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On the contractability criterion of Castelnuovo-Enriques

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Matematica. — *On the contractability criterion of Castelnuovo-Enriques.* Nota di ALEXANDRU LASCU, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Il classico criterio di cui al titolo, caratterizzante le curve irriducibili eccezionali di 1ª specie, viene qui esteso in geometria algebrica astratta e rispetto ad arbitrari morfismi birazionali.

1. The aim of this note is to extend to Abstract Algebraic Geometry the well known criterion of Castelnuovo-Enriques [1] which characterizes the exceptional irreducible curves of the first kind on an algebraic surface. Thus our criterion of [5] is extended to arbitrary birational morphisms and proofs are given here in full.

The problem has recently been considered by Мойшезон [6] for complex analytic varieties. Although the treatment here is quite independent of [6] and concerns abstract algebraic varieties, our result is similar to those of [6].

We shall consider abstract algebraic varieties over a universal domain Ω, of arbitrary characteristic, in the sense of Weil [8].

Definition.—Let Y, X′ be algebraic varieties and Y′ a sub-variety of X′. We shall say that Y′ *is regularly contractable to* Y *within* X′ if there exists a proper birational morphism $X' \xrightarrow{f} X$ which satisfies the following conditions:

1° Y′ is the closed subset of X′ formed by the points where f is not biregular;

2° $f(Y')$ is a subvariety of X isomorphic with Y;

3° each point of $f(Y')$ is simple on X.

We shall then say that $f: X' \to X$ is *a regular contraction* of Y′ to Y.

Note.—Identifying $f(Y')$ with Y we shall write $Y \subset X$. Condition 3° implies that $\dim Y' > \dim Y$, by Z.M.T. (Zariski's Main Theorem), since, in view of 1°, f is not biregular at Y.

THEOREM.—*Let* Y, Y′, X′ *be nonsingular algebraic varieties such that* Y′ *is a subvariety of* Y.

Y′ *is regularly contractable to* Y *within* X′ *if, and only if, the following conditions are satisfied:*

1° *there exists an algebraic vector bundle* E *of base* Y *of rank* $r > 1$ *and an isomorphism* $Y' \xrightarrow{h} P(E)$, *where* P(E) *is the projective bundle associated to* E;

2° $codim_{X'}\, Y' = 1$;

3° *if* l *is the canonical line bundle of* P(E), *then* $h^{-1}(l)$ *is equivalent with the normal bundle of* Y′ *in* X′.

Under these conditions, let $f: X' \to X$ *be a regular contraction of* Y′ *to* Y. *The couple* (Y, X) *is then uniquely determined by* (Y′, X′), *up to an isomor-*

(*) Nella seduta del 22 giugno 1966.

phism; f^{-1} *is a blowing up of* X, *of center* Y; *hence* f *is uniquely determined up to an isomorphism over* X; E *is isomorphic with the normal bundle of* Y *in* X.

Note.—Condition 1° characterizes Y′, while conditions 2° and 3° characterize the embedding of Y′ in X′. Condition 3° is an alternative form of Segre's intersection formula for dilatations [7].

The last part of the theorem concerns the uniqueness of contractions.

The proof of the theorem is an easy consequence of the sequence of lemmas below.

We shall use throughout the following notations: X, Y are non-singular algebraic varieties and Y is a subvariety of codimension greater than 1 of X, $q : \overline{X} \to X$ is such that q^{-1} is the blowing up of X of center Y, defined by the Ideal $\mathfrak{J}$ of Y in O_X, where O_X is the sheaf of local rings of X. It follows that $\overline{Y} = q^{-1}(Y) = P(E)$, where E is the normal bundle of Y in X and $q/\overline{Y} = p$ is the canonical projection of the projective bundle P (E) on Y. Note that q is characterized by the following universal property (similar to those considered in [3]): for every birational morphism $h : Z \to X$ such that $h^{-1}(Y)$ is a hypersurface T of Z, Z is normal and $h/Z \setminus T$ biregular, there exists a canonical birational morphism $k : Z \to \overline{X}$ such that $h = qk$.

