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On Optimal Solutions of 2-person O-sum Games

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Ricerca operativa. — *On Optimal Solutions of 2–person O–sum Games.* Nota di ADI BEN–ISRAEL, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Soluzioni ottimali per giochi « 2–persone, somma–O » vengon qui caratterizzate da certe sottomatrici della matrice di retribuzione.

NOTATIONS AND PRELIMINARIES.

We denote by:

R^n — the n–dimensional real vector space

$R^n_+ = \{x \in R^n : x \geqq 0\}$ the nonnegative orthant in R^n

e = the vector whose components are all 1 and whose dimension is to be determined by the context

$P^n = \{x \in R^n_+ : e^T x = 1\}$.

Let the $m \times n$ real matrix A be the payoff matrix of a 2–person O–sum game which, as is well known, is equivalent to the following pair of dual linear programs:

$$\text{(I)} \quad \max t \qquad A^T x \geqq te \qquad x \in P^m$$

$$\text{(II)} \quad \min u \qquad Ay \leqq ue \qquad y \in P^n$$

where

x is the strategy of the maximizing (row) player

y is the strategy of the minimizing (column) player

and the common optimal value

$$\max t = \min u = v$$

is the *value of the game*.

Let $S = (a_{ij})\ i \in I, j \in J$ be any submatrix of A, i.e. the intersection of the rows of A with indices in I and of the columns with indices in J, where $I \subset \{1, 2, \cdots, m\}$, $J \subset \{1, \cdots, n\}$.

For any vector $x = (x_i) \in R^m$ we denote by

$$x(S) = (x_i) \qquad i \in I$$

$$\bar{x}(S) = (x_i) \qquad i \notin I.$$

Similarly

$$y(S) = (y_j) \qquad j \in J$$

$$\bar{y}(S) = (y_j) \qquad j \notin J,$$

for any vector $y \in R^n$.

(*) Nella seduta del 20 aprile 1968.

We denote by

$$A(S/) = (a_{ij}) \qquad i \in I, \quad J = 1, \cdots, n$$

$$A(/S) = (a_{ij}) \qquad i = 1, \cdots, m, \quad j \in J$$

the submatrices of A consisting of the rows resp columns of A in S.

For any matrix S we denote by

R (S) the range space of S

N (S) the null space of S

S^+ the generalized inverse of S, e.g. [3], [1].

We recall that

(1) $$R(S) = N(S^T)^{\perp}$$

and that

(2) $$x = S^+ s + N(S)$$

is the general solution of

(3) $$Sx = s$$

whenever solvable.

RESULTS.

Optimal strategies of 2-person O-sum games, i.e. optimal solutions x, y of (I), (II), are characterized in the following

THEOREM.—*Assumptions*: (I), (II) *a game with nonzero value* $v \neq 0$

$$x \in R^m_+ \quad , \quad y \in R^n_+ .$$

Conclusions: x, y are optimal strategies if, and only if, there is a submatrix S of the payoff matrix A such that:

(4) $$e \in R(S)$$

(5) $$e \in R(S^T)$$

(6) $$x(S) = \frac{S^{T+} e}{e^T S^+ e} + w, \qquad \text{where} \quad w \in N(S^T)$$

(7) $$\bar{x}(S) = 0$$

(8) $$y(S) = \frac{S^+ e}{e^T S^+ e} + z, \qquad \text{where} \quad z \in N(S)$$

(9) $$\bar{y}(S) = 0$$

(10) $$A(S/)^T x(S) \geqq \frac{1}{e^T S^+ e} e$$

(11) $$A(/S) y(S) \leqq \frac{1}{e^T S^+ e} e .$$

Proof:

