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Almost-periodic solutions of the equation of Schrödinger type. Nota I

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Analisi matematica. — *Almost-periodic solutions of the equation of Schrödinger type* (*). Nota I (**) del Corrisp. LUIGI AMERIO.

RIASSUNTO. — Si assegnano delle condizioni perché l'equazione del tipo di Schrödinger, con operatore e termine noto quasi–periodici, abbia una soluzione quasi–periodica, e delle condizioni perché le autosoluzioni dell'equazione omogenea, con operatore periodico, siano quasi–periodiche.

1.—INTRODUCTION AND STATEMENTS. Let X and Y be two complex Hilbert spaces; we assume $X \subseteq Y$, separable, dense in Y and with a continuous embedding ($\| x \|_Y \leq \sigma \| x \|$, $\sigma > 0$, where $\| \cdot \|_Y$, $(\cdot,\cdot)_Y$ and $\| \cdot \|$, $(\cdot,\cdot)$ denote norm and scalar product in Y and X respectively). Put $J = \{-\infty < t < +\infty\}$ and $\mathcal{A} = \mathcal{L}(X, X)$: hence $\mathcal{A}$ denotes the space of linear and bounded operators A, from X to X, with norm $\| A \|_{\mathcal{A}} = \sup_{\|x\|=1} \| A x \|$.

We call "*equation of Schrödinger type*" the equation [1]

$$\int_J \{ i (x(t), h'(t))_Y + (A(t) x(t) + f(t), h(t)) \} dt = 0, \tag{1,1}$$

where the *unknown function* $x(t)$, *the operator* $A(t)$, the *known term* $f(t)$ and the *test function* $h(t)$ satisfy the following conditions:

i_1) $x(t) \in C(J; X)$;

i_2) $A(t) \in C^1(J; \mathcal{A})$, is *selfadjoint* and verifies the *ellipticity condition*

$$(A(t) x, x) \geq \nu \| x \|^2 \qquad (\nu > 0); \tag{1,2}$$

i_3) $f(t) \in C^1(J; X)$;

i_4) $h(t) \in C(J; X)$, $h'(t) \in C(J; Y)$.

$h(t)$ has, in addition, *compact support* and (1,1) must be true for all test functions $h(t)$.

We denote by $u(t)$ the solutions of the *homogeneous equation*:

$$\int_J \{ i (u(t), h'(t))_Y + (A(t) u(t), h(t)) \} dt = 0. \tag{1,3}$$

(1,1) is the *weak form* which corresponds, for instance, to the following problem. Let Ω be an open, connected and *bounded* or *unbounded* set of the Euclidean space S^m ($\zeta = \{\zeta_1, \cdots, \zeta_m\}$) and consider the equation:

$$i \frac{\partial x(t,\zeta)}{\partial t} + \sum_{j,k}^{1\cdots m} \frac{\partial}{\partial \zeta_j} \left(a_{jk}(t,\zeta) \frac{\partial x(t,\zeta)}{\partial \zeta_k} \right) - a(t,\zeta) x(t,\zeta) = \int_\Omega \Phi(t,\zeta,\xi) x(t,\xi) d\xi + f(t,\zeta) \qquad (t \in J, \zeta \in \Omega). \tag{1,4}$$

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Assume that the coefficients $a_{jk}(t,\zeta)$, $a(t,\zeta)$ are measurable and bounded functions on $J \times \Omega$ with their partial derivatives with respect to t, and that

$$(1,5) \qquad a_{jk}(t,\zeta) = \bar{a}_{kj}(t,\zeta) \quad , \quad \sum_{,k}^{1\cdots m} a_{jk}(t,\zeta)\lambda_j\bar{\lambda}_k \geq \rho \sum_{1}^{m}{}_j |\lambda_j|^2,$$

$$a(t,\zeta) > \rho \ (\rho > 0),$$

where $\bar{\alpha}$ denotes the complex conjugate of α.

The second of (1,5) must be valid for all complex values $\lambda_1, \cdots, \lambda_m$.

Moreover the *kernel* $\Phi(t,\zeta,\xi)$ is supposed to be, $\forall t \in J$, positive semidefinite, selfadjoint and to belong to $L^2(\Omega \times \Omega)$, with its derivative $\Phi_t(t,\zeta,\xi)$.

