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A note on connectedness

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Matematica. — *A note on connectedness.* Nota di ASRIEL EVYATAR e MEIR REICHAW, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Caratterizzazioni di alcuni tipi di connessione per sottoinsiemi di uno spazio di Banach ad infinite dimensioni.

Let Y be a topological space and let $J \subset Y$ be a subset of Y. The questions: when is $Y - J$ connected, locally connected, arcwise connected or locally arcwise connected have been investigated by a number of writers. Some classical answers can be formulated for closed subsets J of a Banach space Y in terms of extension properties of compact fields J (see [6]). Another answer to the above questions follows from results obtained recently in [1]—[5], [9] and [10]. In particular, the results obtained in [1], [2] and [9] imply that if Y is a *separable* infinite dimensional Banach space and $\{K_i\}$ $i = 1, 2, \cdots$ a sequence of compact subsets of Y then $Y - \bigcup_{i=1}^{\infty} K_i$ is a locally arcwise connected, arcwise connected space. The first of these questions is related also to the important problem of finding conditions under which a mapping $f: X \to Y$ maps X onto Y (see [11], Theorem 2, p. 1400). In this paper some conditions are found (Theorem 1) for a set $J \subset Y$ so that the set $Y - J$ turns out to be locally arcwise connected and arcwise connected for locally normed topological (not necessarily linear, see Definition 1) spaces Y. These spaces include connected open subsets Y of infinite dimensional Banach spaces when $J \subset \bigcup_{i=1}^{\infty} K_i$, where K_i are compact sets $i = 1, 2, \cdots$ (Theorem 2) and connected open subsets Y of a Euclidean $(n+2)$ — dimensional space when $J \subset \bigcup_{i=1}^{\infty} K_i$, where K_i $i = 1, 2, \cdots$ are compact spaces of dimension $\leq n$ (corollary 1). Theorem 3 is related to a result of A. Sard (see [12]).

In what follows $B(\varepsilon)$ denotes the open ball $\{x ; \|x\| < \varepsilon\}$ in a normed space, $\overline{P}$ the closure of P, δP the boundary of P, " iff " stands for " if and only if " and " *nbd* " for " neighborhood ". Finally, $[x, z]$ is the closed interval with endpoints x, z, $\widehat{y, z}$ is an arc with endpoints y, z, and a locally complete space is a space Y such that for each point $y \in Y$ there exists a *nbd* of y homeomorphic with a complete metric space.

Definition 1.—A space Y will be called locally normed iff for every point $y \in Y$ there exists an open ball $B(\varepsilon)$ contained in some normed space $X = X(y)$ and a homeomorphism $h = h_y$ defined on $\overline{B(\varepsilon)}$ such that $D = D(y) = h(B(\varepsilon))$ is an open subset of Y, $h(\overline{B(\varepsilon)}) = \overline{D(y)}$ and $h(0) = y$.

(*) Nella seduta dell'8 giugno 1968.

In the sequel $D = D(y) = h(B(\varepsilon))$ and h will stand for the sets and the homeormorphism defined above.

Definition 2.—Let Y be a locally normed space and let $D = h(B(\varepsilon))$ be an open *nbd* of $y \in Y$. Let $A \subset Y$, $K = A \cap \bar{D}$ and let $y_0 \in \bar{D}$. The $\bar{D}$-cone $C(A, y_0, \bar{D})$ is defined as follows: take the point $x \in h^{-1}(K)$ and let $r(x)$ be the ray starting at $x_0 = h^{-1}(y_0)$ and passing through x. Let $z = z(x)$ be the (unique) point of intersection $z \in r(x) \cap \partial B$ (if $x = x_0$, take $z(x) = x$) and let $C_0 = \cup \{[x_0, z(x)]; x \in h^{-1}(K)\}$. We put $C(A, y_0, \bar{D}) = h(C_0)$ and call this set the $\bar{D}$-cone with vertex y_0 passing through A.

Definition 3.—A subset $A \subset Y$ will be called strongly nowhere dense at the point $y \in Y$ (*snd* at y) iff there exists an open *nbd* $D = h(B(\varepsilon))$ of y such that for every $y_0 \in \bar{D}$ the $\bar{D}$-cone $C(A, y_0, \bar{D})$ is nowhere dense in $\bar{D}$. A subset $A \subset Y$ is *snd* iff it is *snd* at every point $y \in Y$. A family F of sets A is said to be *snd* at $y \in Y$ iff there exists an open *nbd* $D = h(B(\varepsilon))$ of y such that for every A of F and every $y_0 \in \bar{D}$ the $\bar{D}$-cone $C(A, y_0, \bar{D})$ is nowhere dense in $\bar{D}$. If F is *snd* at every point $y \in Y$ then F is said to be *snd*.

Examples.—(*a*) If $Y \subset E^{n+2}$ is an open subset of the $(n+2)$—dimensional Euclidean space E^{n+2}, then the family of all sets $A \subset Y$ for which there exists a compact n-dimensional (see [8]) subset $K \subset E^{n+2}$, with $A \subset K$, is *snd*.

