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A new approach to the theory of algebraic numbers

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Logica matematica. — *A new approach to the theory of algebraic numbers.* Nota di ABRAHAM ROBINSON, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Nella teoria degli ideli e adeli, gli ideali di un anello di Dedekind D (per esempio di numeri algebrici o di funzioni algebriche) sono moltiplicativamente isomorfi, a meno di elementi associati, ad un sotto-insieme di un anello, Δ, che è un'estensione di D. L'idea fondamentale della teoria suaccennata fu concepita da Prüfer e sviluppata poi da von Neumann, Chevalley e altri. In questa Nota mostriamo come una teoria di questo tipo può venire rielaborata adoperando modelli non-standard definiti per mezzo di un linguaggio formalizzato.

1. In the theory of idèles and adèles (e.g. refs. [1], [3]), the ideals of a given Dedekind domain D, e.g. within the realm of algebraic numbers or of algebraic functions of one variable are shown to be multiplicatively isomorphic, up to associated elements, to a subset of a ring, Δ, which is an extension of D. In the special cases mentioned, the idea was first realized by Prüfer and, following him, by von Neumann, while the case of general Dedekind domains was treated by Krull (refs. [2], [4], [5]). We propose to show that the theory can be developed conveniently by the use of the notion of a non-standard enlargement of a given mathematical structure which has, in recent years, been applied extensively to Analysis (refs. [6]–[9]). In the present Note, we shall describe our method independently of the above mentioned approaches, leaving the study of the connection with them for a later paper. We also propose to show subsequently that our tools are sufficiently powerful to cope with infinite algebraic extensions of the rational number field and of rational function fields of one variable.

2. Let M be a mathematical structure. Let K be the set of sentences which are formulated in a higher order predicate language which comprises all finite types (or alternatively, which are formulated within a suitable set-theoretic framework) and which are true in M. Consider any binary relation $R(x, y)$ between individuals of M or between entities of other types (e.g. between individuals and sets). An entity will be said to be *in the domain of the first variable* of R if there exists an entity b such that $R(a, b)$ holds in M. $R(x, y)$ will be called *concurrent* if for any finite set of entities, $a_1, \cdots, a_n$ which are in the domain of the first variable of R, there exists an entity b such that $R(a_1, b), \cdots, R(a_n, b)$ all hold in M (and hence, belong to K). It is not difficult to establish the existence of *enlargements* of M; i.e. of extensions *M of M such that (i), M satisfies all sentences of K and (ii) for any concurrent

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relation $R(x,y)$ in M there exists an entity b_R in *M such that $R(a,b_R)$ holds in *M for all entities a which belong to the domain of the first variable of R *in* M. However, referring to (i), it is understood that we interpret the sentences of K in *M in non-standard fashion. That is to say, assertions concerning entities of D other than individuals (e.g. sets, relations between individuals, relations between relations, relations between sets) are to be interpreted (in general) *not* with regard to the totality of such entities in D but with regard to an appropriate sub-class of these, called *internal* (or *admissible*) entities. For example, if N is the structure of the natural numbers and *N is an enlargement of N, then the relation $x < y$, is concurrent. It follows that *N contains numbers which are *infinite*, i.e., larger that all numbers of N. However, it is easy to see that the set of all infinite numbers of *N (still to be called *natural*) does not contain a smallest element. It follows that this set is not internal. For any non empty set of natural numbers in N does possess a smallest element, and this is a fact which can be expressed within K and, accordingly, must be true in *N for all non-empty *internal* sets. On the other hand, the set of prime numbers in *N is an internal set in *N as can be seen by interpreting, in *N, a sentence of K which asserts the existence of this set (for N).

When dealing with a given mathematical theory, it is necessary to *enlarge* all the mathematical structures involved in the theory, simultaneously. This can be done by supposing that these structures have first been embedded in a single structure, M. Thus, when dealing with a ring R, we shall require not only an enlargement *R of R but also a simultaneous enlargement *N of the natural numbers N; for a discussion in ring theory may involve the powers of a ring element, which are given by a mapping from $R \times N$ into R.

3. Let D be a Dedekind domain. By a *proper ideal* we shall mean any ideal other than the zero-ideal, 0, or the entire ring. We shall suppose that D possesses at least one proper ideal, i.e., that it is not a field. By a *prime ideal* we shall mean a *proper* prime ideal. Every proper ideal in D possesses a unique representation as the product of powers of distinct prime ideals.

We embed D and the natural numbers, N, simultaneously in a structure M and consider an enlargement *M of M. M contains enlargements *D and *N of D and N respectively. *D is an integral domain, as can be seen by expressing the fact that D is an integral domain within K and reinterpreting in M. (In the particular case when D is a ring of algebraic numbers, N may be regarded as a substructure of D; and M may then be taken to coincide with D. Accordingly, *M will coincide with *D). For any $a \in D$, $a \neq 0$, and for any proper ideal J in D we write $n = \mathrm{ord}_J(a)$ for the uniquely determined natural number n such that $a \in J^n$ but $a \notin J^{n+1}$.