The problem considered consists in finding a solution $x(t,\zeta)$ satisfying the *initial condition*

$$(1,6) \qquad x(0,\zeta) = x_0(\zeta) \qquad (\zeta \in \Omega)$$

and the *boundary condition*

$$(1,7) \qquad x(t,\zeta)|_{\zeta \in \partial\Omega} = 0 \qquad (t \in J).$$

We assume now

$$Y = L^2(\Omega) \quad , \quad X = H_0^1(\Omega)$$

with the norms

$$(1,8) \qquad \|x\| = \left\{ \int_{\Omega} \left(\sum_{1}^{m}{}_j \left| \frac{\partial x(\zeta)}{\partial \zeta_j} \right|^2 + |x(\zeta)|^2 \right) d\zeta \right\}^{1/2} = \left\{ \sum_{1}^{m}{}_j \left\| \frac{\partial x}{\partial \zeta_j} \right\|_Y^2 + \|x\|_Y^2 \right\}^{1/2}.$$

Hence the embedding of X in Y is continuous.

It results

$$(A(t)x, x) = \int_{\Omega} \left\{ \sum_{j,k}^{1\cdots m} a_{jk}(t,\zeta) \frac{\partial x(\zeta)}{\partial \zeta_j} \frac{\partial \bar{x}(\zeta)}{\partial \zeta_k} + a(t,\zeta)\, x(\zeta)\, \bar{x}(\zeta) \right\} d\zeta +$$

$$+ \int_{\Omega}\int_{\Omega} \Phi(t,\zeta,\xi)\, x(\xi)\, \bar{x}(\zeta)\, d\xi\, d\zeta$$

and (1,2) is satisfied.

Assume now that the derivatives, with respect to t, of the coefficients $a_{jk}(t,\zeta)$, $a(t,\zeta)$ are continuous, as functions of t, uniformly with respect to $\zeta \in \Omega$; assume moreover that, $\forall t \in J$ and $|\tau| \leq 1$, it results

$$|\Phi_t(t+\tau,\zeta,\xi)| \leq \psi(t,\zeta,\xi) \in L^2(\Omega \times \Omega)$$

and

$$\lim_{\tau \to 0} \int_{\Omega}\int_{\Omega} |\Phi_t(t+\tau,\zeta,\xi) - \Phi_t(t,\zeta,\xi)|^2\, d\xi\, d\zeta = 0.$$

Then it is

$$(A'(t)x, x) = \int_\Omega \left\{ \sum_{j,k}^{1\cdots m} \frac{\partial a_{jk}(t,\zeta)}{\partial t} \frac{\partial x(\zeta)}{\partial \zeta_k} \frac{\partial \bar{x}(\zeta)}{\partial \zeta_j} + \frac{\partial a(t,\zeta)}{\partial t} x(\zeta)\bar{x}(\zeta) \right\} d\zeta +$$

$$+ \int_\Omega \int_\Omega \Phi_t(t,\zeta,\xi)\, x(\xi)\bar{x}(\zeta)\, d\xi\, d\zeta,$$

and the operators $A(t), A'(t)$ are $\mathfrak{A}$–continuous.

Setting $x(t) = \{x(t,\zeta); \zeta \in \Omega\}$, $f(t) = \{f(t,\zeta); \zeta \in \Omega\}$, we obtain the weak form (1,1) of our problem: the solution $x(t)$ must satisfy the initial condition $x(0) = \{x_0(\zeta); \zeta \in \Omega\}$.

Let us now recall the *fundamental formulas*, valid for the solutions of (1,1) and (1,3):

a) $\frac{d}{dt}\|x(t)\|_Y^2 = 2\,\mathfrak{I}\,(f(t), x(t))$,

b) $\frac{d}{dt}\{(A(t)x(t), x(t)) + 2\,\mathfrak{R}\,(f(t), x(t))\} = (A'(t)x(t), x(t)) +$ $+ 2\,\mathfrak{R}\,(f'(t), x(t))$,

a') $\|u(t)\|_Y = \|u(0)\|_Y$,

b') (if $A(t) = I$) $\|u(t)\| = \|u(0)\|$.

a') and *b'*) mean that *two principles of conservation of norm* hold: *for the* Y*–norm the principle is always true, for the* X*–norm it is true if* $A(t) = I$ (to which case we can always reduce our problem, by (1,2), if $A(t) =$ const.).

