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Chuan-Chih Hsiung

On the congruence of hypersurfaces

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Geometria. — *On the congruence of hypersurfaces.* Nota (*) di CHUAN-CHIH HSIUNG, presentata (**) dal Socio B. SEGRE.

INTRODUCTION.—The purpose of this paper is to continue our former work [1] to derive a new integral formula for a pair of immersed compact hypersurfaces $\boldsymbol{x}(\mathrm{M})$, $\boldsymbol{x}^*(\mathrm{M})$ in a Euclidean space under a volume-preserving diffeomorphism. By using this integral formula further conditions are found for $\boldsymbol{x}(\mathrm{M})$, $\boldsymbol{x}^*(\mathrm{M})$ to have the same second fundamental form, and a combination of this result with a former one of ours [1] thus gives a new congruence theorem on immersed compact hypersurfaces in a Euclidean space.

In order to simplify the presentation of our work, as in [1] we shall consider tensor products of multivectors and exterior differential forms. Differentiation of multivectors will be taken in the sense of equation (2.3), and differentiation of exterior differential forms will be exterior differentiation; multiplication of matrices will be by the usual row-by-column law. Throughout this paper the range of all Latin indices is always from 1 to n inclusive.

1. LEMMAS.—Let V be a real vector space of dimension $n\,(\geqq 2)$, and G and H two bilinear real-valued functions over $\mathrm{V}\times\mathrm{V}$ so that G and H are completely determined by the values $g_{ik} = \mathrm{G}(\boldsymbol{e}_i, \boldsymbol{e}_k)$ and $h_{ik} = \mathrm{H}(\boldsymbol{e}_i, \boldsymbol{e}_k)$, $1 \leqq i, k \leqq n$, where $\boldsymbol{e}_1, \cdots \boldsymbol{e}_n$, form a basis of the space V. Under a change of basis

$$\boldsymbol{e}_i \rightarrow \boldsymbol{e}_i^* = t_i^k \boldsymbol{e}_k\,, \tag{1.1}$$

the matrices $\| g_{ik} \|$ and $\| h_{ik} \|$ are changed to $\mathrm{T}\| g_{ik} \|{}^t\mathrm{T}$ and $\mathrm{T}\| h_{ik} \|{}^t\mathrm{T}$ respectively, where the repeated index k indicates the summation over its range, $\mathrm{T} = \| t_i^k \|$, and ${}^t\mathrm{T}$ is the transpose of T. Consider the determinant

$$\det(g_{ik} + \lambda h_{ik}) = \det(g_{ik}) + n\lambda \mathrm{P}(g_{ik}, h_{ik}) + \cdots + \lambda^n \det(h_{ik})\,. \tag{1.2}$$

Since $\det(g_{ik} + \lambda h_{ik})$ will be multiplied by $(\det \mathrm{T})^2$ under the change (1.1) of basis, the ratio of any two coefficients in the polynomial in λ on the right side of equation (1.2) is independent of the choice of the basis $\boldsymbol{e}_1 \cdots \boldsymbol{e}_n$. In particular, if G is nonsingular, that is, if $\det(g_{ik}) \neq 0$, the quotient

$$\mathrm{H}_\mathrm{G} = \mathrm{P}(g_{ik}, h_{ik})/\det(g_{ik}) \tag{1.3}$$

depends only on G and H. For example, if $g_{ik} = \delta_{ik}$ ($= 1$ for $i = k$; $= 0$ for $i \neq k$), then $\mathrm{H}_\mathrm{G} = \sum_i h_{ii}/n$.

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Since the construction of H_G is linear in H, it can be generalized to a bilinear function $\mathbf{H}$ over $V \times V$ with values in a vector space W, and $\mathbf{H}_G$ is then an element in W. $\mathbf{H}_G$ can be called the contraction of $\mathbf{H}$ relative to G, or, in the language of tensors, is a vector in W constructed from a covariant tensor $\mathbf{H}$ of order two with values in W relative to a nonsingular covariant tensor G of order two.

