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Note on semi-continuous functions

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NOTE PRESENTATE DA SOCI

Matematica. — *Note on semi-continuous functions.* Nota [(*)] di J. A. JENSEN, presentata dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — In questa Nota è dimostrato che la trasformazione composta $f \circ g$ di una trasformazione univoca continua ed aperta g di un primo spazio topologico su un secondo e di una trasformazione univoca f del secondo in un terzo, è semicontinua (semiaperta) se, e soltanto se, f è semicontinua (semiaperta); e sono stabilite alcune proposizioni sulla semicontinuità e la continuità di trasformazioni univoche da spazi topologici lineari a spazi topologici lineari (per esempio: una tal trasformazione è necessariamente continua se è lineare e semicontinua).

1. INTRODUCTION.—A function $f: E_1 \rightarrow E_2$ where E_1 and E_2 are topological spaces is called *semi-continuous* if and only if, for U open in E_2, then $f^{-1}[U]$ is in $S.O.(E_1)$. $S.O.(E_1)$ is the class of all semi-open sets in E_1 where a set A is *semi-open* if and only if there exists an open set U in E_1 such that $U \subset A \subset U^-$, U^- is the closure of U. The function f is called *open* (*semi-open*) if and only if for every open (semi-open) set A of the domain of f, $f[A]$ is open (semi-open) in the range of f. The inverse f^{-1}, not necessarily a function, is called *semi-open* if and only if for every semi-open set A, $f^{-1}[A]$ is semi-open.

In [1] the proof of the following theorem is given.

THEOREM 1.—If f is a continuous open function from a topological space E_1 into a topological space E_2, then f is semi-open and f^{-1} is semi-open.

This note gives an application of Theorem 1 and some results concerning semi-continuity on linear topological spaces.

2. SEMI-CONTINUITY ON LINEAR TOPOLOGICAL SPACES.

THEOREM 2.—If g is a continuous open function from E_1 onto E_2 and f is a function from E_2 to E_3 where E_1, E_2 and E_3 are topological spaces, then f is semi-continuous (semi-open) if and only if the composition $f \circ g$ is semi-continuous (semi-open).

Proof.—Let U be an open set in E_3 and suppose $f \circ g$ is semi-continuous, then $(f \circ g)^{-1}[U] = g^{-1}[f^{-1}[U]] \in S.O.(E_1)$. Since g is continuous and open, by Theorem 1, $g[g^{-1}[f^{-1}[U]]] = f^{-1}[U] \in S.O.(E_2)$. Therefore, f is semi-continuous.

Suppose f is semi-continuous and U is open in E_3, then, by Theorem 1, $g^{-1}[f^{-1}[U]]$ is semi-open. Therefore, $f \circ g$ is semicontinuous.

(*) Pervenuta all'Accademia il 6 settembre 1967.

If f is a semi-open function on E_2 into E_3 and $A \in S.O.(E_1)$, then, by Theorem 1, $g[A] \in S.O.(E_2)$ and $f[g[A]] = (f \circ g)[A] \in S.O.(E_3)$. Therefore, $f \circ g$ is semi-open.

Suppose $f \circ g$ is semi-open, and let $B \in S.O.(E_2)$ and $A = g^{-1}[B]$, then, by Theorem 1, $A \in S.O.(E_1)$. Therefore, $(f \circ g)[A] = (f \circ g)[g^{-1}[B]] = f[B] \in S.O.(E_3)$.

If E is a linear topological space, then, because of continuity of addition, translation by a member x of E is continuous. Moreover, a translation by x and multiplication by a non-zero scalar t are homeomorphisms since each have a continuous inverse, namely translation by $-x$ and multiplication by $1/t$. Therefore, if a set U is open (closed), so are $x + U$ and tU, for each $x \in E$ and each non-zero scalar t.

THEOREM 3.—If $A \in S.O.(E)$, then $x + A \in S.O.(E)$ and $tA \in S.O.(E)$ for each $x \in E$ and each non-zero scalar t; and if $A \in S.O.(E)$, then $B + A$ is semi-open where B is a subset of E.

Proof.—If $A \in S.O.(E)$, then $U \subset A \subset U^-$ where U is open. Since $(x + U)^- = x + U^-$ and $(tU)^- = tU^-$ for each non-zero scalar t, it follows that $x + A$ and tA are in S.O. (E). $B + A = \cup \{x + A : x \in B\}$ is a union of semi-open sets each $x + A$ is a translation of a semi-open set. Therefore, since the union of an arbitrary collection of semi-open sets is semi-open, $B + A$ is semi-open.

Definition.—A set $M \subset E$ is a *semi-neighborhood* of a point $x \in E$ if and only of there exists $A \in S.O.(E)$ such that $x \in A \subset M$.

By Theorem 3, it follows that a set M is a semi-neighborhood of a point $x \in E$ if and only if $-x + M$ is a semi-neighborhood of $O \in E$; in other words, the semi-neighborhood system at x is the translates by x of members of the semi-neighborhood system at O.

Let F be a linear topological subspace of a linear topological space E. Let E/F be the linear topological quotient space and let Q be the quotient map $Q(x) = x + F$ which is linear, continuous and open. Then the proof of the following theorem requires only a direct application of the preceding results and Theorem 2.

THEOREM 4.—A function T on E/F into a topological space G is semi-continuous (semi-open) if and only if the composition $T \circ Q$ is semi-continuous (semi-open).

THEOREM 5.—If T is a semi-continuous linear transformation from a topological linear space E_1 to a topological linear space E_2, then T is continuous.

Proof.—It is sufficient to show that T is continuous at O. If T is linear, then $T(O) = O$. Let U be any open set about O in E_2. As in [2, p. 34] there exists an open set V which is circled and such that $O \in V$ and $V + V \subset U$. Since T is semi-continuous, there is a semi-open set A such that $O \in A$ and $T[A] \subset V$. Since A is a non-empty semi-open set, it must have non-empty interior A^i. Let $x \in A^i$. Then $O \in -x + A^i$ which is open. However, $T[-x + A^i] = -T(x) + T[A^i] \subset -T(x) + V$ and $-T(x) \in V$ since $T(x) \in V$ and V is circled. Therefore, $T[-x + A^i] \subset V + V \subset U$ and T is continuous.

COROLLARY.—If f is a semi-continuous functional on a linear topological space, then f is continuous.

Since a transformation T on the quotient space E/F is linear if and only if the composition $T \circ Q$ is linear, it follows, by Theorem 4 and Theorem 5, that T is continuous if $T \circ Q$ is linear and semi-continuous.

REFERENCES.

[1] D. R. ANDERSON and J. A. JENSEN, *Semi-continuity on Topological Spaces*, Will appear in «Accademia Nazionale dei Lincei».

[2] J. L. KELLEY and I. NAMIOKA, *Linear Topological Spaces* (Princeton, New Jersey 1963).