The *initial value problem*, $x(0) = x_0$, *for* (1,1), *has one and only one solution*, $x(t)$, $\forall x_0 \in X$; moreover $x(t)$ *depends continuously on* x_0 *and* $f(t)$: *precisely*, $\forall$ *interval* $-T \le t \le T$,

$$\|x(t)\| \le M_T \left\{ \|x_0\| + \|f(0)\| + \int_{-T}^{T} \|f'(\eta)\|\, d\eta \right\} \tag{1,9}$$

where M_T is independent on x_0 and $f(t)$.

In what follows, when we say that a function $z(t)$ is bounded, or uniformly continuous (u.c.), or uniformly weakly continuous (u.w.c.), we *always* mean that this occurs on the whole interval J. The range of $z(t)$ will be indicated by $R_{z(t)}$. Moreover, we shall add the notation of the space where $z(t)$ takes its values, with the exception of the X space: hence $z(t)$ bounded, or u.c., or u.w.c., or almost-periodic (a.p.), or weakly almost-periodic (w.a.p.) means $z(t)$ X–bounded or X–u.c., or X–u.w.c., or X–a.p., or X–w.a.p.

In this paper, we study equations (1,1) and (1,3) with the essential aim to give conditions for the existence of *one* a.p. solution of (1,1), if $A(t)$ and $f(t)$ are a.p., and for the existence of a.p. eigensolutions of (1,3), if $A(t)$ is periodic.

For the first equation we are in the same order of ideas of a preceding paper [2] (concerning the extension of Favard's theorems to abstract equations): we may note, however, that the statements concerning equation (1,1) result notable wider, because of the peculiar properties of such equation.

Let us add that, for studying equation (1,3), we shall use a generalisation of Bochner's fundamental criterion of almost-periodicity (cfr. observation III, at the end of this §).

Let us give now the following *definitions*.

Let $z(t)$ be a bounded function and put:

$$\mu(z) = \operatorname*{Sup}_{J} \| z(t) \| , \tag{1,10}$$

$$\varphi(z\,;v\,,\tau) = \operatorname*{Sup}_{J} | (z(t+\tau) - z(t)\,,v) | \qquad (\forall v \in X\,,\ \tau \in J). \tag{1,11}$$

Let Γ_z be the set (obviously *convex*) of all solutions $x(t)$, bounded and such that

$$\varphi(x\,;v\,,\tau) \leq \varphi(z\,;v\,,\tau) \qquad (\forall v \in X\,,\ \tau \in J). \tag{1,12}$$

Let us enunciate now the statements which will be proved in the following §§ 2, 3, 4.

I. MINIMAX THEOREM.—*Let us assume that there exists a bounded solution,* $z(t)$ *(that is* Γ_z *is not empty).*

Then, if

$$\tilde{\mu} = \operatorname*{Inf}_{\Gamma_z} \mu(x) , \tag{1,13}$$

there exists, in Γ_z, *one and only one solution,* $\tilde{x}(t)$, *such that*

$$\mu(\tilde{x}) = \tilde{\mu} . \tag{1,14}$$

Whe shall call $\tilde{x}(t)$ the *minimal* solution, in Γ_z.

COROLLARY.—$A(t)$ *and* $f(t)$ *periodic, with period* $\omega \Rightarrow \tilde{x}(t)$ *periodic, with period* ω.

II. ALMOST-PERIODICITY THEOREM.—*Let us assume that:*

1) *the operators* $A(t)\,, A'(t)$ *are* $\mathfrak{A}$*–a.p.*;
2) *the functions* $f(t)\,, f'(t)$ *are a.p.*;
3) *there exists a solution,* $z(t)$, *bounded and u.w.c.*

Then the minimal solutions, $\tilde{x}(t)$, *is w.a.p. and* Y*–a.p. Moreover, if* $z(t)$, *bounded, is u.c., then* $\tilde{x}(t)$ *is a.p.*

Let us now enunciate an ALMOST-PERIODICITY THEOREM FOR THE EIGEN-SOLUTIONS $u(t)$:

III.—*Let us assume that:*

1) *the operator* $A(t)$ *is periodic*;
2) *the embedding of* X *in* Y *is compact.*

Then every bounded eigensolution, $u(t)$, *is w.a.p. and* Y*–a.p.*

If $u(t)$ *is bounded and u.c., then* $u(t)$ *is a.p.*

Theorem III gives an extension to equation (1,3), with *periodic operator*, of an interesting property of the solutions of linear ordinary homogeneous systems, with *periodic coefficients*: bounded solution of such systems are in fact a.p., since, by a classical theorem of Liapunov, any periodic system can be *reduced* to one with constant coefficients, by means of a linear periodic non singular transformation.

