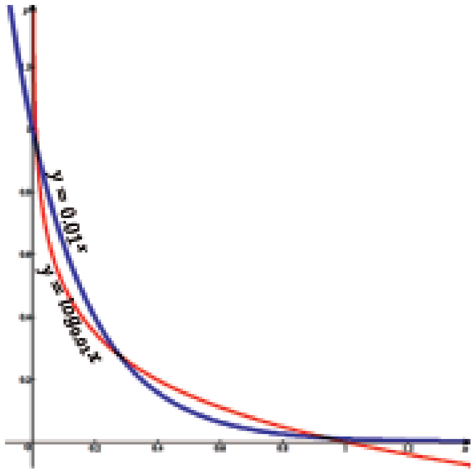

Figure 6. Sketches of two concave graphs with three intersections

In this case, the moral that "visualization graphs can sometimes be misleading" (Eisenberg 2000, p. 100) can be contested: comparing and contrasting different representations (or different sources for the same piece of knowledge) should serve as a monitor for the ways we make use of visual images. In this case, the initial misleading conclusion from the rough sketch was amended by such monitoring which in turn led us to look for a graph that is more visually enlightening and convincing.

We would like to claim also that sometimes, visualization tools can provide us with a systematic means to tackle and solve a problem. Consider, for example, the following version of the three jugs problem (one of its first sources is Coxeter & Greitzer 1967, p. 89).

The problem presents us with three jugs with the capacity of holding 8, 5 and 3 liters respectively. At the beginning, the largest jug is full and the other two are empty, and we are requested to end up with two jugs holding exactly 4 liters each without using any measuring instrument, just by pouring liquid between the jugs.

Once we realize that the only measuring instruments are the jugs themselves, we are limited to two "operations": to empty a jug and/or to fill it. We assume that no liquid is wasted during the pouring, thus, at any time of the pouring process, the sum of the volumes contained by the three jugs should be 8.

This problem can be solved by trial and error. Table 1 describes an example of successive pouring steps which solve the problem (each line in the table shows the amount held by each of the jugs, and by looking at successive steps one can deduce what was poured from where to where).

Table 1. Steps of the solution for the three jugs problem

8-liter jug	5-liter jug	3-liter jug
8	0	0
5	0	3
5	3	0
2	3	3
2	5	1
7	0	1
7	1	0
4	1	3
4	4	0

This is indeed a satisfactory solution. However, is it the only one? Is it the shortest one? Is there a "method" to solve the problem other than by trial and error? A visual display of information that helps answer this question in a systematic way is called a ternary diagram, a barycentric diagram or a de Finetti diagram, in honor of its proposer. This diagram consists of an equilateral triangle of height 8 which serves as a "coordinate system" as shown in Figure 7:

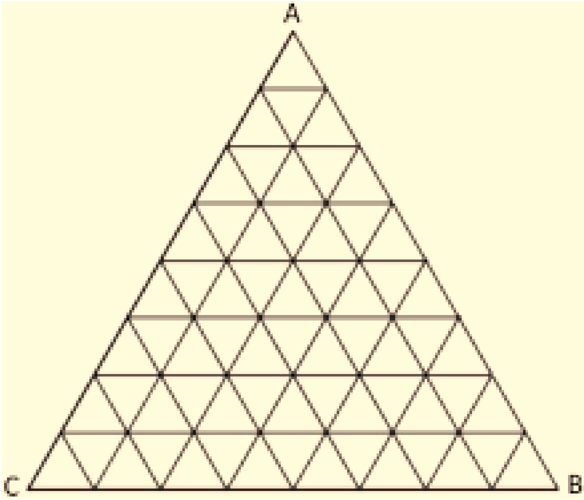

Figure 7. de Finetti diagram

The distances from a point to the sides of the triangle will constitute its "coordinates", as follows: the distance of a point to the sides AB, AC and CB will be the first, second and third coordinate respectively. Thus the triangular coordinates of points A, B and C are (0, 0, 8), (0, 8, 0) and (8, 0, 0) respectively. Also the coordinates of points H are (2, 1, 5) and those of point G are (4, 4, 0) as shown in Figure 8:

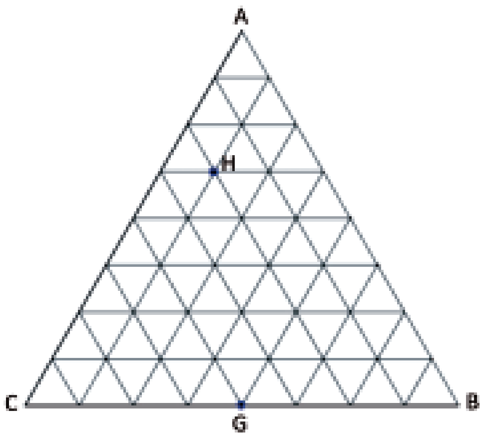

Figure 8. H(2,1,5) and G(4,4,0) plotted in a de Finetti diagram

Viviani's Theorem states that in an equilateral triangle, the sum of the distances from any interior point to the sides is equal to the length of the triangle's altitude. Thus, in our case, the sum of the three coordinates will always be 8, which makes this an appropriate precondition to represent different steps of the pouring of the jugs: the distance from AB will represent the amount of liquid in the 8-liter jug, the distance from AC the amount in the 5-liter jug and the distance from CB the amount in the 3-liter jug.

