Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Olaf Tamaschke

A Generalization of the Second Isomorphism Theorem in Group Theory

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **45** (1968), n.3-4, p. 135–141.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1968_8_45_3-4_135_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1968.

Matematica. *A Generalization of the Second Isomorphism Theorem in Group Theory.* Nota (*) di OLAF TAMASCHKE, presentata dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — Siano H e K sottogruppi di un gruppo G soddisfacenti alla HK = KH. Indichiamo con $HK/K := \langle KhK \mid h \in H \rangle$ e con $S(H/H \cap K) := \langle H \cap KhK \mid h \in H \rangle$ i semigruppi generati dagli assegnati sottoinsiemi di G con riferimento alla moltiplicazione fra « complessi ». HK/K è un semigruppo di Schur su HK, $S(H/H \cap K)$ è un semigruppo di Schur su H e risulta $HK/K \cong S(H/H \cap K)$. Se K è normale in G, questo risultato si riduce esattamente al secondo teorema sugli isomorfismi nella teoria dei gruppi.

Let G be a group, H a subgroup of G, and K a normal subgroup of G. Then the Second Isomorphism Theorem states:

(1) $H \cap K$ *is a normal subgroup of* H.

(2) K *is a normal subgroup of* HK.

(3) *The factor group* $H/H \cap K$ *is isomorphic to the factor group* HK/K.

We weaken the hypotheses of this theorem.

Let G be a group, and let H and K be subgroups of G such that

$$HK = KH.$$

Is there any isomorphy that can be stated for a factor structure of H modulo $H \cap K$ and a factor structure of HK modulo K? Which are the factor structures of such a hypothetical statement, and which is their notion of isomorphy?

First let us discuss a factor structure of HK modulo K. By HK/K we denote the semigroup with respect to subset (i.e. " complex ") multiplication which is generated by the double cosets KgK, $g \in HK$, that is every element of HK/K is the product of a finite number of double cosets KgK, $g \in HK$. We call HK/K the *double coset S–semigroup* of HK modulo K. (The " *S* " in that notation will be explained later.) Obviously, HK/K is a group if an only if K is a normal subgroup of HK in which case HK/K coincides with the factor group of HK modulo K. Therefore the double coset *S*–semigroup HK/K seems to be a suitable generalization of the factor group HK/K in the Second Isomorphism Theorem.

Our next aim is to find an appropriate factor structure of H modulo $H \cap K$. We denote by $S(H/H \cap K)$ the semigroup with respect to subset multiplication which is generated by all the intersections

$$H \cap KgK \quad , \quad g \in HK.$$

We note that each double coset KgK with $g \in HK$ can be written as KhK with a suitable $h \in H$.

(*) Pervenuta il 1° ottobre 1968.

LEMMA 1. *The following statements hold.*

(1) $$H = \bigcup_{h \in H} (H \cap KhK).$$

(2) $$H \cap KhK = H \cap Kh' K \quad \textit{or} \quad (H \cap KhK) \cap (H \cap Kh' K) = \varnothing \quad \textit{for all } h, h' \in H.$$

(3) $$(H \cap KhK)^{-1} := \{g^{-1} \mid g \in H \cap KhK\} = H \cap Kh^{-1} K \quad \textit{for all } h \in H.$$

(4) $$(H \cap X)(H \cap Y) = H \cap XY \quad \textit{for all} \quad X, Y \in HK/K.$$

Proof. Statements (1), (2), (3) are trivial. We prove (4).