The following two lemmas will be needed in the proof of our main theorem.

LEMMA 1.1.—*Let* V *be a real vector space of dimension* n ($\geqq 2$), *and* G *and* H *symmetric positive definite bilinear real-valued functions over* $V \times V$ *completely determined by the values* $g_{ik} = G(\mathbf{e}_i, \mathbf{e}_k)$ *and* $h_{ik} = H(\mathbf{e}_i, \mathbf{e}_k)$, $1 \leqq i, k \leqq n$, *where* $\mathbf{e}_1, \cdots, \mathbf{e}_n$ *form a basis of the space* V. *Denote*

$$g = \det(g_{ik}) \quad , \quad h = \det(h_{ik}) . \tag{1.4}$$

Then

$$H_G \geqq (h/g)^{1/n} , \tag{1.5}$$

where the equality holds when and only when $h_{ik} = \rho g_{ik}$ *for a certain* ρ.

This lemma is due to Gårding [2].

LEMMA 1.2.—*Let* $\boldsymbol{f}_1, \cdots, \boldsymbol{f}_n$ *be* n *mutually orthogonal unit vectors, and* $\xi_1, \cdots, \xi_n, \eta_i^1, \cdots, \eta_i^n$, $i = 1, \cdots, n$, *linear differential forms at a point* p *of a Riemannian manifold* M *of dimension* n ($\geqq 2$) *imbedded in a Euclidean space* E^{n+m} *of dimension* $n + m$ *for* $m \geqq 1$. *Denote*

$$\left\{ \begin{array}{lll} \boldsymbol{f} = \left\| \begin{array}{c} \boldsymbol{f}_1 \\ \vdots \\ \boldsymbol{f}_n \end{array} \right\| \quad , & \xi = \| \xi_1, \cdots, \xi_n \| , & \\ \boldsymbol{w} = \xi \boldsymbol{f} \quad , & \eta = \| \eta_i^k \| , & 1 \leqq i, k \leqq n . \end{array} \right. \tag{1.6}$$

Then at the point p *of the manifold* M

$$\eta \boldsymbol{f} \boldsymbol{w}^{n-1} - (n-1) \boldsymbol{f} \xi \eta \boldsymbol{f} \boldsymbol{w}^{n-2} = 0 . \tag{1.7}$$

This lemma is due to Chern and Hsiung; for its proof see [1, p. 283].

2. HYPERSURFACES IN EUCLIDEAN SPACE.—Let M be a C^2–Riemannian manifold of dimension n ($\geqq 2$), and consider an immersion $\boldsymbol{x}: M \rightarrow E^{n+1}$, that is, a C^2–mapping $\boldsymbol{x}$ of M into a Euclidean space E^{n+1} of dimension $n+1$, such that the induced linear mapping $\boldsymbol{x}_*$ on the tangent spaces of M is univalent everywhere. Then $\boldsymbol{x}(p)$, $p \in M$, is a vector in E^{n+1} and will be called the position vector of the hypersurface $\boldsymbol{x}(M)$. Let $\mathbf{e}_1, \cdots, \mathbf{e}_n$ be mutually orthogonal unit vectors in the tangent space of M at a point p so that they form a frame. Since the mapping $\boldsymbol{x}_*$ is univalent, we identify $\mathbf{e}_i$ with $\boldsymbol{x}_*(\mathbf{e}_i)$. Let $\omega^1 \cdots \omega^n$ be the coframe dual to $\mathbf{e}_1 \cdots \mathbf{e}_n$ so that the volume element of M is

$$dV = \omega^1 \wedge \cdots \wedge \omega^n . \tag{2.1}$$

Let e_{n+1} be the unit normal vector of $x(M)$ at $x(p)$, and introduce the matrices

$$\omega = \| \omega^1, \cdots, \omega^n \| \quad , \quad e = \left\| \begin{matrix} e_1 \\ \vdots \\ e_n \end{matrix} \right\| . \tag{2.2}$$

