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The projective transformation in a Finster space

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Geometria differenziale. — *The projective transformation in a Finsler space.* Nota di H. D. PANDE, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — Si considerano le trasformazioni proiettive (cioè che conservano le geodetiche) di uno spazio di Finsler, si studia il modo di alterarsi dei parametri di una congruenza per una tale trasformazione, e si dimostra che la connessione indotta su una varietà immersa in quello spazio subisce pure una trasformazione proiettiva.

1. INTRODUCTION.

Let F_n be an n-dimensional Finsler space equipped with the positively homogeneous metric function $F(x, \dot{x})$ of degree one in its directional arguments. The entities ([1], page 13, (3.1)).

$$g_{ij}(x, \dot{x}) \overset{\text{def}}{=} \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x}) \quad (1), (2) \tag{1.1}$$

form the covariant components of the metric tensor of F_n. They are symmetric directional arguments and

$$g^{ij}(x, \dot{x})\, g_{jk}(x, \dot{x}) = \delta^i_k \begin{cases} 1 & \text{if } k = i \\ 0 & \text{if } k \neq i . \end{cases} \tag{1.2}$$

The covariant components of the unit vector along the direction of the element of support $(x^i, \dot{x}^i)$ are given by ([1], page 69 (1.15′))

$$l_i(x, \dot{x}) = \dot{\partial}_i F(x, \dot{x}). \tag{1.3}$$

The covariant derivative of a vector $X^i(x, \dot{x})$, depending on the element of support, with respect to x^k in the sense of Cartan is given by [1], Chap. III

$$X^i_{|k}(x, \dot{x}) = (\partial_k X^i) - (\dot{\partial}_j X^i)\, G^j_k + X^j \Gamma^{*i}_{jk}, \tag{1.4}$$

where

$$G^j_k(x, \dot{x}) \overset{\text{def}}{=} \dot{\partial}_k G^i(x, \dot{x}), \tag{1.5a}$$

$$2\, G^i(x, \dot{x}) \overset{\text{def}}{=} \gamma^i_{jk}(x, \dot{x})\, \dot{x}^j \dot{x}^k, \tag{1.5b}$$

$\gamma^i_{jk}(x, \dot{x})$ being the Christoffel symbols of second kind ([1], page 59, (27) and (23)) and $\Gamma^{*i}_{jk}(x, \dot{x})$ are the Cartan connection coefficients symmetrical

(*) Nella seduta del 9 dicembre 1967.

(1) $\partial_i = \partial/\partial x^i$ and $\dot{\partial}_i = \partial/\partial \dot{x}^i$; Latin indices run from 1 to n.

(2) Numbers in brackets refer to the references at the end of the paper.

in their lower indices and homogeneous of degree zero in their directional arguments. We have ([1], Ch. III)

$$G^i_{jk}(x, \dot{x})\, x^j = \Gamma^{*i}_{jk}(x, \dot{x})\, \dot{x}^j = G^i_k(x, \dot{x}), \tag{1.6}$$

where $G^i_{jk}(x, \dot{x}) \overset{\text{def}}{=} \dot{\partial}_k G^i_j(x, \dot{x})$.

Let $\lambda_{(a)}\ \{a = 1, 2, \cdots, n\}$ be the unit tangents of n–congruences of an orthogonal ennuple. The subscript 'a' in the parenthesis simply distinguishes one congruence from the other. The covariant and contravariant components of $\lambda_{(a)}$ will respectively be denoted by $\lambda^i_{(a)}$ and $\lambda_{(a)i}$. Since n–congruences are mutually orthogonal, we have [2]

$$g_{ij}(x, \dot{x})\, \lambda^i_{(a)}\, \lambda^j_{(b)} = \delta_{ab}, \tag{1.7}$$

where the Kronecker delta $\delta_{ab} = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{if } a \neq b \end{cases}$. We have the Ricci coefficients of rotation, given by [2, 3]

$$\gamma_{abc}(x, \dot{x}) \overset{\text{def}}{=} \lambda^i_{(a)|j}\, \lambda_{(b)i}\, \lambda^j_{(c)}, \tag{1.8}$$

where the symbol $|$ denotes the covariant derivative with respect to x^k in the sense of Cartan and

$$\mu^i_{(m)}(x, \dot{x}) \overset{\text{def}}{=} \sum_h \gamma_{mhm}\, \lambda^i_{(h)}. \tag{1.9}$$

The geometric entities $\mu^i_{(m)}(x, \dot{x})$ are called the first curvature vector of a curve of a congruence in Finsler space [3].