I. Take any $\varnothing \neq Y \subseteqq HK$ such that $KY = Y$. Then

$$Y = \bigcup_{\substack{h \in H \\ Kh \subseteqq Y}} Kh,$$

$$H \cap Y = \bigcap_{\substack{h \in H \\ Kh \subseteqq Y}} H \cap Kh = \bigcap_{\substack{h \in H \\ Kh \subseteqq Y}} (H \cap K) h.$$

For every $h \in H$ the set $H \cap Kh = (H \cap K) h$ is the set of all representatives from H of the coset Kh. Therefore $H \cap Y$ is the set of all representatives from H for all of the cosets $Kh \subseteqq Y$, and hence

$$K(H \cap Y) = Y.$$

II. Take any $\varnothing \neq X \subseteqq HK$ such that $KXK = X$. Then, because $K(H \cap X) = X$, we obtain

$$K(H \cap X)(H \cap Y) = X(H \cap Y) = XK(H \cap Y) = XY.$$

Therefore $(H \cap X)(H \cap Y)$ contains a complete set of representatives from H for all the cosets $Kh \subseteqq XY$. Obviously

$$(H \cap K)(H \cap X)(H \cap Y) = (H \cap X)(H \cap Y)$$

holds which implies that $(H \cap X)(H \cap Y)$ contains *all* representatives from H for all the cosets $Kh \subseteqq XY$. Hence, by what we have proved in I,

$$(H \cap X)(H \cap Y) = H \cap XY.$$

All elements $X, Y \in HK/K$ have the property $KXK = X$ and $KY = Y$. Thus we have proved Lemma 1.

If we set $X = KhK$ and $Y = Kh' K$ with $h, h' \in H$ in statement (4) of Lemma 1, then we obtain

$$(H \cap KhK)(H \cap Kh'K) = \bigcup_{g \in (KhK)(Kh'K)} (H \cap KgK) \quad \text{for all } h, h' \in H.$$

Since every element of $S(H/H \cap K)$ is the product of a finite number of the sets $H \cap KhK$, $h \in H$, we have proved

LEMMA 2. *Every element of* $S(H/H \cap K)$ *is the union of some of the sets* $H \cap KhK$, $h \in H$.

Lemmas 1 and 2 tell us that $S(H/H \cap K)$ is a semigroup of a special type. For the convenience of the reader we recall the definition of that class of semigroups.

The set $\overline{G} := \{X \mid \varnothing \neq X \subseteq G\}$ is a semigroup with respect to the subset multiplication

$$(X, Y) \to XY := \{xy \mid x \in X \quad and \quad y \in Y\}.$$

DEFINITION 1 ([3], p. 74). *A subsemigroup* T *of* $\overline{G}$ *is called a* Schur-semigroup (*in short*: *S*-semigroup) *on* G *if it has a unit element and if there exists a set* $\mathfrak{T} \subseteq \overline{G}$ *such that*

(1) $$G = \bigcup_{\mathfrak{C} \in \mathfrak{T}} \mathfrak{C}.$$

(2) $$\mathfrak{S} = \mathfrak{C} \text{ or } \mathfrak{S} \cap \mathfrak{C} = \varnothing \text{ for all } \mathfrak{S}, \mathfrak{C} \in \mathfrak{T}.$$

(3) $$\mathfrak{C}^{-1} := \{g^{-1} \mid g \in \mathfrak{C}\} \in \mathfrak{T} \text{ for all } \mathfrak{C} \in \mathfrak{T}.$$

(4) $$X = \bigcup_{\substack{\mathfrak{C} \in \mathfrak{T} \\ \mathfrak{C} \cap X \neq \varnothing}} \mathfrak{C} \text{ for all } X \in T.$$

(5) T *is generated by* $\mathfrak{T}$, *that is every element of* T *is the product of a finite number of elements of* $\mathfrak{T}$.

Note that $\mathfrak{T}$ is uniquely determined by T and the axioms (1)-(5). Therefore we call the elements of $\mathfrak{T}$ the T-*classes* of G.

Thus Lemmas 1 and 2 show that $S(H/H \cap K)$ is an *S*-semigroup on H with the set $\{H \cap KhK \mid h \in H\}$ as the set of all $S(H/H \cap K)$-classes of H. Obviously, the double coset *S*-semigroup HK/K is an *S*-semigroup on HK (and that is the reason for having chosen the term double coset *S*-semigroup).