Observation I.—If $A(t) = I$, *then the hypothesis of (weak or strong) uniform continuity of the bounded solution* $z(t)$, *or* $u(t)$, *can be eliminated* (cfr. §§ 3, 4).

Observation II.—By (1,11) it follows that $z(t)$ *u.w.c.* $\Rightarrow \tilde{x}(t)$ *u.w.c.*

Setting, moreover,

$$\varphi(x\,;\tau) = \operatorname*{Sup}_{J} \|x(t+\tau) - x(t)\| \qquad (\tau \in J) \tag{1,15}$$

we have, by (1,11),

$$\varphi(x\,;\tau) \leq \varphi(z\,;\tau). \tag{1,16}$$

Hence $z(t)$ *u.c.* $\Rightarrow \tilde{x}(t)$ *u.c.*

Observation III.—For proving theorem III we shall use the following generalisation of Bochner's criterion. For clarity's sake, let us recall, at first, the way for obtaining such criterion.

Let B *be a Banach space, and let* K *be the Banach space of all continuous and bounded functions* $f(t)$, *from* J *to* B $(K = C(J\,;B) \cap L^{\infty}(J\,;B))$, *with norm corresponding to uniform convergence*: if $\tilde{f}$ is the point of K which corresponds to the function $f(t)$, it will therefore be

$$\tilde{f} = \{f(t)\,;t \in J\} \quad , \quad \|\tilde{f}\|_K = \operatorname*{Sup}_{J} \|f(t)\|_B\,. \tag{1,17}$$

Let us now consider, together with $f(t)$, the set of the translates $f(t+s)$, $\forall s \in J$. If $\tilde{f}(s) = \{f(t+s)\,;t \in J\}$ we have defined an application, that we shall call *Bochner's transform*, $s \to \tilde{f}(s)$, from J to K; furthermore $\tilde{f}(0) = \tilde{f}$.

The range $R_{\tilde{f}(s)}$ of Bochner's transform has the following properties.

α) $R_{\tilde{f}(s)}$ *is a spherical line*: in fact

$$\|\tilde{f}(s)\|_K = \operatorname*{Sup}_{J} \|f(t+s)\|_B = \operatorname*{Sup}_{J} \|f(t)\|_B = \|\tilde{f}(0)\|_K\,; \tag{1,18}$$

β) $R_{\tilde{f}(s)}$ *is described in such a way that the "principle of conservation of distances" holds*: in fact, by (1,18),

$$\|\tilde{f}(s+\tau) - \tilde{f}(s)\|_K = \|\tilde{f}(\tau) - \tilde{f}(0)\|_K = \operatorname*{Sup}_{J} \|f(t+\tau) - f(t)\|_B\,; \tag{1,19}$$

γ) $f(t)$ *a.p.* $\Longleftrightarrow \tilde{f}(s)$ *a.p., with the same ε–almost-periods.*

α), β) and γ) are obvious. Very deep is property

δ) $\tilde{f}(s)$ *a.p.* $\Longleftrightarrow R_{\tilde{f}(s)}$ *relatively compact (r.c.).*

By γ) and δ) we deduce Bochner's criterion: $f(t)$ *a.p.* $\Longleftrightarrow R_{\tilde{f}(s)}$ *r.c.*

Let us now prove the following generalisation of δ):

δ') *$\tilde{f}(s)$ a.p. $\Longleftrightarrow$ there exists a relatively dense sequence $\{s_n\}$ such that the sequence $\{\tilde{f}(s_n)\}$ is r.c.*

(For instance, $f(t)$ is a.p. if the sequence $\{\tilde{f}(n)\}$ $(n = 0, \pm 1, \pm 2, \cdots)$ is r.c.).

The condition is obviously necessary, since $f(t)$ a.p. $\Rightarrow$ any sequence $\{\tilde{f}(s_n)\}$ r.c.

Let us now prove that the condition is sufficient. For that, we shall prove, at first, that, $\forall \varepsilon > 0$, the set $\{\tau\}_\varepsilon$, of the ε–a.p., is relatively dense (r.d.).

