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Normal Partial Isometries Closed under Multiplication on Unitary Spaces

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Algebra lineare. — *Normal Partial Isometries Closed under Multiplication on Unitary Spaces.* Nota di IVAN ERDELYI, presentata [(*)] dal Socio B. SEGRE.

RIASSUNTO. — Limitatamente alle matrici non-singolari, le isometrie godono notoriamente di normalità e, rispetto al prodotto, formano un gruppo. Fuori dalla non-singolarità, occorrono condizioni supplementari affinché si conservino e la normalità e la chiusura; alcune fra queste formano l'oggetto della presente Nota.

INTRODUCTION.

The isometries on unitary spaces are invertible operators closed under multiplication. Their straightforward extensions to the non-invertible case are the normal partial isometries. Since the spectral numbers still remain extreme points of the unit circle joined by the isolated origin, and the exponential correspondance with Hermitian operators enjoys a powerful generalization [1], the closure under multiplication is lost beyond the nonsingular field.

It is the purpose of this paper to extend the isometries to particular classes of normal partial isometries closed under multiplication on finite-dimensional unitary spaces, E^n.

A partial isometry V, in terms of Moore-Penrose's concept of the generalized inverse V^+ (see e.g. some of the original papers [2, 3, 4]) is expressible by

$$V^* = V^+ \quad (* \text{ for the conjugate transpose}),$$

and normality, in addition, implies [(1)]

$$VV^+ = V^+V. \tag{1}$$

Some properties are referred to in the following discussion. These include:

– each of the following four conditions is necessary and sufficient that a linear transformation V be a partial isometry:

$$V = VV^*V \quad , \quad V^* = V^*VV^*,$$

V^*V is a projection on the carrier $R(V^*)$ of V,
VV^* is a projection on the range $R(V)$ of V;

(*) Nella seduta del 10 febbraio 1968.

(1) The group-inverse concept for square matrices [5, pp. 120–123], allows a defining the normal partial isometries on a unitary space by the sole functional relation: $A^* = A^{\#}$ ($\#$ for the group-inverse).

– if λ is an eigenvalue and x the corresponding eigenvector of a partial isometry V, then (2) [6, pp. 459–460]

$$|\lambda| = \frac{\|V^* V x\|}{\|x\|} ; \tag{2}$$

– for any conformable unitary U and V,

$$(UAV)^+ = V^* A^+ U^*, \tag{3}$$

A normal partial isometry is the direct sum of a unitary and a zero operator. In fact, if V is normal its null-space is invariant under both V and V*, and therefore so is its orthogonal complement. If moreover, V is a partial isometry then V is isometric on that orthogonal complement, and hence, in the finite-dimensional case, the restriction of V to the initial space is unitary.

Furthermore, the nonzero eigenvalues of a normal partial isometry V have modulus 1 and, by property (2), its eigenvectors can only be located on the initial space R (V*) and on the null-space N (V).

A useful characterization of normal partial isometries involves the EPr matrices introduced by H. Schwerdtfeger [7], as a generalization of normality. A square matrix A of rank r is called EPr if A and A* have the same null-spaces. The EPr matrices are recognizable by an amount of noticeable properties proved by M. H. Pearl and I. J. Katz [8, 9, 10, 11, 12]. It is easy to ascertain that a normal partial isometry is equivalent to an EPr partial isometry.

MULTIPLICATIVE PROPERTIES.

The product of matrices, in general, preserves neither normality, nor isometry (3). It is well-known that if two normal matrices commute their product is normal. This condition suffices for transmitting normal partial isometry through multiplication. In fact, A is normal if and only if there exists a polynomial p such that $A^* = p(A)$. Hence it follows that for two commuting normal operators A and B, the operators A , A*, B , B* commute pairwise. The product $P = AB$ of two commuting normal partial isometries, besides being normal, is a partial isometry since, by a succession of factor permutations, there follows:

$$PP^* P = AB \cdot B^* A^* \cdot AB = AA^* A \cdot BB^* B = AB = P.$$

However, this condition is not necessary.

(2) This formulation was suggested to the author by A. Ben–Israel.

(3) Conditions for transmission of partial isometry through multiplication are to be found in [6, pp. 464–466; 13].

When the matrix factors A and B have equal ranks, a more powerful condition is available:

THEOREM 1. *The product*

$$P = AB$$

of two normal partial isometries A *and* B, *both of rank* r, *is a normal partial isometry of rank* r *if and only if they have equal ranges*

$$R(A) = R(B). \tag{4}$$

Proof. if:

Since A and B are normal partial isometries, $AA^* = A^*A$ and $BB^* = B^*B$ are idempotent and, in addition, the hypotheses on the equal ranks and common range imply

$$AA^* = BB^*. \tag{5}$$

We then have successively:

$$PP^* = A \cdot BB^* \cdot A^* = A \cdot AA^* \cdot A^* = (AA^*)^2 = AA^*,$$

$$P^*P = B^* \cdot A^*A \cdot B = B^* \cdot B^*B \cdot B = (BB^*)^2 = BB^*,$$

and hence $PP^* = P^*P$ is a projection. Thus, P is a normal partial isometry of rank r.