For the generalization of the Second Isomorphism Theorem which we are going to establish we take the *S*-semigroup $S(H/H \cap K)$ as a factor structure of H modulo $H \cap K$. Now we have both factor structures of our still hypothetical Isomorphism Theorem. We will deal now with the relevant notion of ismorphism.

Let F be a group, Σ an *S*-semigroup on F, and $\mathfrak{S}$ the set of all Σ-classes of F (that is $\mathfrak{S}$ plays the same rôle for Σ as $\mathfrak{T}$ does for T).

DEFINITION 2 ([3], Definition 2.1). *A mapping* φ *of* T *into* Σ *is called a* homomorphism *of the S-semigroup* T *on* G *into the S-semigroup* Σ *on* F *if it has the following properties.*

(1) $$(XY)^{\varphi} = X^{\varphi} Y^{\varphi} \text{ for all } X, Y \in T.$$

(2) *For every* T-*class* $\mathfrak{C}$ *of* G *there exists a* Σ-*class* $\mathfrak{S}$ *of* F *such that*

$$\mathfrak{C}^{\varphi} = \mathfrak{S} \quad and \quad (\mathfrak{C}^{-1})^{\varphi} = \mathfrak{S}^{-1}.$$

(3) $$X^{\varphi} = \bigcup_{\substack{\mathfrak{C} \in \mathfrak{T} \\ \mathfrak{C} \subseteq X}} \mathfrak{C}^{\varphi} \text{ for all } X \in T.$$

A homomorphism $\varphi : T \to \Sigma$ is called an *isomorphism* if φ is a bijective mapping.

To return to our problem, let us look at the mapping

$$\varphi : X \to H \cap X \qquad (X \in HK/K).$$

We want to show that φ is an isomorphism of the double coset *S*–semigroup HK/K onto the *S*–semigroup S(H/H ∩ K). By Lemma 1 (4)

$$(XY)^{\varphi} = X^{\varphi}\, Y^{\varphi} \text{ holds for all } X, Y \in HK/K.$$

Every element $X \in HK/K$ is the product

$$X = (Kh_1 K) \cdots (Kh_x K)$$

of a finite number of double cosets modulo K. Hence

$$X^{\varphi} = (Kh_1 K)^{\varphi} \cdots (Kh_x K)^{\varphi} = (H \cap Kh_1 K) \cdots (H \cap Kh_x K)$$

is an element of the *S*–semigroup S(H/H ∩ K) by the definition of S(H/H ∩ K). Therefore φ really is a mapping of HK/K into S(H/H ∩ K). Furthermore, Definition 2 (2) holds for φ because

$$(KhK)^{\varphi} = H \cap KhK \quad \text{and} \quad ((KhK)^{-1})^{\varphi} = (H \cap KhK)^{-1}.$$

Definition 2 (3) is satisfied as well since

$$X^{\varphi} = H \cap \bigcup_{g \in X} KgK = \bigcup_{g \in X} (H \cap KgK) = \bigcup_{g \in X} (KgK)^{\varphi}.$$

Thus we have proved that φ is a homomorphism of the double coset *S*–semigroup HK/K onto the *S*–semigroup S(H/H ∩ K). Finally, the arguments of the proof of Lemma 1 (4) show that

$$\psi : Y \to KY \quad (Y \in S(H/H \cap K))$$

is the inverse mapping of φ. Therefore φ is an isomorphism. Now we are able to state the intended generalization of the Second Isomorphism Theorem.

THEOREM 1. *Let* G *be a group, and assume that* H *and K are subgroups of* G *such that* HK = KH *holds. Then*

(1) *The semigroup* S(H/H ∩ K) *with respect to the subset multiplication which is generated by the set* $\{H \cap KhK \mid h \in H\}$ *is an S–semigroup on* H *contained in* $\overline{H/H \cap K} := \{\varnothing \neq Z \subseteq H \mid (H \cap K)\, Z\, (H \cap K) = Z\}$.