1. *If*:

$$e^T x = e^T x(S) \quad , \quad \text{by (7)}$$

$$= \frac{e^T S^{T+} e}{e^T S^+ e} \quad , \quad \text{by (6), (4), } w \in N(S^T) \text{ and (1)}$$

$$= 1 \quad , \quad \text{since } S^{T+} = S^{+T}.$$

$$\therefore \quad x \in P^m.$$

Similarly: $y \in P^n$.
Finally

$$x^T Ay = x(S)^T Sy(S), \quad \text{by (7), (9)}$$

$$= \frac{e^T S^+ SS^+ e}{(e^T S^+ e)^2} \quad , \quad \text{by (6), (8)}$$

$$= \frac{1}{e^T S^+ e} \quad , \quad \text{since } S^+ SS^+ = S^+$$

which, together with (10) and (11), proves the optimality of x, y, and moreover that the value of the game is:

$$v = \frac{1}{e^T S^+ e} \,. \tag{12}$$

2.—*Only if*:

Let x, y be optimal strategies, and let S be the submatrix of A defined as the intersection of the rows of A with equality in

$$Ay \leqq ve$$

and of the columns of A with equality in:

$$A^T x \geqq ve.$$

The complementary slackness theorem of linear programming guarantees that

$$x^T (ve - Ay) = 0$$

$$y^T (A^T x - ve) = 0$$

therefore

$$\bar{x}(S) = 0 \quad , \quad \bar{y}(S) = 0, \qquad \text{proving (7), (9),}$$

and

$$S^T x(S) = ve \tag{13}$$

$$Sy(S) = ve \tag{14}$$

which proves (4), (5).

Using (2) we get from (14) that

(15) $$y(S) = vS^{+}e + z, \qquad z \in N(S)$$

Therefore

(16) $$\begin{aligned} 1 &= e^{T} y \\ &= e^{T} y(S), \qquad \text{by (9)} \\ &= ve^{T} S^{+} e, \qquad \text{by (15), } z \in N(S) \text{ and (5)} \end{aligned}$$

which proves (12).

(8) follows from (15) and (12).

(6) is similarly proved.

(10) and (11) follow from (12) and the optimality of x, y.

The following corollaries similarly characterize *basic optimal strategies*, i.e. vertices of the polyhedra of optimal solutions. They follow from the theorem, by using the well known characterization of vertices, e.g. [2]:

x_0 is a vertex of the nonempty polyhedron

$$K = \{x : Ax = b, x \geqq 0\}$$

if, and only if, $x_0 \in K$ and the columns of A corresponding to the positive components of x_0 are linearly independent.

COROLLARY 1.—*Assumptions: Same as in the theorem.*

Conclusions: x is a basic optimal strategy, and y is an optimal strategy if, and only if, there is a submatrix S of A such that:

(4′) S has full row rank (i.e. the rows of S are l.i.; (4) clearly holds).

(5) $$e \in R(S^{T})$$

(6′) $$x(S) = \frac{S^{T+} e}{e^{T} S^{+} e} \qquad \text{(note that } N(S^{T}) = \{0\})$$

(7)–(11) as in the theorem.

Similarly, submatrices S with full column rank characterize the basic optimal strategies y.

Combining the above results we obtain (a variant of) the well known characterization of basic optimal strategies given by Shapley and Snow ([4] theorem 2):

COROLLARY 2.—*Assumptions: Same as in the theorem.*

Conclusions: x , y are basic optimal strategies if, and only if, there is a nonsingular submatrix S_0 of A such that:

$$x(S_0) = \frac{(S_0^{-1})^T e}{e^T S_0^{-1} s} \tag{6''}$$

$$\bar{x}(S_0) = 0 \tag{7''}$$

$$y(S_0) = \frac{S_0^{-1} e}{e^T S_0^{-1} e} \tag{8''}$$

$$\bar{y}(S_0) = 0 \tag{9''}$$

$$A(S/)^T x(S) \geqq \frac{1}{e^T S_0^{-1} e} e \tag{10''}$$

$$A(/S) y(S) \leqq \frac{1}{e^T S_0^{-1} e} e . \tag{11''}$$

The matrix S_0 is any maximal nonsingular submatrix of S in the theorem, for which (7′′) and (9′′) hold.

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