2. LEMMA 1.—*Let* V *be a normal algebraic variety*, U *a hypersurface of* V *and* f: $V \to X$ *a birational morphism such that* f *maps biregularly* $V \setminus U$ *onto* $X \setminus Y$ *and, additionally*, $f(U) = Y$. *There exists then a canonical birational morphism* $g : V \to \overline{X}$ *such that* $f = qg$. *If* f *is complete over an open nonempty set of* Y, *then* $g(U)$ *contains an open set of* $\overline{Y}$. *If moreover* f *is proper, then* $g(U) = \overline{Y}$.

Proof.—In view of the universal property of q there exists a birational morphism $g : V \to \overline{X}$ such that $f = qg$. It follows $g(U) \subset \overline{Y}$. If f is complete over an open nonempty set G of Y, then g in complete over $q^{-1}(G)$ and so $q^{-1}(G) \subset g(V)$. As f maps biregularly $V \setminus U$ onto $X \setminus Y$, g maps biregularly $V \setminus U$ onto $\overline{X} \setminus \overline{Y}$; hence $q^{-1}(G) \subset g(U)$. If f is proper, then we have $G = Y$, i.e., $q^{-1}(G) = \overline{Y}$.

LEMMA 2.—*Let* U *be a simple subvariety of* V *and* $f : V \to X$ *a birational morphism which maps biregularly* $V \setminus U$ *onto* $X \setminus Y$ *and is such that* $f(U) = Y$. *If* $\dim U > \dim Y$, *then* $\operatorname{codim}_V U = 1$.

Proof.—Suppose $r = \operatorname{codim}_V U > 1$. As U is simple on V, there exists an open set G of V such that $U \cap G \neq \emptyset$ and $U \cap G = H_1 \cdots H_r$, where H_i are hypersurfaces of G $(1 \leq i \leq r)$. Owing to the local character of the problem we can replace V by G, U by $U \cap G$ and, consequently, X by $f(G)$ and Y by $f(U \cap G)$. We can therefore suppose $U = H_1 \cdots H_r$; $f(H_i) = K_i$ is a simple hypersurface of X, in view of the biregularity of f in $V \setminus U$. It follows: $Y = f(U) \subseteq K_1 \cap \cdots \cap K_r$. Hence there exists an irreducible component Z of $K_1 \cap \cdots \cap K_r$ such that $Y \subseteq Z$. But $\dim Z \geq \dim X - r = \dim V - r = \dim U$ and $\dim U > \dim Y$. This shows that $Y \neq Z$, which contradicts the hypothesis $U = H_1 \cdots H_r$, since f maps biregularly $V \setminus U$ onto $X \setminus Y$.

COROLLARY.—*Under the hypotheses of Lemma 2, if f is proper and* V *normal there exists a canonical birational proper morphism* $g : V \to \bar{X}$, *such that* $f = qg$ *and* $g(U) = \bar{Y}$, *which maps* $V \setminus U$ *onto* $\bar{X} \setminus \bar{Y}$ *biregularly.*

LEMMA 3.—*Let* $f : X' \to X$ *be a regular contraction of* Y' *to* Y *and* X' *non-singular. There exists then a canonical isomorphism* $g : X' \to \bar{X}$ *such that* $f = qg$.

Proof.—Taking in the corollary above $U = Y'$, $V = X'$, it remains only to show that g is an isomorphism. By Z. M. T., it is sufficient to prove that g/Y' is injective.