(2) *The mapping*

$$\varphi : X \to H \cap X$$

is an isomorphism of the double coset S–semigroup HK/K *on* HK *onto the S–semigroup* S(H/H ∩ K) *on* H, *and*

$$\psi : Y \to KY$$

is its inverse.

In general, $S(H/H \cap K) = H/H \cap K$ will not be true. We ask for conditions that this equality hold. For that end another concept is needed.

DEFINITION 3 ([3], Definition 1.9). *Let* T *be an S–semigroup on the group* G, *and* $\mathfrak{T}$ *the set of all* T*–classes of* G. *Let* N *be a subgroup of* G *such that*

(1) $$N = \bigcup_{\substack{\mathfrak{C} \in \mathfrak{T} \\ \mathfrak{C} \cap N \neq \emptyset}} \mathfrak{C}.$$

(2) $$N\mathfrak{C} = \mathfrak{C}N \text{ for all } \mathfrak{C} \in \mathfrak{T}.$$

Then N *is called a* T–normal *subgroup of* G.

If we apply Definition 3 to the double coset *S*–semigroup $HK/H \cap K$ then a subgroup N of HK is $HK/H \cap K$–normal if and only if

(1) $$H \cap K \leq N,$$

(2) $$N\,(H \cap K)\,g\,(H \cap K) = (H \cap K)\,g\,(H \cap K)\,N \quad \text{for all } g \in HK.$$

THEOREM 2. *Under the hypotheses of Theorem* 1 *the following are equivalent.*

(1) $$S(H/H \cap K) = H/H \cap K.$$

(2) K *is an* $HK/H \cap K$*–normal subgroup of* HK.

Proof. Assume that (1) holds. Then, by Definition 2,

$$(KhK)^{\varphi} = H \cap KhK = (H \cap K)\,h\,(H \cap K) \quad \text{for all } h \in H.$$

For every $g \in HK$ there exist $k \in K$ and $h \in H$ such that $g = kh$. Therefore, using the arguments of the proof of Lemma 1 (4),

$$K\,(H \cap K)\,g\,(H \cap K) = Kh\,(H \cap K) = K\,(H \cap K)\,h\,(H \cap K) =$$
$$= K\,(H \cap KhK) = KhK = KgK.$$

The arguments of the proof of Lemma 1 (4) can also be applied to the cosets hK instead of the cosets Kh, and $g \in HK$ can be written as $g = h'\,k'$ with $h' \in H$ and $k' \in K$. Hence

$$(H \cap K)\,g\,(H \cap K)\,K = (H \cap K)\,h'\,K = (H \cap K)\,h'\,(H \cap K)\,K =$$
$$= (H \cap Kh'K)\,K = Kh'K = KgK.$$

It follows that K is an $HK/H \cap K$–normal subgroup of HK, i.e. (2) holds. Conversely, (2) implies (1) by the Second Isomorphism Theorem for *S*–semigroups ([3], Theorem 2.13).

Let us finish this paper with comments on Theorem 1.

I. Theorem 1 shows that the property of an *S*–semigroup to be a double coset *S*–semigroup is not invariant under isomorphisms since $S(H/H \cap K)$

is, in general, not a double coset S–semigroup though it is isomorphic to the double coset S–semigroup HK/K. Yet it is easy to see that every homomorphic image of a double coset S–semigroup into any double coset S–semigroup is again a double coset S–semigroup. The point is that though $S(H/H \cap K)$ is contained in $\overline{H/H \cap K}$ the mapping $\varphi : X \to H \cap X$ need not yield a homomorphism in the sense of Definition 2 of the double coset S–semigroup HK/K into the double coset S–semigroup $H/H \cap K$.