The initial stage of the jugs in the problem can be then represented by point C, and the final stage by point G(4, 4, 0).

A few observations are in place:

– Given the capacity of the jugs, the domain of our problem (i.e. the points whose three coordinates represent a possible situation for the three jugs) is not the whole triangle ABC (only one jug has the capacity of 8 liters) but the parallelogram highlighted in Figure 9.

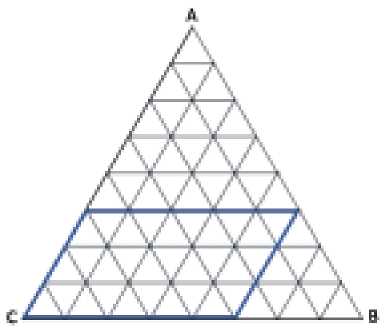

Figure 9. The domain of representation of the three jugs problem

– Any action of pouring leaves the content of one of the jugs unchanged, whereas it brings one of the other two jugs to its maximum or minimum capacity (full or empty). Thus only points on the perimeter of the parallelogram represent legitimate states. The change in the volume of liquid is represented in the diagram by moving along a "coordinate line", that is a line parallel to one of the sides of the triangle (because it leaves unchanged the contents of one jug), until we reach the border of the domain.

A solution would consist of departing from C, towards G, according to the two rules above. The solution found above by trial and error can be described as the following steps in the diagram (Figure 10):