Then we have the matrix equations

$$dx = \omega e \quad , \quad de = \Omega e + \theta e_{n+1} , \tag{2.3}$$

where

$$\Omega = \| \omega_i^k \| \quad , \quad \theta = \left\| \begin{matrix} \omega_1^{n+1} \\ \vdots \\ \omega_n^{n+1} \end{matrix} \right\| , \qquad 1 \leqq i , k \leqq n , \tag{2.4}$$

ω_i^k, ω_i^{n+1} being linear differential forms in the bundle induced over M from the principal bundle of E^{n+1}. Exterior differentiation of equations (2.3) gives

$$d\omega = \omega \wedge \Omega \quad , \quad \omega \wedge \theta = 0 , \tag{2.5}$$

$$d\Omega = \Omega^2 + \theta \wedge \tilde{\Omega} \quad , \quad d\theta = \Omega \wedge \theta , \tag{2.6}$$

where $\tilde{\Omega}$ is defined by

$$\tilde{\Omega} = \| \omega_{n+1}^1 , \cdots , \omega_{n+1}^n \| , \tag{2.7}$$

$$de_{n+1} = \tilde{\Omega} e . \tag{2.8}$$

It is well-known that the matrix Ω gives the connection form of the induced Riemannian metric by x, and that the second fundamental form II(p) of the hypersurface $x(M)$ at the point $x(p)$ is the negative of the scalar product (dx , de_{n+1}) of the two vectors dx and de_{n+1} in the Euclidean space E^{n+1}. From equations (2.2), (2.3), (2.7), (2.8) it follows immediately that

$$(dx , de_{n+1}) = \sum_i \omega^i \omega_{n+1}^i , \tag{2.9}$$

where the multiplication of linear differential forms is commutative in the ordinary sense. In the explicit form the second equation of (2.5) can be written as

$$\omega^i \wedge \omega_i^{n+1} = 0 , \tag{2.10}$$

which enables us to put

$$\omega_i^{n+1} = b_{ij} \omega^j , \tag{2.11}$$

where b_{ij} is symmetric in i, j, that is,

$$b_{ij} = b_{ij} . \tag{2.12}$$

Since $(e_i , e_{n+1}) = 0$, $1 \leqq i \leqq n$, by exterior differentiation and equations (2.3), (2.4), (2.7), (2.8) we can easily have

$$\omega_{n+1}^i = - \omega_i^{n+1} . \tag{2.13}$$

Substituting equations (2.11), (2.13) in equation (2.9) we thus reduce the second fundamental form of the hypersurface $\boldsymbol{x}(\mathrm{M})$ at the point $\boldsymbol{x}(p)$ to the form

$$\mathrm{II} = b_{ij}\,\omega^i\,\omega^j. \tag{2.14}$$

On the other hand, using equations (2.3), (2.11), (2.14) we obtain the second differential of $\boldsymbol{x}$ in the ordinary sense:

$$d^2\boldsymbol{x} = (d\omega^i + \omega^k\,\omega^i_k)\,\boldsymbol{e}_i + \mathrm{II}\,\boldsymbol{e}_{n+1}, \tag{2.15}$$

from which it follows that

$$(\boldsymbol{e}_{n+1}, d^2\boldsymbol{x}) = \mathrm{II}. \tag{2.16}$$

Bu putting

$$(\boldsymbol{x}, \boldsymbol{e}_{n+1}) = y_{n+1}, \tag{2.17}$$

we have the one-rowed matrix

$$y_{n+1}\,{}'\theta = \| y_{n+1}\,b_{ij}\,\omega^j \|. \tag{2.18}$$

Let $\boldsymbol{x}^{\perp}(p)$ be the orthogonal projection of the position vector $\boldsymbol{x}(p)$ onto the direction along the normal vector e_{n+1} of $\boldsymbol{x}(\mathrm{M})$ at the point $\boldsymbol{x}(p)$. Then we have

$$\boldsymbol{x}^{\perp} = y_{n+1}\,\boldsymbol{e}_{n+1}, \tag{2.19}$$

and therefore the following quadratic differential form:

$$\mathrm{Q} = (\boldsymbol{x}^{\perp}, d^2\boldsymbol{x}) = y_{n+1}\,\mathrm{II}. \tag{2.20}$$