II. Theorem 1 also shows that the notion of S–semigroup has a sort of " categorical " property in the following sense. Given any group H and any subgroup D of H, then the S–semigroups on H which are contained in $\overline{H/D}$ yield, up to a certain degree, information on the possible embeddings of H into a group G such that

$$G = HK \quad \text{and} \quad H \cap K = D$$

holds for a subgroup K of G. In fact, not all S–semigroups on H contained in $\overline{H/D}$ are relevant to that sort of embedding, but only those which are isomorphic to double coset S–semigroups, namely isomorphic to $S(H/H \cap K)$ for a possible embedding of H in the described sense.

III. Our remark II points to applications of Theorem 1 in the following direction. Let G be a transitive permutation group on a set Ω. Let G_α denote the stabilizer in G of a letter $\alpha \in \Omega$. Assume further that H is a transitive subgroup of G. This means that

$$G = HG_\alpha = G_\alpha H$$

holds. Thus we have the situation of Theorem 1, and the double coset S–semigroup G/G_α is isomorphic to the S–semigroup $S(H/H \cap G_\alpha)$.

As for the meaning of the double coset S–semigroup G/G_α as a sort of " endomorphism ring " for the transitive permutation group G we refer the reader to [4], Section 10. There the isomorphy class $[G/G_\alpha]$ has been introduced as the *type* of the transitive permutation group G.

To indicate how the applications of Theorem 1 will work it should be noted that every subgroup U of G such that $G_\alpha \leq U \leq G$ is mapped (in the sense of [3], Proposition 2.2) by the isomorphism $\varphi : X \to H \cap X$ onto the subgroup $H \cap U$ of H which has the properties

$$H \cap U = \bigcup_{h \in H \cap U} (H \cap G_\alpha h G_\alpha) \quad \text{and} \quad H \cap G_\alpha \leq H \cap U \leq H.$$

Thus the transitive permutation group G is primitive if and only if there does not exist any subgroup V of H such that

$$V = \bigcup_{v \in V} (H \cap G_\alpha v G_\alpha) \quad \text{and} \quad H \cap G_\alpha < V < H.$$

Then we call $S(H/H \cap G_\alpha)$ a *primitive* S–semigroup.

Also the number of G/G_α–classes of G is equal to the number of orbits of G_α. For instance, G is two-fold transitive if and only if G has exactly two G/G_α–classes. Therefore, if H has no primitive S–semigroups contained in $\overline{H/H\cap G_\alpha}$ other than the trivial one which is defined by the subsets $H\cap G_\alpha$ and $H\setminus(H\cap G_\alpha)$, then G must be either imprimitive or two-fold transitive.

Thus some of the properties of the transitive permutation group G can be decided internally within the smaller transitive group H by the properties of the existing S–semigroups on H contained in $\overline{H/H\cap G_\alpha}$.

What we have indicated in III is just the *method of Schur* in a general form for abitrary groups which, incidentally, justifies our notation of Schur-semigroup. In fact, Theorem 1 is just the straightforward generalization of SCHUR's Theorem of the "transitivity module" of G_α (cf. [6], Theorem 24.1) to arbitrary groups and S–semigroups, a fact which the author wishes to acknowledge expressis verbis.

REFERENCES.

[1] SCHUR I., *Zur Theorie der einfach transitiven Permutationsgruppen.* « Sitz.–Ber. Preuss. Akad. Wiss. », Phys.–Math. Kl., *1933*, 598–623.

[2] TAMASCHKE O., *An Extension of Group Theory.* Istituto Nazionale di Alta Matematica. « Symposia Mathematica », *1*, 5–13 (1968).

[3] TAMASCHKE O., *An Extension of Group Theory to S–semigroups.* « Math. Z. », *104*, 74–90 (1968).

[4] TAMASCHKE O., *On Permutation Groups.* « Annali di Matematica ». To appear.

[5] TAMASCHKE O., *On the Theory of Schur-rings.* « Annali di Matematica ». To appear.

[6] WIELANDT H., *Finite Permutation Groups.* New–York – London: Academic Press 1964.