(*b*) If Y is an open subset of an infinite dimensional Banach space Y_1 then the family of all subsets $A \subset Y$ for which there exists a compact subset K of Y_1 with $A \subset K$ is *snd*.

LEMMA 1.—*If* Y *is a locally normed space,* $y \in Y$ *and* $A \subset Y$ *is snd at* y *then for every open nbd* U *of* y *there exists an open nbd* $D = D(y) = h(B(\varepsilon))$ *of* y *such that* $\bar{D} \subset U$, $\partial D = \bar{D} - D \neq \varnothing$, *and such that for every point* $y_0 \in \bar{D}$ *the set* $C(A, y_0, \bar{D}) \cap \partial D$ *is nowhere dense in* ∂D.

Proof.—By definition 3 and definition 1 there exists an open *nbd* $D = h(B(\varepsilon))$ with $\bar{D} \subset U$, $\partial D = \bar{D} - D \neq \varnothing$. Suppose now to the contrary that there exists a point $y_0 \in \bar{D}$ such that $\overline{C(A, y_0, \bar{D})} \cap \partial D$ contains an open (in ∂D) subset $L \neq \varnothing$. Then $\varnothing \neq h^{-1}(L) \subset \partial(B(\varepsilon))$. Thus the set $\cup \{[h^{-1}(y_0), x]; x \in h^{-1}(L)\}$ contains an open subset of $B(\varepsilon)$. Hence $C(A, y_0, \bar{D})$ is not nowhere dense in $\bar{D}$, contradicting the assumption that A is *snd* at y.

LEMMA 2.—*Let* $F = \{A_i\}_{i=1,2,\ldots}$ *be a sequence of subsets of a locally normed space* Y *and let* $D = h(B(\varepsilon))$ *be an open nbd of* $y \in Y$, *such that* $C(A_i, y_0, \bar{D})$ *is nowhere dense in* $\bar{D}$ *for every* $y_0 \in \bar{D}$. *Let* $S = \{y_j\}_{j=1,2,\ldots}$ *be a sequence of points contained in* $\bar{D} - \bigcup_{i=1}^{\infty} A_i$. *Then the set of all points* $z_0 \in \partial D$ *such that*

$$C(S, z_0, \bar{D}) \cap \left(\bigcup_{i=1}^{\infty} A_i\right) \neq \varnothing \tag{1}$$

is of the first category in ∂D.

Proof.—For each y_j we have as in Lemma 1, that $C(A_i, y_j, \bar{D}) \cap \partial D$ is nowhere dense in ∂D. Thus $\bigcup_{i,j} C(A_i, y_j, \bar{D}) \cap \partial D$ is of the first category in ∂D and the Lemma holds.

THEOREM 1.—*If* Y *is a locally normed connected and locally complete space and* $F = \{A_i\}_{i=1,2,\ldots}$ *is a sequence of sets which is snd then* $Y - \bigcup_{i=1}^{\infty} A_i$ *is a locally arcwise connected and arcwise connected space.*

Proof.—Let $y \in Y$ be a given point and let U be an arbitrary open *nbd* of y. Applying Lemma 1 and Lemma 2 one obtains an open *nbd* $D = D(y) = h(B(\varepsilon))$ with $\bar{D} \subset U$ such that for every two points y_1 and y_2 of $\bar{D} - \bigcup_{i=1}^{\infty} A_i$ the set of all points $y_0 \in \partial D$ for which $C(S, y_0, \bar{D}) \cap (\bigcup_{i=1}^{\infty} A_i) \neq \varnothing$ is of the first category in ∂D, where $S = \{y_1, y_2\}$ is the sequence consisting of the two points y_1 and y_2. Since by Lemma 1 $\partial D \neq \varnothing$ it follows by the local completeness of Y that there exists a point z_0 for which (1) does not hold. Hence there exist two arcs $R_1 = \widehat{y_1, z_0}$ and $R_2 = \widehat{y_2, z_0}$ contained in $\bar{D} - \bigcup_{i=1}^{\infty} A_i$ (we have even $R_1 \cap R_2 = \{z_0\}$). Thus

(2) for each open *nbd* U of y there exists an open *nbd* $D \subset \bar{D} \subset U$ such that every two points y_1 and y_2 of $D - \bigcup_{i=1}^{\infty} A_i$ can be joined by an arc in U.