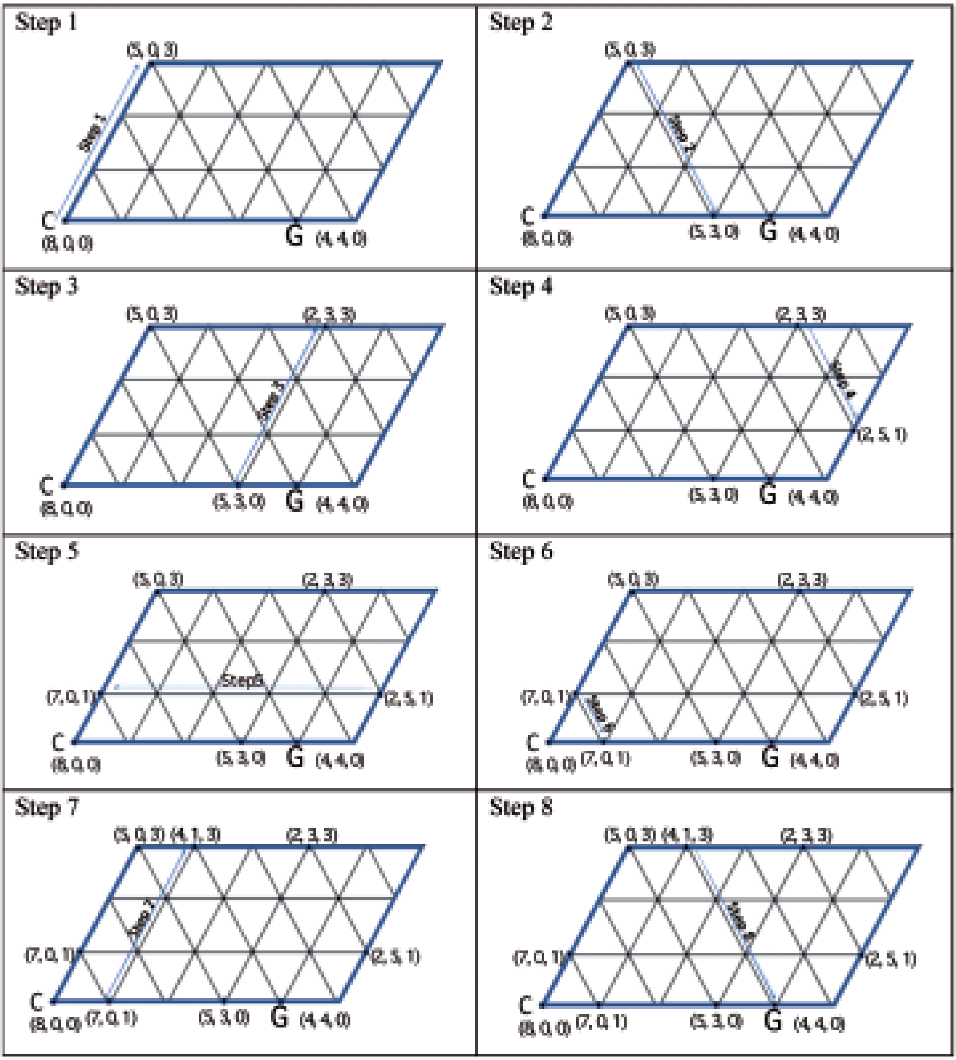

Figure 10. Steps in the systematic solution of the three jugs problem

The de Finetti diagram supports a systematic method for solving this and similar problems. It is easy to see, at any stage, what are possible moves and where is the goal. Also, it is immediately obvious when a move takes you back to a situation you have already been in. By this method one can find other solutions to the same problem, compare the number of steps, and make sure one exhausts the possibilities. Thus, in some cases, visual methods can and do achieve the generality and the systematicity we seek in mathematics.

5. – Visualization and Proof

“The mathematicians insisted that proofs are crucial to ensure that a result is true. The high school teachers demurred, pointing out that students no longer considered traditional, axiomatic proofs to be as convincing as, say, visual arguments.” (Horgan 1993, p. 103)

In a similar vein, Hanna (1990) argues for stressing explanatory proofs, whose main goal is not only to allow students to follow the deductive links in a chain, but also to provide insight, to support understanding and to make connections to previous experiences.

Much has been investigated about the teaching and learning of proof - see, for example, just two seminal undertakings: a special issue of ZDM (2008) and the 19th ICMI Study (Hanna & de Villiers 2012). In particular, the link between proofs and visualization has been studied and many beautiful examples have been put forward in a series entitled “proofs without words” (e.g. Nelsen 1993 and 2000).

The following is such a proof taken from Alsina and Nelsen (2006, pp. 39-40):

To prove that $S = |S_x - S_y|$ in the following Figure 11.

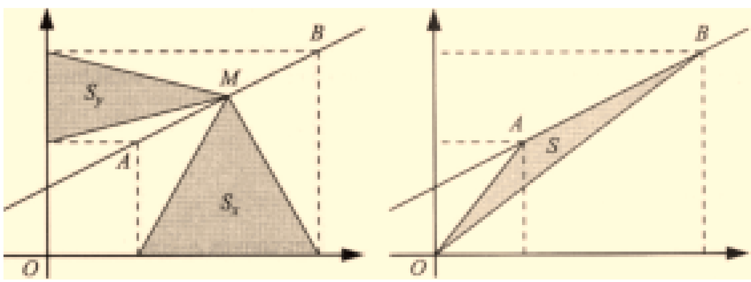

Figure 11. Statement of the theorem to be visually proven

The proof step by step is shown in Figure 12:

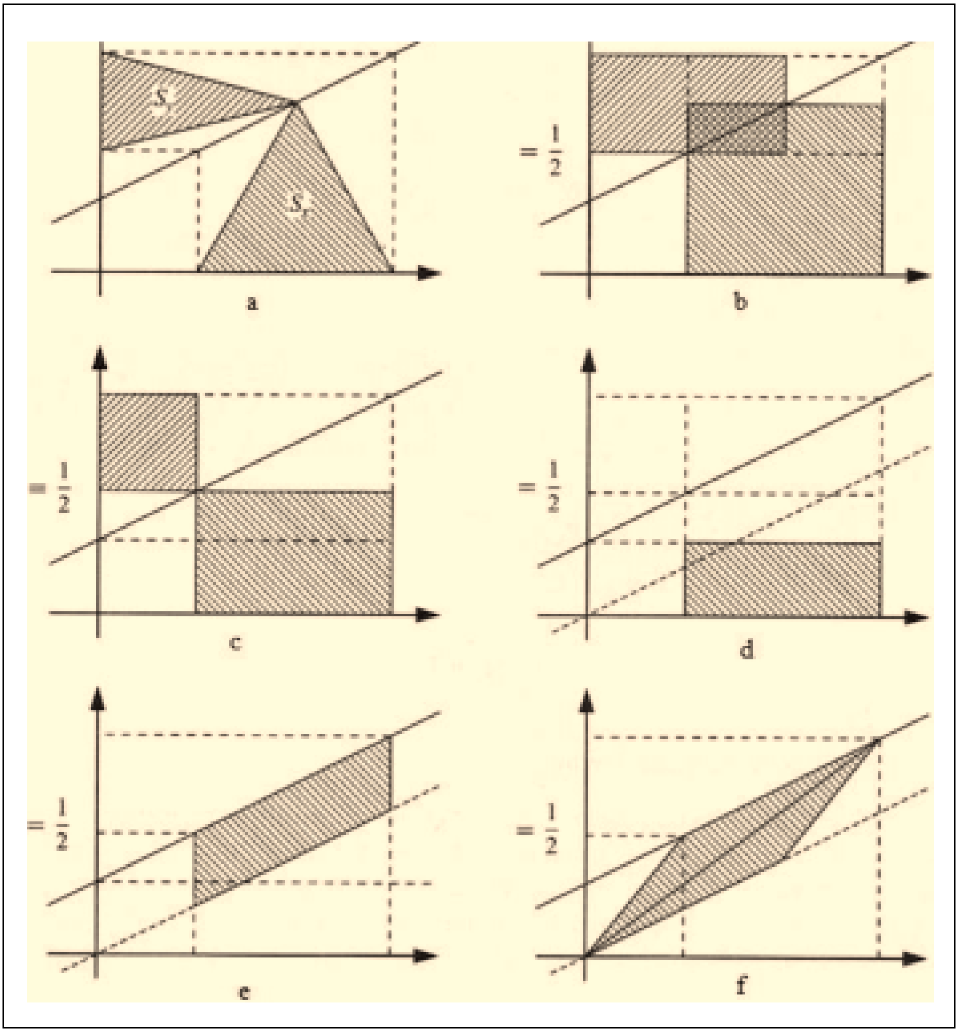

Figure 12. The visual proof step by step

This visual proof is based on the following:

From a to b: The areas of the rectangles are twice the areas of the triangles.

From b to c: The difference of areas is unchanged when we subtract equal areas from the two rectangles.

From *c* to *d*: We again subtract equal areas as follows: The given straight line bisects three rectangles into pairs of congruent right angle triangles. The shaded rectangle above the straight line is made up of the large triangle less the two small triangles. Similarly, the shaded rectangle below the straight line has the same area – it too is made up of the same large triangle less the two smaller ones.

From *d* to *e*: The rectangle has the same area as the parallelogram.

Indeed for many students, the visual transformation can be not only part of a deductive chain but also an explanatory sequence of steps that convincingly leads from the premise to the statement to be proven.

There is much to be discussed here from both the mathematical and the pedagogical points of view. However, I would like to call the attention to what may be considered an inaccurate description of these types of proofs when it is said of them that they are "without words". As an exercise, we invite the reader to carefully follow step by step the above proof and to notice the amount of verbal description (see above) needed (out loud or in our minds) in order to follow it.

Thus, in order to profit and learn from the potential explanatory power of visual proofs, they should be accompanied by verbal descriptions, comments, and arguments that make clear what is visually provided. We claim that in many cases, it is only during or after the verbal explanations are explicitly developed, that the global and convincing nature of these proofs becomes apparent and makes sense.

The complementarity between visually powerful displays of mathematical ideas and the verbal description or arguments produced to accompany them has been noticed and advocated by researchers (Zaskis et al. 1996, Mudaly 2010). This also became very apparent to us when working with visually impaired students. We concluded then that: "Verbal explicitness may avoid the too common phenomenon of two persons looking at the same object and seeing different things while being completely unaware of that" (Figueiras and Arcavi 2014).

In advocating the blending of the visual and the verbal ways of communication and of doing mathematics, we borrow an interesting idea from the field of the arts: *ekphrasis.* Simply stated, this construct refers to the verbal representation of a visual representation, which may add not only explicitness but also "rhetorical vividness" to what an image may depict.

6. – Conclusion

In this paper, we attempted to focus on several aspects of visualization which are still the object of widespread interest and study.

In particular we proposed tentative partial answers to some of Presmeg's questions through morals from rather simple examples, as follows.

Promoting the pedagogical strengths of visualization should first and foremost include a thorough search of enlightening examples which relate and are relevant to the classroom curriculum and which should involve alternative ways to approach concepts, problems and thinking strategies. The examples we brought are just a few illustrations which among other things should involve and stimulate students invocation of their common sense (the fraction calculation), should refine the visualization of what at a first glance can be considered as hard to visualize (two concave curves that cross each other three times), should present a systematic way to solve a problem (de Finetti diagrams) and should integrally blend the strengths of other representations (e.g. verbalizations of the visual). This will in itself support the practice of changing registers as natural and rewarding in terms of the sense making they may nurture.

The examples above can also serve as springboards for generalizing. For example, one can attempt to visualize $\frac{1}{n}+\frac{1}{n-1}\cdot\frac{1}{n}=\frac{1}{n-1}$, for several values of n, or one may attempt to use de Finetti diagrams for other values of the original jug problem.

These proposals need to be accompanied by long term empirical follow ups, in which visualization and its many functions (including bypassing the potential limitations and difficulties) are thoroughly scrutinized from the early years of mathematics education onwards.

These studies should be enhanced by a prolific collection of enriching examples that we need to continue developing as the repertoire upon which the many still unanswered questions should be approached.

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