3. INTEGRAL FORMULAS.—Consider two immersions $\boldsymbol{x}, \boldsymbol{x}^*$ of a C^2–Riemannian manifold M of dimension n ($\geqq 2$) into a Euclidean space E^{n+1} and a diffeomorphism f as given by the commutative diagram

$$\begin{array}{ccc} \mathrm{M} & \xrightarrow{\ \boldsymbol{x}\ } & \boldsymbol{x}(\mathrm{M}) \subset \mathrm{E}^{n+1} \\ & {\scriptstyle \boldsymbol{x}^*}\searrow & \downarrow f \\ & & \boldsymbol{x}^*(\mathrm{M}) \subset \mathrm{E}^{n+1}. \end{array}$$

Then § 2 can be applied to the hypersurface $\boldsymbol{x}(\mathrm{M})$, and for the corresponding quantities and equations for the hypersurface $\boldsymbol{x}^*(\mathrm{M})$ we shall use the same symbols and numbers with a star respectively. Suppose that f is volume-preserving, that is, by definition it maps the volume element of one immersed hypersurface into that of the other. As a consequence of this definition f exists only if M is oriented, and f is then orientation-preserving. Now over the abstract manifold M there are two induced Riemannian metrics with the same volume element, namely, $(d\boldsymbol{x}(p), d\boldsymbol{x}(p))$ and

$$(d\boldsymbol{x}^*(p), d\boldsymbol{x}^*(p)) = (d(fo\boldsymbol{x})(p), d(fo\boldsymbol{x})(p)).$$

Thus the notion of frames $e_1 \cdots e_n$ having measure 1 and an orientation coherent with that of M has a sense in both metrics. At a point $p \in M$ any such frame can be obtained from a fixed one by a linear transformation of determinant 1. The condition for the frames $e_1 \cdots e_n$ to be of measure 1 is

$$(3.1) \qquad (e_1 \wedge \cdots \wedge e_n, e_1 \wedge \cdots \wedge e_n) = 1 .$$

Differentiating equation (3.1) and using the second equation of (2.3) we can easily obtain

$$(3.2) \qquad \operatorname{Tr} \Omega \equiv \sum_i \omega_i^i = 0 .$$

For an $(m \times n)$–matrix $a = \| a_{ij} \|$ and an $(n \times p)$–matrix $b = \| b_{ik} \|$, whose elements are vectors in the space E^{n+1}, we shall use the notation (a, b) to denote the matrix of real numbers given by

$$(3.3) \qquad (a, b) = \left\| \sum_j (a_{ij}, b_{jk}) \right\| .$$

In order to derive a new integral formula for two compact hypersurfaces $x(M)$, $x^*(M)$ under a volume-preserving diffeomorphism f, we have to construct some exterior differential forms globally defined over the manifold M. For this purpose we introduce the following matrices:

$$(3.4) \qquad G = (e, {}^te) = {}^tG = \| g_{ik} \| ,$$

$$(3.5) \qquad \Lambda = \| \omega_1, \cdots, \omega_n \| = \omega G ,$$

$$(3.6) \qquad v = {}^t\theta e \quad , \quad v^* = {}^t\theta^* e ,$$

$$(3.7) \qquad Y = (x, {}^te) = \| y_1, \cdots, y_n \| ,$$

$$(3.8) \qquad r = Ye ,$$

$$(3.9) \qquad \Psi = r v^{*\,n-1} = \psi e_1 \wedge \cdots \wedge e_n .$$

Since ψ is an exterior differential form of degree $n-1$ globally defined over M, for a compact manifold M Stokes's theorem gives

$$(3.10) \qquad \int_M d\psi = 0 .$$

Making use of equations (3.7), (2.3), (3.4), (3.5), (2.17) we obtain

$$(3.11) \qquad dY = (dx, {}^te) + (x, d\,{}^te) = \Lambda + Y\,{}^t\Omega + y_{n+1}\,{}^t\theta ,$$

from which and equation (3.8) it follows that

$$(3.12) \qquad dr = [\Lambda + Y(\Omega + {}^t\Omega) + y_{n+1}\,{}^t\theta]\, e + Y\theta\, e_{n+1} .$$

