Edgar R. Lorch, Hing Tong

Uniform approximation of Baire functions by continuous functions

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **41** (1966), n.1-2, p. 47–48.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1966_8_41_1-2_47_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1966.

Topologia. — *Uniform approximation of Baire functions by continuous functions.* Nota di EDGAR R. LORCH e HING TONG (*), presentata (**) dal Socio B. SEGRE.

RIASSUNTO. — Dato un insieme $\mathcal{S}$, ad ogni topologia metrica compatta di $\mathcal{S}$ corrisponde un insieme di funzioni di Baire definite su $\mathcal{S}$ rispetto a quella; ma diverse topologie possono condurre alle medesime funzioni, il che dà luogo a peculiarità espresse da una serie di proposizioni qui soltanto enunciate. Su questi risultati è stato basato il Corso tenuto dal primo dei due Autori presso l'Istituto Matematico dell'Università di Roma, Corso che apparirà più tardi in forma ciclostilata.

Let $\mathcal{S}$ be a set and τ a compact (Hausdorff) metric topology on $\mathcal{S}$. Let C_τ represent the algebra of real-valued τ-continuous functions and let I_τ represent the algebra of bounded real-valued τ-Baire functions. The present paper studies a variety of phenomena which arise from the fact that many distinct topologies lead to the same Baire functions. Two topologies τ and τ' will be called coherent if and only if $I_\tau = I_{\tau'}$. The notion of coherence sets up an equivalence relation among topologies in an obvious way. If τ_0 is some fixed topology, the equivalence class determined by τ_0 will be called T. Thus $T = \{\tau : I_\tau = I_{\tau_0}\}$. The set $\mathcal{S}$ with topology τ will often be denoted by $(\mathcal{S}, \tau)$. A Baire equivalence $\Lambda : (\mathcal{D}, \sigma) \to (\mathcal{S}, \tau)$ is a bijection which along with Λ^{-1} carries Baire sets into Baire sets.

The propositions below imply, roughly speaking, that in a suitable coherent topology any Baire function may be approximated uniformly by continuous functions; that for any two Baire sets of the same cardinality and same co-cardinality (cardinality of complements) there is a homeomorphism in a coherent topology of $\mathcal{S}$ on itself which interchanges sets and also their complements; that for any ordinal α, $0 \leq \alpha < \Omega$, and any Baire set, the question as to whether there exists a coherent topology in which that set has Baire order α depends exclusively on its cardinality and co-cardinality.

LEMMA 1. – *Let $\mathcal{D}$ and $\mathcal{S}$ be two sets with topologies σ and τ respectively. Suppose there exists a coherent topology τ' (on $\mathcal{S}$) such that the spaces $(\mathcal{D}, \sigma)$ and $(\mathcal{S}, \tau')$ are homeomorphic. Then $(\mathcal{D}, \sigma)$ and $(\mathcal{S}, \tau)$ are Baire equivalent. Conversely, suppose that $(\mathcal{D}, \sigma)$ and $(\mathcal{S}, \tau)$ are Baire equivalent. Then there exists a coherent topology τ' (on $\mathcal{S}$) such that $(\mathcal{D}, \sigma)$ and $(\mathcal{S}, \tau')$ are homeomorphic.*

LEMMA 2. – *Let $\mathcal{S}$ be the disjoint union of the τ_0-Baire sets $\mathfrak{M}_1, \cdots, \mathfrak{M}_s$. Then there exists a coherent (compact metric) topology τ in which the sets $\mathfrak{M}_i$ are open and closed.*

(*) These results were obtained with the help of N.S.F. grants G.P. 5455, G.P. 3850, and G.P. 1618.

(**) Nella seduta del 22 giugno 1966.

THEOREM A.—*Let* φ *be a Baire function and let* $\varepsilon > 0$ *be a positive number. Then there exists a coherent topology* $\tau \in T$ *and a function* f *which is* τ*–continuous such that for all* $x \in \mathcal{E}$, $|\varphi(x) - f(x)| < \varepsilon$.

THEOREM B.—*Let* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$ *be two Baire sets in the space* $\mathcal{E}$ *under the compact metric topology* τ_0. *Suppose that* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$ *have the same cardinality and that* $\mathfrak{M}_1 - \mathfrak{M}_1 \cap \mathfrak{M}_2$ *and* $\mathfrak{M}_2 - \mathfrak{M}_1 \cap \mathfrak{M}_2$ *have the same cardinality. Then there exists a coherent compact metric topology* τ *and a* τ*–homeomorphism of* $\mathcal{E}$ *onto itself which maps* $\mathfrak{M}_1$ *onto* $\mathfrak{M}_2$ *and* $\mathcal{E} - \mathfrak{M}_1$ *onto* $\mathcal{E} - \mathfrak{M}_2$.

THEOREM C.—*Let* $\mathfrak{M}$ *be a Baire set and let* α *be an ordinal number. Suppose there exists a Baire set* $\mathfrak{N}$ *of the same cardinality and co-cardinality as* $\mathfrak{M}$ *which has Baire order* α. *Then there exists a coherent compact metric topology* τ *in which the Baire order of* $\mathfrak{M}$ *is* α.