Let us suppose $\bar{y} \in \bar{Y}$, $p(\bar{y}) = y$ and $r = \operatorname{codim}_X Y$. There exists then r hyper-surfaces $H_1, \cdots, H_{r-1}$ of X such that:

(a) Y is a simple subvariety of H_i $(1 \leq i \leq r-1)$;

(b) if $\bar{H}_i$ is the hypersurface of X corresponding by q to H_i and $\bar{F}_y = p^{-1}(y)$, then $\bar{H}_1 \cdots \bar{H}_{r-1} \cdot \bar{F}_y = \{\bar{y}\}$. Replacing X by a suitable neighborhood of y, we can find $\varphi_1, \cdots, \varphi_{r-1} \in \Omega(X)$ such that $(\varphi_i) = H_i$ $(1 \leq i \leq r-1)$. Then, if $\bar{\varphi}_i = \varphi_i \circ q$, $\varphi'_i = \varphi_i \circ f$, we have $(\bar{\varphi}_i) = \bar{H}_i + \bar{Y}$ [4] and so $(\varphi'_i) = H'_i + Y'$, since the hypersurfaces of X', $\bar{X}$ correspond by g biregularly. Consider now $\Gamma' = H'_1 \cdots H'_{r-1}$, $F'_y = f^{-1}(y)$, $\bar{\Gamma} = g(\Gamma')$. As $g(H'_i) = \bar{H}_i$, we have $\bar{\Gamma} = g(\Gamma')$. Since $F'_y = g^{-1}(\bar{F}_y)$, we can apply the projection formula and get $g(\Gamma' \cdot F'_y) = \bar{\Gamma} \cdot \bar{F}_y$, i.e., $g(\Gamma' \cdot F'_y) = \{\bar{y}\}$. There exists therefore a unique point $y' \in Y'$ such that $\Gamma' \cdot F'_y = \{y'\}$. We can now prove that $g^{-1}(\bar{y}) = \{y'\}$. Indeed, suppose contrariwise that there exists an $y'' \neq y'$ such that $g(y'') = \bar{y}$ and so, evidently, $y'' \in F'_y$. Hence $y'' \notin \Gamma'$, because $\Gamma' \cdot F'_y = y'$. Therefore $y'' \notin H'_i$, with a suitable i $(1 \leq i \leq r-1)$. As $(\varphi'_i) = H'_i + Y'$, we see that $\varphi'_i \in O(y'', X')$. Let $\operatorname{ord}_{y''}(\psi)$ be the order of $\psi \in O(y'', X')$ [4] at y'', i.e., the least integer α for which $\psi \in \boldsymbol{m}^\alpha$, where $\boldsymbol{m} = \boldsymbol{m}(y'', X')$ is the maximal ideal of $O(y'', X')$. By the divisor's formula (Theorem 3, [4]) we have $\operatorname{ord}_{y''}(\varphi'_i) = m(y'', H'_i) + m(y'', Y')$, where $m(y'', H)$, $m(y'', Y)$ are the multiplicities of y'' on H', Y' respectively. Hence $\operatorname{ord}_{y''}(\varphi'_i) = 1$, since y'' is simple on Y' and $y'' \notin H'_i$. Similarly $\operatorname{ord}_{\bar{y}}(\varphi_i) = m(\bar{y}, \bar{Y}) + m(\bar{y}, \bar{H}_i) = 2$, i.e., $\varphi_i \in \boldsymbol{n}^2$ where $\boldsymbol{n} = \boldsymbol{m}(\bar{y}, \bar{X})$. But, since $g(y'') = \bar{y}$, $O(y'', X')$ dominates $O(\bar{y}, \bar{X})$ by the canonical morphism $g^* : \Omega(\bar{X}) \to \Omega(X')$. This proves that $g^*(\boldsymbol{n}) \subseteq \boldsymbol{m}$ and so $g^*(\boldsymbol{n}^2) \subseteq \boldsymbol{m}^2$. Thus we get $\varphi'_i \in \boldsymbol{m}^2$, which contradicts the hypothesis $\operatorname{ord}_{y''}(\varphi'_i) = 1$.

3. Lemma 3 proves the " only if " part of the theorem. It proves also the uniqueness part of the theorem but for the fact that (Y, X) is uniquely determined by (Y', X'). We shall now prove this remaining assertion.

