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D. R. Anderson, J. A. Jensen

Semi-continuity on topological spaces

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Matematica. — *Semi-continuity on topological spaces.* Nota di D. R. ANDERSON e J. A. JENSEN, presentata (*) dal Corrisp. G. SCORZA DRAGONI.

RIASSUNTO. — In questa Nota sono contenuti alcuni risultati concernenti la semicontinuità in ispazi topologici. Secondo il teorema conclusivo, una trasformazione univoca da uno spazio di Hausdorff localmente compatto ad uno spazio metrico nel quale nessuno degli insiemi costituiti dai singoli punti sia aperto è aperta e continua se e soltanto se essa e la sua inversa sono semiaperte.

1. INTRODUCTION.—As in [1], let S.O. (E) denote the class of all semi-open sets in a topological space E where a set A is *semi-open* if and only if there exists an open set U in E such that $U \subset A \subset U^-$ where U^- is the closure of U. This note gives some results concerning semi-continuity on topological spaces.

2. SEMI-CONTINUOUS FUNCTIONS.

Definition 1.—A function f: $E_1 \to E_2$ where E_1 and E_2 are topological spaces is called *semi-continuous* if and only if, for U open in E_2, then $f^{-1}[U] \in$ S.O. (E_1).

Definition 2.—A function f on a topological space into a topological space is called *open* (*semi-open*) if and only if for every open (semi-open) set A of its domain $f[A]$ is open (semi-open] in its range. The inverse f^{-1}, not necessarily a function, is called *semi-open* if and only if for every semi-open set A, $f^{-1}[A]$ is semi-open.

THEOREM 1.—*If f is a continuous open function from a topological space E_1 to a topological space E_2, then f is semi-open and f^{-1} is semi-open.*

Proof.—If $A \in$ S.O. (E_1), then $U \subset A \subset U^-$ where U is open and $f[U] \subset f[A] \subset f[U^-] \subset f[U]^-$. Since $f[U]$ is open, $f[A] \in$ S.O. $[E_2]$ and f is semi-open.

Suppose $A \in$ S.O. (E_2), then $U \subset A \subset U^-$ where U is open. However, $f^{-1}[U] \subset f^{-1}[A] \subset f^{-1}[U] = f^{-1}[U]^-$ since f is both open and continuous. Therefore, $f^{-1}[A] \in$ S.O. (E_1).

The following theorems show that with appropriate conditions on the spaces E_1 and E_2, f will be a continuous open function if and only if both f and f^{-1} are semi-open.

THEOREM 2.—*Let f: $E \to D$ with E a topological space and D a metric space such that for any $x \in D$, $\{x\}$ is not open. If f is semi-open and semi-continuous then f is continuous.*

(*) Nella seduta del 21 giugno 1967.

Proof.—We suppose f is not continuous. Then there is a net x_α and $x \in E$ such that $x_\alpha \to x$ but $f(x_\alpha) \to f(x)$. There is an $\varepsilon > 0$ and a net x'_α such that for any α, $d(f(x'_\alpha), f(x)) > 2\varepsilon$ and $x'_\alpha \to x$. Let $B(f(x'_\alpha), \varepsilon)$ be the open ball centered at $f(x'_\alpha)$ and of radius ε. Then $U = \cup_\alpha B(f(\alpha'_\alpha), \varepsilon)$ is open and as f is semi-continuous, $f^{-1}[U]$ is semi-open. But x is a limit point of $f^{-1}[U]$ as $f^{-1}[U]$ contains $\{x'_\alpha\}$. Then $f^{-1}[U] \cup \{x\}]$ is semi-open. As f is semi-open $A = f[f^{-1}[U] \cup \{x\}]$ is semi-open. However, $\{f(x)\} = A \cap B(f(x), \varepsilon)$ and it is easy to see that since $B(f(x), \varepsilon)$ is open, $\{f(x)\}$ must be open, which is impossible.

THEOREM 3.—*Let* $f: E_1 \to E_2$ *where* E_1 *is Hausdorff and locally compact and* E_2 *is a Hausdorff space such that for any* $x \in E_2$, $\{x\}$ *is not open. If* f *is semi-open and continuous and* f^{-1} *is semi-open then* f *is open.*

Proof.—Suppose f is not open. There is an open set $G \subset E_1$ such that $f[G]$ is not open. There is an $x \in G$ such that $f(x)$ is not in the interior of $f[G]$. As E_1 is both Hausdorff and locally compact there is a compact neighborhood N of x such that $N \subset G$. As f is continuous, $f[N]$ is compact. As E_2 is Hausdorff, $f[N]$ is closed. Let U be the complement of $f[N]$. As $f(x)$ is not an interior point of $f[N]$, $U \cup \{f(x)\}$ is semi-open. Since f^{-1} is semi-open, $A = f^{-1}[U \cup \{f(x)\}]$ is semi-open. Let H be the interior of N. Since H is open it is easy to see that $A \cap H$ is semi-open. As f is semi-open, $f[A \cap H]$ is semi-open. However, $f[A \cap H] = \{f(x)\}$ which must be open as well (any singleton semi-open set is open). This contradicts the hypothesis.

THEOREM 4.—*Let* $f: E \to D$ *where* E *is Hausdorff and locally compact and* D *is a metric space such that for any* $x \in D$, $\{x\}$ *is not open. Then,* f *is open and continuous if and only if* f *and* f^{-1} *are semi-open.*

Proof.—The theorem is easily proved from Theorems 1, 2 and 3.

REFERENCES.

[1] NORMAN LEVINE, *Semi-open sets and semi-continuity in topological spaces*, «Amer. Math. Monthly», 70 36–41 (1963).

[2] J. L. KELLEY and I. NAMIOKA, *Linear topological spaces* (Princeton, New Jersey, 1963).