2. PROJECTIVE TRANSFORMATION.

The differential equation of a geodesic

$$\frac{d^2 x^i}{ds^2} + \Gamma^{*i}_{jk}(x, dx/ds)\, \frac{dx^j}{ds}\, \frac{dx^k}{ds} = 0 \tag{2.1}$$

assumes the following form by the transformation of its parameter s to t [4]:

$$\dot{x}^l \left\{ \frac{d^2 x^i}{dt^2} + \Gamma^{*i}_{jk}(x, \dot{x})\, \dot{x}^j \dot{x}^k \right\} - \dot{x}^i \left\{ \frac{d^2 x^l}{dt^2} + \Gamma^{*l}_{jk}(x, \dot{x})\, \dot{x}^j \dot{x}^k \right\} = 0, \tag{2.2}$$

where

$$\Gamma^{*i}_{jk}(x, \dot{x}) = \Gamma^{*i}_{kj}(x, \dot{x}). \tag{2.3}$$

The equation (2.2) remains unchanged if we replace the Cartan connection coefficient $\Gamma^{*i}_{jk}(x, \dot{x})$ by a new symmetric coefficient $\overline{\Gamma}^{*i}_{jk}(x, \dot{x})$, given by

$$\overline{\Gamma}^{*i}_{jk}(x, \dot{x}) \overset{\text{def}}{=} \Gamma^{*i}_{jk}(x, \dot{x}) + 2\, \delta^i_{(j} p_{k)} + p_{jk}\, \dot{x}^i, \tag{2.4}$$

where $p_k(x, \dot{x})$ is a covariant vector, positively homogeneous of degree zero in its directional arguments and

$$p_{jk}(x, \dot{x}) \overset{\text{def}}{=} \dot{\partial}_j p_k(x, \dot{x}). \tag{2.5}$$

Definition 2.1. Let F_n and $\overline{F}_n$ be two spaces with fundamental tensors $g_{ij}(x, \dot{x})$ and $\bar{g}_{ij}(x, \dot{x})$ at the corresponding points. Then the spaces are said to be in geodesic correspondence if their geodesics are the same and we shall call (2.4) a " projective change " of the Cartan function $\Gamma^{*i}_{jk}(x, \dot{x})$.

Contracting (2.4) with respect to the indices i and j, we get

$$\overline{\Gamma}^{*h}_{hk}(x, \dot{x}) = \Gamma^{*h}_{hk}(x, \dot{x}) + (n+1)\, p_k. \tag{2.6}$$

Differentiating (2.6) with respect to $\dot{x}^l$, we obtain

$$\dot{\partial}_l \overline{\Gamma}^{*h}_{hk}(x, \dot{x}) = \dot{\partial}_l \Gamma^{*h}_{hk}(x, \dot{x}) + (n+1)\, p_{lk}. \tag{2.7}$$

THEOREM 2.1. *If* $\mu^i_{(a)}$ *and* $\mu_{(a)i}$ *are the contravariant and covariant components of the first curvature vector of a curve of congruence then the following geometric entities are invariant under the projective change*:

$$T^i_{(a)} \overset{\text{def}}{=} \mu^i_{(a)} - \frac{1}{n+1}\{2\, g^{ki}\, \Gamma^{*h}_{hk} + g^{kj}\, \dot{\partial}_j \Gamma^{*h}_{hk}\, \dot{x}^i\} \tag{2.8}$$

and

$$T_{(a)i} \overset{\text{def}}{=} \mu_{(a)i} - \frac{1}{n+1}\{2\, \Gamma^{*h}_{hi} + g^{kj}\, g_{li}\, \dot{x}^l\, \dot{\partial}_j \Gamma^{*h}_{hk}\}. \tag{2.9}$$