If follows that $Y - \bigcup_{i=1}^{\infty} A_i$ is locally arcwise connected. Let now z and z^* be arbitrary points of Y. Since Y is connected there exists by (2) a simple chain $D_j = D(z_j) = h(B(\varepsilon_j))$ $j = 1, 2, \cdots, n$ of open neighborhoods (see [7], p. 108) with $z_1 = z$ and $z_n = z^*$ such that $D(z_j) - \bigcup_{i=1}^{\infty} A_i$ is arcwise connected, $j = 1, 2, \cdots, n$. Moreover since Y is locally complete, and the sequence $\{A_i\}$ is *snd* it follows that for every $j = 1, 2, \cdots, n-1$ one has $(D(z_j) - \bigcup_{i=1}^{\infty} A_i) \cap (D(z_{j+1}) - \bigcup_{i=1}^{\infty} A_i) \neq \varnothing$. Hence $Y - \bigcup_{i=1}^{\infty} A_i$ is arcwise connected. Theorem 1 is proved.

COROLLARY 1.—*Let* $Y \subset E^{n+2}$ *be an open, connected subset of a* $(n+2)$*—dimensional Euclidean space* E^{n+2} *and let* $\{A_i\}$ *and* $\{K_i\}$ *be sequences of sets of* E^{n+2} *with* $A_i \subset K_i$ *and* K_i *compact and at most n–dimensional (in the sense of Menger–Urysohn, see* [8]). *Then* $Y - \bigcup_{i=1}^{\infty} A_i$ *is a locally arcwise connected and arcwise connected set.*

Proof.—The space Y is a locally normed connected and locally complete space and the sequence $\{A_i\}$ is *snd*. It remains to apply Theorem 1.

We prove now

THEOREM 2.—*Let* Y *be an open connected subset of an infinite dimensional Banach space* Y_1 *and let* $\{A_i\}$ *and* $\{K_i\}$ *be sequences of sets of* Y_1 *with* $A_i \subset K_i$ *and* K_i *compact. Then* $Y - \bigcup_{i=1}^{\infty} A_i$ *is a locally arcwise connected and arcwise connected set.*

Proof.—Since Y_1 is infinite dimensional and K_i are compact $i = 1, 2, \cdots$ the sequence $\{A_i\}$ is *snd*. Y being connected and locally complete, it remains to apply Theorem 1.

COROLLARY 2.—*Let* $\{y_n\}_{n=1,2,\ldots}$ *be a sequence of points in an open connected subset* Y *of an infinite dimensional Banach space* Y_1 *and let* $\{Q_{n,i}\}_{i=1,2\cdots}$ *be a sequence of finite dimensional planes passing through* y_n. *Then* $Y - \bigcup_{n,i} Q_{n,i}$ *is a locally arcwise connected and arcwise connected set.*

Proof.—Represent each $Q_{n,i}$ as a countable union $Q_{n,i} = \bigcup_{j=1}^{\infty} K_{n,i,j}$ of compact subsets of Y_1 and apply Theorem 2.

The following example—communicated to the authors by A. Ran—shows that Theorem 2 does not have a natural generalization to infinite dimensional linear topological locally convex spaces.

Example.—Let $Y = l_2$ be the Hilbert space of all points $y = (y_1, y_2, \cdots)$ y_i—real numbers and $\sum_{i=1}^{\infty} y_i^2 < \infty$, with the *weak* topology. Let $A = \{a_i\}_{i=1,2,\ldots}$ be a dense (in the norm topology) sequence in Y and let $\overline{B}(a_i, 1) = \overline{B}_i = \{y ; \| y - a_i \| \leq 1\}$ be the closed ball of radius 1 and center a_i, $i = 1, 2, \cdots$ Take any two points $x_0 \neq y_0$ of Y. The set $Y - (\{x_0\} \cup \{y_0\})$ can be covered by the sequence $\{K_i\}$ of compact (in the weak topology) sets where $K_i = \overline{B}_i$, $i = 1, 2, \cdots$. But obviously $\{x_0\} \cup \{y_0\}$ is not connected.

We end the paper with the following

THEOREM 3.—*Let* $f : X \to Y$ *be a mapping (not necessarily continuous) of a second—countable topological space* X *into an infinite dimensional Banach space* Y *and let* Z *be the set of all points* $z_0 \in X$ *such that there exists an open set* $U(z_0) = U$ *with* $f(U)$ *contained in a finite dimensional plane* $Q = Q(z_0)$ *(depending on* z_0 *and* U*). Then* $f(Z)$ *is of the first category in* Y *and* $Y - f(Z)$ *is locally arcwise connected and arcwise connected.*

Proof.—Since X is second—countable one can find a countable family $U_n = U(z_n)$, $z_n \in Z$ covering Z. For every point z_n, the set $f(U_n)$ is contained in a finite dimensional plane $Q_n = Q(z_n)$. Thus $f(U_n)$ can be covered by a countable family of compact sets $K_{n,i}$ $i = 1, 2, \cdots$. Hence $f(Z) \subset \bigcup_{n,i} K_{n,i}$ and it follows that $f(Z)$ is of the first category in Y. By Theorem 2, $Y - f(Z)$ is also locally arcwise connected and arcwise connected.

Remark.—As easily seen Theorem 3 holds when Y is an arbitrary connected, open subset of an infinite dimensional Banach space.

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