Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Complementary Bounds for Eigenvalues

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **46** (1969), n.2, p. 164–165.

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<http://www.bdim.eu/item?id=RLINA_1969_8_46_2_164_0>

Matematica. — *Complementary Bounds for Eigenvalues.* Nota di Kurt Kreith, presentata (*) dal Socio M. Picone.

Riassunto. — Per taluni problemi di frontiera concernenti un'equazione lineare a derivate parziali del second'ordine autoaggiunta, con arbitrario numero di variabili reali, dipendente linearmente da un parametro, si dà una limitazione inferiore ed una superiore per i primi autovalori del parametro.

The purpose of this note is to apply an identity due to M. Picone to establish upper and lower bounds for the first eigenvalue of the self-adjoint elliptic equation

$$L[w] \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial w}{\partial x_i}\right) + cw = \lambda p w$$

satisfying mixed boundary conditions on a smooth bounded domain $G \subset R^n$. We assume positivity of $p(x)$ and sufficient regularity of the coefficients of L in G.

Theorem.—*Let $u(x)$ and $v(x)$ be positive and of class C^2 in G and satisfy*

$$(1) \qquad \frac{\partial u}{\partial \nu} + \sigma u = 0 \quad \text{on} \quad \partial G,$$

$$(2) \qquad \frac{\partial v}{\partial \nu} + \tau v = 0 \quad \text{on} \quad \partial G,$$

respectively, with $-\infty < \tau(x) \leq \sigma(x) \leq \infty$ (1). *Let* λ_1 *and* μ_1 *be the first eigenvalues of* $L[w] = \lambda p w$ *with the boundary conditions* (1) *and* (2), *respectively. Then*

$$(3) \qquad \lambda_1 \geq \inf_{x \in G} \frac{L[v]}{pv}$$

and

$$(4) \qquad \mu_1 \leq \sup_{x \in G} \frac{L[u]}{pu}.$$

Proof.—If $v(x) \neq 0$ in G, then a direct calculation yields the following Picone-type identity:

$$\sum_j \frac{\partial}{\partial x_j}\left[u \sum_i a_{ij}\frac{\partial u}{\partial x_i} - \frac{u^2}{v}\sum_i a_{ij}\frac{\partial v}{\partial x_i}\right] = u \sum_{i,j}\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) -$$

$$- \frac{u^2}{v}\sum_{i,j}\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial v}{\partial x_i}\right) + \sum_{i,j} a_{ij}\left(\frac{\partial u}{\partial x_i} - \frac{u}{v}\frac{\partial v}{\partial x_i}\right)\left(\frac{\partial u}{\partial x_j} - \frac{u}{v}\frac{\partial v}{\partial x_j}\right).$$

(*) Nella seduta dell'8 febbraio 1969.

(1) We follow the usual conventions in using $\sigma(x_0) = +\infty$ to denote the boundary condition $u(x_0) = 0$.

Integrating over G and applying Green's theorem yields

$$\int_{\partial G} u^2 \left[\frac{1}{u}\frac{\partial u}{\partial \nu} - \frac{1}{v}\frac{\partial v}{\partial \nu}\right] ds \geq - \int_{G} \left[u \mathrm{L}[u] - \frac{u^2}{v}\mathrm{L}[v]\right] dx .$$

Case 1.—If $\mathrm{L}[u] = \lambda_1 u$, then

(5) $$0 \geq \int_{\partial G} u^2 [\tau - \sigma] \, ds \geq - \int_{G} p u^2 \left[\lambda_1 - \frac{\mathrm{L}[v]}{pv}\right] dx .$$

Case 2.—If $\mathrm{L}[v] = \mu_1 v$, then

(6) $$0 \geq \int_{\partial G} u^2 [\tau - \sigma] \, ds \geq - \int_{G} p u^2 \left[\frac{\mathrm{L}[u]}{pu} - \mu_1\right] dx .$$

The estimates (3) and (4) follow readily from (5) and (6).

Remarks. For the case $\mathrm{L} = -\Delta$, a special case of (3) and (4) are due to Barta [1] who derived the first estimates of this type. Swanson [2] has used related techniques to prove (3) for self-adjoint elliptic operators in an unbounded domain for the special case $\sigma \equiv +\infty$ on ∂G. The present techniques can readily be generalized to include unbounded domains by imposing additional " boundary condition at ∞ " of the form required in [2]. Protter and Weinberger [3] have used different techniques to prove more general versions of (3) and (4) which apply to non-self-adjoint problems. The identity underlying the proof of the Theorem is a special case of an identity first used by Picone [4] to establish Sturmian comparison theorems for elliptic equations.

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