By equations (3.6), (2.3) and the second equation of (2.6)* a similar calculation gives

$$(3.13) \qquad dv^* = -\,{}^t\theta^*\,({}^t\Omega^* + \Omega)\, e - {}^t\theta^*\, \theta\, e_{n+1} .$$

Using equations (3.11), (3.12) and noting that

$$d(\boldsymbol{v}^{*n-1}) = (n-1)(d\boldsymbol{v}^*)\,\boldsymbol{v}^{*n-2}, \tag{3.14}$$

from equation (3.9) we have

$$d\Psi = [\Lambda + \mathrm{Y}({}'\Omega + \Omega) + y_{n+1}\,{}'\theta]\,\boldsymbol{e}\,\boldsymbol{v}^{*n-1} - (n-1)\,\boldsymbol{r}\,{}'\theta^*({}'\Omega^* + \Omega)\,\boldsymbol{e}\,\boldsymbol{v}^{*n-2} + \\ + \mathrm{Y}\theta\,\boldsymbol{e}_{n+1}\,\boldsymbol{v}^{*n-1} - (n-1)\,\boldsymbol{r}\,{}'\theta^*\,\theta\boldsymbol{e}_{n+1}\,\boldsymbol{v}^{*n-2}. \tag{3.15}$$

Since $d(\boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n)$ contains no term in $\boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n$ in consequence of equation (3.2), equating the terms in $\boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n$ on both sides of equation (3.15) we thus obtain

$$(d\psi)\,\boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n = (\Lambda + y_{n+1}\,{}'\theta)\,\boldsymbol{e}\boldsymbol{v}^{*n-1} + \\ + \mathrm{Y}\,[({}'\Omega + \Omega)\,\boldsymbol{e}\boldsymbol{v}^* - (n-1)\,\boldsymbol{e}\,{}'\theta^*({}'\Omega^* + \Omega)\,\boldsymbol{e}]\,\boldsymbol{v}^{*n-2}. \tag{3.16}$$

By putting $\boldsymbol{f} = \boldsymbol{e}$, $\xi = {}'\theta^*$, $\eta = \Omega$ and $\eta = {}'\Omega^*$ respectively in Lemma 1.2 of of § 1 we can reduce equation (3.16) to

$$(d\psi)\,\boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n = [\Lambda + y_{n+1}\,{}'\theta + \mathrm{Y}({}'\Omega - {}'\Omega^*)]\,\boldsymbol{e}\boldsymbol{v}^{*n-1}. \tag{3.17}$$

Thus the integral formula (3.10) implies that for a pair of compact immersed hypersurfaces $\boldsymbol{x}(\mathrm{M})$, $\boldsymbol{x}^*(\mathrm{M})$ in the space E^{n+1} under a volume-preserving diffeomorphism f, the integral, over the manifold M, of the coefficient of $\boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n$ on the right side of equation (3.17) is zero.

4. THEOREMS.—Let M and M* be two n-dimensional ($n \geqq 2$) C^2-Riemannian manifolds with fundamental tensors G and G* respectively, and $f: \mathrm{M} \to \mathrm{M}^*$ a C^2-mapping. Then on M there are two connections: the Levi-Civita connection defined by its Riemannian metric and the connection induced by the mapping f from the Levi-Civita connection of M*. The difference of these two connections is a tensor field Δ of contravariant order 1 and covariant order 2, and the construction in § 1 gives a vector field Δ_{G}. f is called an isometry, if $\mathrm{G}^* = \mathrm{G}$, and an almost isometry relative to G if $\Delta_{\mathrm{G}} = 0$. It is obvious that an isometry is also an almost isometry, since in this case $\Delta = 0$.