Proof. If $\lambda^i_{(a)|\bar{k}}$ denotes the covariant derivative of $\lambda^i_{(a)}$ in the sense of Cartan under the projective change, then we have

$$\lambda^i_{(a)|\bar{k}} = \partial_k \lambda^i_{(a)} + \lambda^j_{(a)}\, \overline{\Gamma}^{*i}_{jk}(x, \dot{x}). \tag{2.10}$$

Hence we get in consequence of (2.4)

$$\lambda^i_{(a)|\bar{k}} - \lambda^i_{(a)|k} = \lambda^j_{(a)}\{2\,\delta^i_{(j}p_{k)} + p_{jk}\,\dot{x}^i\}. \tag{2.11}$$

With the help of equations (2.6) and (2.7) the above equation assumes the form

$$\lambda^i_{(a)|\bar{k}} - \lambda^i_{(a)|k} = \frac{1}{n+1}[\lambda^j_{(a)}\{\delta^i_j(\overline{\Gamma}^{*h}_{hk} - \Gamma^{*h}_{hk}) + \delta^i_k(\overline{\Gamma}^{*h}_{hj} - \Gamma^{*h}_{hj}) + + \dot{x}^i\,\dot{\partial}_j(\overline{\Gamma}^{*h}_{hk} - \Gamma^{*h}_{hk})\}]. \tag{2.12}$$

Multiplying (2.12) by $\lambda^k_{(a)}$ and using the orthogonality condition (1.7), we obtain

$$\bar{\mu}^i_{(a)} - \frac{1}{n+1}\{g^{ki}\,\overline{\Gamma}^{*h}_{hk} + g^{ij}\,\overline{\Gamma}^{*h}_{hj} + g^{kj}\,\dot{x}^i\,\dot{\partial}_j\overline{\Gamma}^{*h}_{hk}\} = = \mu^i_{(a)} - \frac{1}{n+1}\{g^{ki}\,\Gamma^{*h}_{hk} + g^{ij}\,\Gamma^{*h}_{hj} + g^{kj}\,\dot{x}^i\,\dot{\partial}_j\Gamma^{*h}_{hk}\}, \tag{2.13}$$

where

$$\overline{\mu}^{i}_{(a)} \overset{\text{def}}{=} \lambda^{i}_{(a)|\bar{k}} \lambda^{k}_{(a)} . \tag{2.14}$$

Again multiplying (2.12) by the product $\lambda^{k}_{(a)} g_{il}$ and using (1.7), we get

$$\overline{\mu}_{(a)l} - \frac{1}{n+1} \{ 2 \overline{\Gamma}^{*h}_{hl} + g^{kj} g_{il} \dot{x}^{i} \dot{\partial}_{j} \overline{\Gamma}^{*h}_{hk} \} = \tag{2.15}$$

$$= \mu_{(a)l} - \frac{1}{n+1} \{ 2 \Gamma^{*h}_{hl} + g^{kj} g_{il} \dot{x}^{i} \dot{\partial}_{j} \Gamma^{*h}_{hk} \} ,$$

where

$$\overline{\mu}_{(a)l} \overset{\text{def}}{=} \lambda^{i}_{(a)|\bar{k}} \lambda^{k}_{(a)} g_{il} . \tag{2.16}$$

THEOREM 2.2. *The necessary and sufficient condition, that a curve of congruence whose unit tangent is $\lambda_{(l)}$ satisfies the relation $\mu_{(l)} = 0$, is that*

$$\Gamma^{*j}_{ij} (x , \dot{x}) - \lambda^{j}_{(l)} \partial_{j} \lambda_{(l)i} = 0 \tag{2.17}$$

holds.