Since $\{\tilde{f}(s_n)\}$ is r.c., there exist k values (depending on ε): $\tilde{f}(s_{1,0}), \cdots, \tilde{f}(s_{k,0})$, such that it results, $\forall n$,

$$\tilde{f}(s_n) \in \bigcup_j^{1\cdots k} (\tilde{f}(s_{j,0}), \varepsilon)$$

(where $(\tilde{f}, \varepsilon)$ denotes the open sphere with centre $\tilde{f}$ and radius ε).

Let us now divide the sequence $\{\tilde{f}(s_n)\}$ into k subsequences $\{\tilde{f}(s_{j,n})\}$ such that, $\forall j$,

$$\|\tilde{f}(s_{j,n}) - \tilde{f}(s_{j,0})\|_K < \varepsilon, \tag{1,20}$$

that is, by (1,19),

$$\|\tilde{f}(s_{j,n} - s_{j,0}) - \tilde{f}(0)\|_K < \varepsilon. \tag{1,21}$$

Hence, by (1,19),

$$\tau_{j,n} = s_{j,n} - s_{j,0} \tag{1,22}$$

is an ε–a.p.

Let us now prove that $\bigcup_j^{1\cdots k} \{\tau_{j,n}\}$ is a r.d. sequence. Let $d > 0$ be an inclusion length for the r.d. sequence $\{s_n\}$ and put

$$m = \min_{1 \le j \le k} \{-s_{j,0}\} \quad , \quad M = \max_{1 \le j \le k} \{-s_{j,0}\}, \tag{1,23}$$

$$l = M - m + d. \tag{1,24}$$

Consider an interval $a \vdash\!\dashv a + l$, a arbitrary. The interval $a - m \vdash\!\dashv a - m + d$ contains one point, s_{j_1,n_1}, at least, of $\{s_n\}$: hence, by (1,23),

$$a - m + m \le s_{j_1,n_1} - s_{j_1,0} \le a - m + d + M,$$

that is, by (1,22),

$$a \le \tau_{j_1,n_1} \le a + l$$

and the thesis is proved.

We have now to prove that $\tilde{f}(s)$ is continuous, that is, by (1,19), that $f(t)$ is u.c.

Setting $\Delta = \{-d \leq \eta \leq d\}$, let $Z = C(\Delta ; B)$ be the space of all functions $z(\eta)$ continuous from Δ to B: hence

(1,25) $$z = \{z(\eta) ; \eta \in \Delta\} \quad , \quad \| z \|_Z = \max_{\Delta} \| z(\eta) \|_B .$$

Put $z_n = \{f(\eta + s_n) ; \eta \in \Delta)$ and observe that, since

$$\| z_n - z_m \|_Z \leq \| \tilde{f}(s_n) - \tilde{f}(s_m) \|_K ,$$

the sequence $\{z_n\}$ is, as $\{\tilde{f}(s_n)\}$, r.c.

Since $f(t)$ is continuous, it follows that the functions $f(\eta + s_n)$ are equally-continuous on Δ: hence to every $\varepsilon > 0$ there corresponds δ_ε, $0 \leq \delta_\varepsilon \leq \frac{d}{2}$, such that

$$\eta', \eta'' \in \Delta , |\eta'' - \eta'| \leq \delta_\varepsilon \Rightarrow \| f(\eta' + s_n) - f(\eta'' + s_n) \|_B \leq \varepsilon , \qquad \forall n.$$

Taken an arbitrary $\bar{t} \in J$, there exists $s_{\bar{n}} \in \bar{t} - \frac{d}{2} \mid\!-\!\mid \bar{t} + \frac{d}{2}$; therefore $\bar{t} = \bar{\eta} + s_{\bar{n}}$, with $|\bar{\eta}| \leq \frac{d}{2}$. Suppose now $|t - \bar{t}| \leq \delta_\varepsilon$ and set $t = \eta + s_{\bar{n}}$: it results $|\eta| \leq d$, and $|\eta - \bar{\eta}| = |t - \bar{t}| \leq \delta_\varepsilon$. It follows

$$\| f(t) - f(\bar{t}) \|_B = \| f(\eta + s_{\bar{n}}) - f(\bar{\eta} + s_{\bar{n}}) \|_B \leq \varepsilon .$$

Hence $f(t)$ is u.c. and δ') is proved.

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[1] Cfr. J. L. LIONS et E. MAGENES, *Problèmes aux limites non homogènes et applications*, Dunod, Paris.

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