Only if:

Let

$$A = U \left[\begin{array}{c:c} W & \\ \hdashline & O \end{array}\right] U^*, \tag{6}$$

$$B = V \left[\begin{array}{c:c} W_1 & \\ \hdashline & O \end{array}\right] V^* \tag{7}$$

be the unitary decompositions of A and B, where the couples U, V and W, W_1 are n by n and r by r unitary matrices, respectively. If we denote by

$$U^*V = \left[\begin{array}{c:c} Q & S_1 \\ \hdashline S_2 & R \end{array}\right] \tag{8}$$

the conformable partitioned form of the unitary U^*V, the product P is expressible as

$$P = U \left[\begin{array}{c:c} WQW_1 & \\ \hdashline & O \end{array}\right] V^*.$$

Since P is supposed to have rank r, Q is nonsingular. Then, by property (3), the generalized inverse of the product P is given by

$$P^+ = V \left[\begin{array}{c:c} (WQW_1)^{-1} & \\ \hdashline & O \end{array}\right] U^*.$$

The normality condition (1) for P, $PP^+ = P^+P$, gives

$$U \left[\begin{array}{c:c} I_r & \\ \hdashline & O \end{array}\right] U^* = V \left[\begin{array}{c:c} I_r & \\ \hdashline & O \end{array}\right] V^*,$$

or

$$\left[\begin{array}{c|c} I_r & \\ \hline & O \end{array}\right] U^* V = U^* V \left[\begin{array}{c|c} I_r & \\ \hline & O \end{array}\right] \tag{9}$$

where I_r is the *rth* order identity matrix.

Now, we introduce the partitioned form (8) for $U^* V$ into the foregoing relation (9) and, after some elementary algebra, we obtain

$$S_1 = O_{r,n-r} \quad , \quad S_2 = O_{n-r,r}.$$

Thus $U^* V$ is block diagonal

$$U^* V = \left[\begin{array}{c|c} Q & \\ \hline & R \end{array}\right] \tag{10}$$

and, necessarily, Q and R are unitary of appropriate sizes. From (10), we derive

$$V = U \left[\begin{array}{c|c} Q & \\ \hline & R \end{array}\right] \quad , \quad V^* = \left[\begin{array}{c|c} Q^* & \\ \hline & R^* \end{array}\right] U^*$$

and hence by (7),

$$B = U \left[\begin{array}{c|c} Q & \\ \hline & R \end{array}\right] \cdot \left[\begin{array}{c|c} W_1 & \\ \hline & O \end{array}\right] \cdot \left[\begin{array}{c|c} Q^* & \\ \hline & R^* \end{array}\right] U^* = U \left[\begin{array}{c|c} QW_1 Q^* & \\ \hline & O \end{array}\right] U^*$$

or

$$B = U \left[\begin{array}{c|c} W_2 & \\ \hline & O \end{array}\right] U^* \tag{11}$$

where $W_2 = QW_1 Q^*$ is unitary.

Condition (5) follows from (6) and (11),

$$AA^* = BB^* = U \left[\begin{array}{c|c} I_r & \\ \hline & O \end{array}\right] U^*,$$

and this leads to the equality (4) of the ranges.

We note that in the hypotheses of Theorem 1,

$$R(A) = R(B) = R(AB).$$

COROLLARY. *If* A, B *and* AB *are normal partial isometries of rank r, then* BA *is a normal partial isometry of rank r. Moreover,* $R(AB) = R(BA)$.

The transmission of normal partial isometry through multiplication admits a straightforward extension for any finite number of factors, in terms of the following

THEOREM 2. *Let* $A_1, A_2, \cdots, A_m$ *be m normal partial isometries of rank r. Any partial product*

$$P_{ij} = A_i A_{i+1}, \cdots, A_j, \qquad 1 \leq i < j \leq m,$$

is a normal partial isometry of rank r if and only if each factor has the same range:

$$R(A_1) = R(A_2) = \cdots = R(A_m). \tag{12}$$

Proof. When $m = 2$, it reduces to Theorem 1. Assume that the statements of Theorem 2 are true for $m = q$, and consider

$$P_{i,q+1} = P_{iq} A_{q+1}$$

where A_{q+1} is a normal partial isometry of rank r.

By Theorem 1, $P_{i,q+1}$ is a normal partial isometry of rank r if and only if

$$P_{iq} P_{iq}^* = A_{q+1} A_{q+1}^* .$$

On the other hand, the range equalities (12) equivalent to

$$A_1 A_1^* = A_2 A_2^* = \cdots = A_m A_m^* ,$$

lead us to the following sequence of relations:

$$P_{iq} P_{iq}^* = P_{i,q-2} A_{q-1} (A_q \cdot A_q^*) A_{q-1}^* P_{i,q-2}^* = P_{i,q-2} A_{q-1} (A_{q-1} A_{q-1}^*) A_{q-1}^* P_{i,q-2}^* =$$
$$= P_{i,q-2} (A_{q-1} A_{q-1}^*)^2 P_{i,q-2}^* = P_{i,q-2} A_{q-1} \cdot A_{q-1}^* P_{i,q-2}^* = P_{i,q-1} P_{i,q-1}^* ,$$

and, further on

$$P_{iq} P_{iq}^* = P_{i,q-1} P_{i,q-1}^* = P_{i,q-2} P_{i,q-2}^* = \cdots = A_i A_i^* .$$

Thus, by induction, the proof is complete.

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