LEMMA 4.—*Let* $f : X' \to X$, $f_1 : X' \to X_1$ *be two regular contractions of* Y' *within* X'. *The birational correspondence* $t : X \to X_1$ *defined by* f *and* f_1 *is then an isomorphism.*

Proof. (*by induction on* $\dim X' = n$).—For $n = 1$ lemma 4 is trivial. Let $n > 1$, $f(Y') = Y$, $f_1(Y') = Y_1$ and suppose that the lemma is true for the dimension $n-1$.

If $\dim Y_1 = 0$, then by Z.M.T. t is regular. In this case, if $\dim Y > 0$, then (by Lemma 2) $\operatorname{codim}_X Y = 1$ which contradicts the hypothesis that is a contraction. Hence dim Y = 0 and so t^{-1} is biregular, again by Z.M.T. Thus we can suppose dim Y and dim $Y_1 \geq 1$. By Lemma 3 f and f_1 are monoidal transformations of centers Y, Y_1 respectively. Let $y \in Y$ and H be a hypersurface of X transversal to Y at y on X. Then, in view of the local character of the problem, we can suppose $H \cdot Y = Z$ where Z is a subvariety of Y. Since f is a monoidal transformation of X centered in Y, $f^{-1}(H) = H'$ and f/H' is a monoidal transformation of H, of center Z. It follows $Z' = f^{-1}(Z) \subset H'$ and $H' \cdot Y' = Z'$. Since f_1 is a monoidal transformation, this shows that f_1/H' is a monoidal transformation of $f(H') = H_1$; hence we have either $Y_1 \subset H_1$ and $f_1(Z') = Y_1$ or $Y_1 \not\subset H_1$ and $Y_1 \cdot H_1 = f_1(Z')$. By the hypothesis of induction $t/H = t_H$ induces an isomorphism $H \tilde{\rightarrow} H_1$, hence $t_H(Z) = f_1(Z')$. It follows that dim Y $\geq$ dim Y_1 and, by symmetry, dim $Y_1 \geq$ dim Y; hence dim Y = dim Y, which shows that $f_1(Z') \neq Y_1$ i.e. $Y_1 \not\subset H_1$. Thus we see that $f^{-1}(y) = f_1^{-1}(y_1)$, where $y_1 = t_H(y)$. Therefore the fibres of f/y' and those of f_1/y' coincide. This shows that t is bijective on Y and so, by Z.M.T., t is regular.

LEMMA 5.—*Let* Y, Y′, X′ *be nonsingular algebraic varieties and suppose that* Y′ *is a subvariety of* X′. *If the conditions* 1°–3° *of the Theorem are satisfied, then there exists a regular contraction* $f: X' \to X$ *of* Y′ *to* Y.

Proof.—We shall construct X and f piece-wise. Indeed, owing to Lemma 4, these " pieces " can be canonically pieced together to get the variety X. By definition we take $X' \setminus Y'$ as an open set of X and $f/X' \setminus Y' = 1$. Let $y' \in Y'$. According to condition 1° we can identify Y′ with P (E). Let $p: Y' \to Y$ be the canonical projection and $y = p(y')$. Then $p^{-1}(y) = P(E_y)$, where E_y is the fiber of E in y. Let r be the rank of E and $P_1, \cdots, P_r \subset P(E_y)$ the hyperplanes corresponding to a system of coordinates in E_y. Taking a local system of coordinates of E in an open neighborhood U of $y \in Y$ in Y, we get such a system $P_1(u), \cdots, P_r(u)$ in every point $u \in U$. By 2° and 3° there exists $\psi_i \in O(y', X')$ such that $(\psi_i) = Y' + H'_i$ in an open neighborhood U′ of y' in X′ and, additionally, for every $u' \in U' \cap Y'$, $p^{-1}(p(u')) \subset U'$ and $H'_i \cdot p^{-1}(p(u')) = P_i(p(u'))$. It follows that, in U′, we have $H'_1 \cap \cdots \cap H'_r \cap Y' = \emptyset$ because, for every $u' \in U'$ and $u = p(u')$, we have $\bigcap_1^r P_i(u) = \emptyset$. We can therefore reduce U′ to an open neighborhood of $U' \cap Y'$ in X′ such that $H'_1 \cap \cdots \cap H'_r = \emptyset$ in U′. We can now evidently suppose $p(U' \cap Y') = U$, and so restrict our construction to U′. Replacing X′ by U′, Y′ by $Y' \cap U'$ and Y by U we get the following conditions:

(a) $(\psi_i) = Y' + H'_i \ (1 \leq i \leq r)$;

(b) $H'_i \cdot Y' = P(E_i)$, where $E_i \ (1 \leq i \leq r)$ are subbundles of E corresponding to a system of coordinates of E;

(c) $\bigcap_1^r H'_i = \emptyset$.

Similarly, by restriction to suitable chosen open sets, we can suppose, that there exists $\omega_1, \cdots, \omega_s \in \Omega(X')$ and $\alpha_1, \cdots, \alpha_t \in \Omega(Y)$ such that $\Omega(\omega_1, \cdots, \omega_s) = \Omega(X')$ are everywhere defined on X′ and null on Y′, and that $\Omega(\alpha_1, \cdots, \alpha_t) = \Omega(Y)$ and $\alpha_1, \cdots, \alpha_t$ are everywhere defined on Y. We may moreover suppose that there exists $\beta_1, \cdots, \beta_t \in \Omega(X')$ which are regular at every point of X′ and such that $\beta_i/Y' = \alpha_i$ $(1 \leq i \leq s)$, where $\Omega(Y)$ is identified with a subfield of $\Omega(Y')$ by the canonical injection $p^* : \Omega(Y) \to \Omega(Y')$ associated to p.

Consider now the locus X'_i of $(\beta, \omega, \psi, \varphi^{(i)})$ in the affine space S^N $(N = t + s + 2r)$, where $\{\varphi^{(i)}\} = \{\varphi_1^{(i)}, \cdots, \varphi_r^{(i)}\}$ with $\varphi_l^{(i)} = \psi_l/\psi_i$ $(1 \leq i \leq r)$. X'_i is isomorphic with the open set $X' \setminus H'_i$ of X′. We shall identify X'_i with this set. Let X be the locus of (β, ω, ψ) in S^M $(M = t + s + r)$ and $f_i : X'_i \to X$ the birational mapping defined by $f_i((\beta, \omega, \psi, \varphi^{(i)})) = (\beta, \omega, \psi)$. It is easily seen that f_i $(1 \leq i \leq r)$ is everywhere regular on X'_i and biregular in $X'_i \setminus Y'_i$, where $Y'_i = X'_i \cap Y'$. But $X' = \bigcup_1^r X'_i$, since $\bigcap_1^r H'_i = \emptyset$. Hence f_i $(1 \leq i \leq r)$ defines a birational morphism $f : X' \to X$. It is obvious that $f(Y') = Y$, where Y is identified with the locus of $(\alpha, 0, 0)$ in S^M. Since f_i is biregular in $X' \setminus Y'_i$, f is biregular in $X' \setminus Y'$.