THEOREM 4.1.—*Let* M *be a* C^2*-Riemannian manifold of dimension* $n\,(\geqq 2)$, $\boldsymbol{x}, \boldsymbol{x}^* : \mathrm{M} \to \mathrm{E}^{n+1}$ *two immersed compact hypersurfaces, in a Euclidean space* E^{n+1} *of dimension* $n+1$, *with fundamental tensors* G, G* *and positive definite second fundamental forms* II, II*, *whose coefficient tensors being denoted by* B, B*, *respectively, and* $f : \boldsymbol{x}(\mathrm{M}) \to \boldsymbol{x}^*(\mathrm{M})$ *a volume-preserving diffeomorphism. Then* II $=$ II*, *if*

$$\det \mathrm{B} = \det \mathrm{B}^*, \tag{4.1}$$

$$\mathrm{G}_{\mathrm{B}^*} \geqq \mathrm{G}_{\mathrm{B}}, \tag{4.2}$$

and f *is an almost isometry relative to* B*, *that is,*

$$\Delta_{\mathrm{B}^*} = 0. \tag{4.3}$$

Proof.—In order to prove this theorem we have to find $d\psi$, which is the coefficient of $e_1 \wedge \cdots \wedge e_n$ on the right side of equation (3.17). For this purpose we observe that each term on the right side of equation (3.17) is of the same type as the form $\pi e v^{*n-1}$, where

$$\pi = \|\pi_1, \cdots, \pi_n\| \tag{4.4}$$

is a one-rowed matrix of linear differential forms. By using equations (2.2), (2.4)*, (3.6) we have

$$\pi e v^{*n-1} = \Big(\sum_i \pi_i e_i\Big)\Big(\sum_j \omega_j^{*n+1} e_j\Big)^{n-1} = \tag{4.5}$$

$$= (\Sigma\, \varepsilon_{i_1 \dots i_n} \pi_{i_1} \wedge \omega_{i_2}^{*n+1} \wedge \cdots \wedge \omega_{i_n}^{*n+1})\, e_1 \wedge \cdots \wedge e_n,$$

where $\varepsilon_{i_1 \dots i_n}$ is equal to $+1$ or -1 according as $i_1, \cdots, i_n$ form an even or odd permutation of $1, \cdots, n$, and is equal to zero otherwise, and the summation is extended over all $i_1, \cdots, i_n$ from 1 to n. Assuming

$$\pi_i = h_{ij}\, \omega^j, \tag{4.6}$$

$$H = h_{ij}, \qquad i, j = 1, \cdots, n, \tag{4.7}$$

and using equations (2.11)*, (2.1), (1.2), (1.3), (2.14)* and elementary properties of determinants, from equation (4.5) we can easily obtain

$$\pi e v^{*n-1} = n!\, H_{B^*} (\det B^*)\, (e_1 \wedge \cdots \wedge e_n)\, dV. \tag{4.8}$$

By putting $\pi = \Lambda$ and $\pi = y_{n+1}{}'\theta$ in equation (4.8), and recalling $\omega_i = g_{ij}\,\omega^j$ and equation (2.14), we therefore have

$$\Lambda e v^{*n-1} = n!\, G_{B^*} (\det B^*)\, (e_1 \wedge \cdots \wedge e_n)\, dV, \tag{4.9}$$

$$y_{n+1}{}'\theta\, e v^{*n-1} = n!\, y_{n+1}\, B_{B^*} (\det B^*)\, (e_1 \wedge \cdots \wedge e_n)\, dV. \tag{4.10}$$