Let Ω be the set of ideals in D, and let $^*\Omega$ be the corresponding entity in *M. That is to say, $^*\Omega$ is denoted by the same symbol as Ω in the vocabulary of K. The elements of $^*\Omega$ are the *internal* ideals of *D. An internal ideal $A \in {}^*\Omega$ is *standard*, by definition, if there exists an ideal B in D, such that $A = {}^*B$, i.e. such that A is denoted by the same symbol as B in the vocabulary of K. The notation $n = \text{ord}_J(a)$ still has a meaning for all proper internal ideals in *D and for all $a \in {}^*D$, $a \neq 0$, but n may now be a finite *or infinite* natural number.

We define the monad μ of *D as the intersection of all proper standard ideals in *D. Thus $\mu = \bigcap_{\nu} {}^*J_{\nu}$, where J_{ν} varies over the proper ideals of D and $^*J_{\nu}$ is the corresponding ideal in *D in each case. μ is an ideal in D, since it is an intersection of ideals; but it can be shown that μ is not internal. An equivalent definition for μ is $\mu = \bigcap_{\nu} {}^*P_{\nu}^{n}$, where P_{ν} varies over the prime ideals of D and n varies over the finite natural numbers. The following facts can now be established without difficulty.

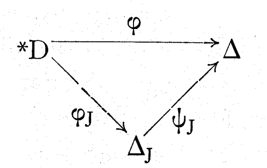
(i) μ does not contain any element of D other than zero. For if $a \in D$, $a \neq 0$, then there exists a prime ideal P in D such that $a \notin P^n$ for a sufficiently large finite n. Hence $a \notin {}^*P^n$, $a \notin \mu$.

(ii) Let B be any ideal in D. Then $a \in {}^*B \cap D$ if and only if $a \in B$. For the relation $a \in B$ can be expressed within K for any $a \in D$ and so, for such $a \in {}^*B$ if and only if $a \in B$.

(iii) There exists an internal proper ideal J in *D (i.e., $J \in {}^*\Omega$, $J \neq 0$, $J \neq {}^*D$) such that $J \subset \mu$.

For consider the relation $R(x, y)$ which is defined by the following expression: "x is a proper ideal, and y is a proper ideal which is included in x". One verifies that $R(x,y)$ is concurrent in D (and in M). Accordingly, by one of the basic properties of M, there exists an internal proper ideal J in *D such that $J \subset {}^*J_{\nu}$, for any proper ideal J_{ν} in D. Hence $J \subset \mu = \bigcap_{\nu} {}^*J_{\nu}$, as asserted.

4. Let $\Delta = {}^*D/\mu$, so that Δ is the quotient ring of *D with respect to μ and, for any internal proper ideal $J \subset \mu$ in *D, such as exists according to 2. (iii) above, let $\Delta_J = {}^*D/J$. Denote by φ, φ_J the canonical (homomorphic) mapping from *D onto Δ, Δ_J, respectively. For J as considered there are mappings ψ_J from Δ_J onto Δ with kernel $\varphi_J(\mu)$ and we have the commutative diagrams



By a standard results on Dedekind domains, the quotient ring of D with respect to a proper ideal is a principal ideal ring. Formulating this fact within K and reinterpreting in *M, we conclude that the *internal* ideals of any quotient ring of *D with respect to an *internal* ideal are principal. The internal ideals of $^*D/J$ are just the images $\varphi_J(A)$ of the internal ideals A of *D. Moreover, a homomorphic image of a principal ideal is principal. Hence, the images $\varphi(A) = \psi_J(\varphi_J(A))$ are principal ideals in Δ for all internal ideals A in *D. Conversely, if F is a principal ideal in Δ, $F = (a)$, say, let $a = \varphi(b)$, $b \in {}^*D$.

Then $F = \varphi((b))$, so that F is the image of an internal ideal. We conclude that an ideal in Δ is principal if and only if it is the image of an internal ideal in *D under φ.

Moreover, if A and B are ideals in D, then $\varphi((A, B)) = (\varphi(A), \varphi(B))$. It follows that the domain of principal ideals in Δ is closed under the operation of taking the greatest common divisor. We conclude that, if an ideal F in Δ has a finite base $F = (a_1, \cdots, a_n)$, then F is actually principal since $F = (\cdots(a_1), (a_2)), (a_3)), \cdots, (a_n))$. We conclude further that for any a and b in Δ, $a \neq 0$, $b \neq 0$, there exists a greatest common divisor d; i.e., d divides a and b, and any d' which divides a and b divides also d.

The basis of a principal ideal in Δ, $J = (a)$, where a is not a divisor of zero, is uniquely defined up to associated elements of Δ, i.e., up to multiplication by a unit of Δ. If A_1 and A_2 are distinct ideals of D, $A_1 \neq A_2$, then $^*A_1 \neq {}^*A_2$ and $\varphi(^*A_1) \neq \varphi(^*A_2)$ in view of 3, (ii) above. Thus, φ induces a one-to-one multiplicative mapping Φ from the proper ideals of D into the classes of associated elements of Δ.

The mapping φ is one-to-one on D: for, if $a \neq b$ for elements a and b of D, then $a - b \neq 0$ and so $\varphi(a - b) \neq 0$ by 2, (i) above, and hence $\varphi(a) - \varphi(b) \neq 0, \varphi(a) \neq \varphi(b)$. Thus, φ provides an injection of D into Δ; in other words, Δ may be regarded as an extension of D. Accordingly, Δ satisfies the requirements mentioned at the beginning of this Note. A more detailed description of the structure of Δ will be given in a subsequent paper (1).

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