Let $y \in Y$. We shall show that y is simple on X. In view of the hypothesis $b)$ above, $p^{-1}(y) \cdot \prod_{j=1}^{r-1} H'_j = \{y'\}$ where y' is a point. Hence $\prod_{j=1}^{r-1} H'_j$ has a unique irreducible component Z′ at y' and Z′ is transversal to Y′ at y' on X′. Let Y″ be the unique irreducible component of Z′. Y′ containing y' and $H_j = f(H'_j)$, $(1 \leq j \leq r)$. If $Z = f(Z')$, then Z is an irreducible component of multiplicity one of $\prod_{j=1}^{r-1} H_j$ on X because f maps $X' \setminus Y'$ onto $X \setminus Y$ biregularly. Consider now a generic point of $S = S^M$, (A, B, C), where $(A) = (A_i)_{1 \leq i \leq t}$ $B = (B_i)_{1 \leq i \leq s}$ and $C = (C_i)_{1 \leq i \leq r}$. Let $D_l^{(i)} = C_l/C_i$ and $S'_i \subset S_i^N$ be the locus of $(A, B, C, D_l^{(i)})$. Then $(A, B, C, D^{(i)}) \to (A, B, C)$ defines a birational morphism $F_i : S'_i \to S$ and, piecing together these maps, we get a monoidal transformation $F : S' \to S$ of S having the center T defined by $C_1 = \cdots = C_r = 0$. X′ is a subvariety of S′ and $F/X' = f$. Put $T' = F^{-1}(T)$ and let $\xi_i \in \Omega(S)$ be the coordinate function of $\Omega(S)$ for which (ξ_i) is the hyperplane L_i defined by $C_i = 0$ $(1 \leq i \leq r)$; L_i corresponds to a hypersurface L'_i of S′ and $F^{-1}(L_i) = L'_i + T'$. Hence, if $\xi'_i = \xi_i \circ F \in \Omega(S')$, then $(\xi'_i) = L'_i + T'$. ξ'_i induces on X the function $\chi_i = \psi_i \circ f$. This shows that $(\chi_i) = H_i$, because $(\psi_i \circ f) = H_i$. It follows that $H_i = L_i \cdot X$, since $(\chi_i) = (\xi_i) \cdot X$ by a well known formula [8]. Similarly $F^{-1}(L_i) \cdot X' = (F^{-1}((\xi_i))) \cdot X' = (\xi'_i) \cdot X' = (\psi_i) = H'_i + Y'$. The cycle $Z' \cdot (H'_r + Y')$ is defined on X′ and equal to $Z' \cdot Y'$, since $H'_1 \cdots H'_r = 0$. But, as we have just seen, $H'_r + Y' = F^{-1}(L_r) \cdot X'$. This proves that $Z' \cdot F^{-1}(L_r)$ is defined on S′ and that $(Z' \cdot F^{-1}(L_r))_{S'} = (Z' \cdot Y')_{X'}$ by the associativity intersection formula. Note that Y″ is the unique irreducible component of $Z' \cdot F^{-1}(L_r)$ which meets $f^{-1}(y)$, hence also $F^{-1}(y)$, since $Z' \subset Y'$ and $F^{-1}(y) \cap Y' = f^{-1}(y)$. It is evident that $L_r \cap Z = H_r \cap Z = Y$. On the other hand, $Z = F(Z')$. This shows that we can apply the projection formula with respect to $F : S' \to S$ to $Z' \cdot F^{-1}(L_r)$. Thus we get $i(Y, L_r \cap Z; S) = 1$, since $i(Y'', F^{-1}(L_r) \cap Z'; S') = 1$. As Z is the unique irreducible component of $H_1 \cdots H_{r-1}$ containing y, we can suppose $Z = (H_1 \cdots H_{r-1})_X$. This shows that $Z = (L_1 \cdots L_{r-1} \cdot X)_S$, because $H_i = L_i \cdot X$ $(1 \leq i \leq r-1)$. Therefore $L_r \cdot Z = L_1 \cdots L_r \cdot X$. We get thus finally $i(Y, L_1 \cdots L_r \cdot X; S) = 1$, which proves that y is simple on X by the well known multiplicity one criterion.

Note.—It is now quite simple to see that E is isomorphic with the normal bundle N of Y in X. Indeed, since $f: X' \to X$ is a regular contraction of Y′ to Y with X′, it follows that f is a monoidal transformation of center Y of X. Then $Y' = P(N)$, and so the normal bundle N′ of Y′ in X′ contracts regularly to N. But, in view of Condition 3°, $N' \overset{\sim}{\to} l$ and l is regularly contractable to E. Hence, by Lemma 4, $N \overset{\sim}{\to} E$.

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