Proof. In consequence of (1.8) and (1.9), we get, using the relation $\mu_{(l)} = 0$

$$\{ \partial_{j} \lambda_{(l)i} - \lambda_{(l)m} \Gamma^{*m}_{ij} \} \lambda^{i}_{(h)} \lambda^{j}_{(l)} \lambda_{(h)} = 0 , \tag{2.18}$$

which yields, in view of (1.7) that

$$\{ (\partial_{j} \lambda_{(l)i}) \lambda^{j}_{(l)} - \Gamma^{*j}_{ij} \} \lambda^{i}_{(h)} \lambda_{(h)} = 0. \tag{2.19}$$

Conversely, if the condition (2.17) is satisfied, it is obvious that

$$\mu^{i}_{(l)} = 0. \tag{2.20}$$

3. PROJECTIVE CHANGE OF THE INDUCED CONNECTION IN THE SUBSPACE OF A FINSLER SPACE.

An m-dimensional subspace F_m of a Finsler space F_n is represented parametrically by the equations

$$x^{i} = x^{i} (u^{\alpha}) \text{ (3)}, \tag{3.1}$$

We suppose that the variables u^{α} form a co-ordinate system of F_m. We further assume that the functions (3.1) are of class c^4 ([1], page 155) and we put

$$B^{i}_{\alpha} \overset{\text{def}}{=} \partial x^{i} / \partial u^{\alpha}. \tag{3.2}$$

The induced connection parameters are expressed as in [1], page 159

$$\Gamma^{*\alpha}_{\beta\gamma} = B^{\alpha}_{i} (B^{i}_{\beta\gamma} + \Gamma^{*i}_{hk} B^{hk}_{\beta\gamma}), \tag{3.3}$$

(3) Latin indices run from 1 to n, Greek indices from 1 to n.

where

(3.4) $$B^i_{\beta\gamma} \overset{\text{def}}{=} \frac{\partial^2 x^i}{\partial u^\beta \, \partial u^\gamma},$$

and

(3.5) $$B^{hk}_{\alpha\beta} = B^h_\alpha \, B^k_\beta .$$

THEOREM 3.1. *When the connection* $\Gamma^{*i}_{jk}(x, \dot{x})$ *of a Finsler space undergoes a projective change, the same is true of the induced connection of any subspace and*

(3.6) $$\delta^\alpha_\beta \, \Phi_\gamma = B^\alpha_h \, p_k \, B^{hk}_{\beta\gamma},$$

where the vectors $\Phi_\gamma(u, \dot{u})$ *determine the projective transformation in the subspace.*

Proof. Substituting the value of $\Gamma^{*i}_{jk}(x, \dot{x})$ from (2.4), the equation (3.3) reduces to the form

(3.7) $$\Gamma^{*\alpha}_{\beta\gamma} + B^\alpha_i \, (2 \, \delta^i_{(h} p_{k)} + p_{hk} \, \dot{x}^i) \, B^{hk}_{\beta\gamma} = B^\alpha_i \, (B^i_{\beta\gamma} + \overline{\Gamma}^{*i}_{hk} \, B^{hk}_{\beta\gamma}).$$

If we put

(3.8) $$\overline{\Gamma}^{*\alpha}_{\beta\gamma}(u, \dot{u}) \overset{\text{def}}{=} \Gamma^{*\alpha}_{\beta\gamma} + 2 \, \delta^\alpha_{(\beta} \Phi_{\gamma)} + \Phi_{\beta\gamma}(u, \dot{u}) \, \dot{u}^\alpha,$$

where $\Phi_\gamma(u, \dot{u})$ are the covariant vectors positively homogeneous of zero degree in their directional argument, then the equation (3.7) becomes

(3.9) $$\overline{\Gamma}^{*\alpha}_{\beta\gamma} = B^\alpha_i \, (B^i_{\beta\gamma} + \overline{\Gamma}^{*i}_{hk} \, B^{hk}_{\beta\gamma}),$$

provided we have

(3.10)a $$\delta^\alpha_\beta \, \Phi_\gamma = B^\alpha_i \, p_k \, \delta^i_h \, B^{hk}_{\beta\gamma}$$

and

(3.10)b $$\Phi_{\beta\gamma} \, \dot{u}^\alpha = B^\alpha_i \, B^{hk}_{\beta\gamma} \, p_{hk} \, \dot{x}^i .$$

From (3.10)a it is clear that $(m - n)$ independent vectors $\Phi_\alpha(u, \dot{u})$ exist such that there is no change in the induced connection.

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