Since $\omega = \omega^*$, from the first equations of (2.5), (2.5)* we have

$$\omega \wedge (\Omega - \Omega^*) = 0, \tag{4.11}$$

so that we can write

$$\omega_i^k - \omega_i^{*k} = a_{ij}^k\, \omega^j. \tag{4.12}$$

Equations (4.11), (4.12) imply

$$a_{ij}^k\, \omega^i \wedge \omega^j = 0, \tag{4.13}$$

which gives the symmetry of a_{ij}^k in the subscripts i, j, that is,

$$a_{ij}^k = a_{ij}^k. \tag{4.14}$$

From the properties of the forms Ω, Ω^* and the definition of the tensor Δ it follows that the components of Δ are a_{ij}^k. For each fixed k denote

$$\Delta^k = a_{ij}^k, \qquad i, j = 1, \cdots, n. \tag{4.15}$$

On the other hand, a use of equations (3.5), (3.5)*, (3.7), (4.12) yields readily the matrix

$$(4.16) \qquad Y({}^t\Omega - {}^t\Omega^*) = \| y_k \overset{k}{a}_{1j}, \cdots, y_k \overset{k}{a}_{nj}\, \omega^j \| .$$

By putting $H = y_k \Delta^k$ in equation (4.8) we obtain

$$(4.17) \qquad Y({}^t\Omega - {}^t\Omega^*)\, \boldsymbol{ev}^{*\,n-1} = n!\, y_k \overset{k}{\Delta}_{B^*} (\det B^*)\, (\boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n)\, dV,$$

since $\overset{k}{\Delta}_{B^*} = 0$ due to condition (4.3). Thus equations (4.17), (4.6), (4.7) reduce equation (3.17) to

$$(4.18) \qquad d\psi = n!\, (G_{B^*} + y_{n+1} B_{B^*})\, (\det B^*)\, dV.$$

Hence for a pair of compact immersed submanifolds $\boldsymbol{x}(M)$, $\boldsymbol{x}^*(M)$ under a volume-preserving diffeomorphism f with $\Delta_{B^*} = 0$ the integral formula (3.10) becomes

$$(4.19) \qquad \int_M (G_{B^*} + y_{n+1} B_{B^*})\, (\det B^*)\, dV = 0 .$$

In particular, when the two hypersurfaces $\boldsymbol{x}(M)$, $\boldsymbol{x}^*(M)$ are identical, by definition $B_{B^*} = 1$ and the formula (4.19) is reduced to

$$(4.20) \qquad \int_M (G_B + y_{n+1})\, (\det B)\, dV = 0 .$$

Subtracting equation (4.20) from equation (4.19) and noticing condition (4.1) we obtain

$$(4.21) \qquad \int_M [(G_{B^*} - G_B) + y_{n+1} (B_{B^*} - 1)]\, (\det B)\, dV = 0 .$$

Since by the assumption of the theorem the compact hypersurface $\boldsymbol{x}(M)$ has positive definite second fundamental form II, we can choose the common origin of the position vectors $\boldsymbol{x}(p)$ for all points $p \in M$ in the Euclidean space E^{n+1} to be in $\boldsymbol{x}(M)$ so that $y_{n+1} > 0$. Moreover, by Lemma 1.1 of § 1 we have

$$(4.22) \qquad B_{B^*} - 1 \geqq 0 ,$$

where the equality holds when and only when, for a certain ρ, $B = \rho B^*$, from which it follows that $II = II^*$ due to condition (4.1). Thus by condition (4.2) the integrand on the left side of equation (4.21) is nonnegative, and the validity of equation (4.21) gives immediately that the equality in (4.22) holds. Hence the theorem is proved.

By combining Theorem 4.1 and a former one of ours [1] and noticing equation (2.20) we are readily led to

THEOREM 4.2.—*Under the same assumptions as those in Theorem 4.1, the diffeomorphism f is a rigid motion, if the following further conditions are satisfied:*

$$B_{G^*} \geqq B_G \,, \tag{4.23}$$

and f is an almost isometry relative to G^*, *that is,*

$$\Delta_{G^*} = 0 \,. \tag{4.24}$$

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