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A Direct Method for Selfadjoint Systems of Second Order Differential Equations

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Matematica. — *A Direct Method for Selfadjoint Systems of Second Order Differential Equations.* Nota di KURT KREITH, presentata (*) dal Socio M. PICONE.

RIASSUNTO. — In questa Nota è estesa ai sistemi di equazioni differenziali ordinarie autoaggiunte un'identità dovuta a M. Picone per le equazioni Sturm–Liouville. Fondandosi su tale estensione si perviene a stabilire taluni teoremi oscillatori per le soluzioni dei sistemi di equazioni considerate.

Sturm theory for selfadjoint systems of second order ordinary differential equations has been studied by several authors [1], [2] who use the calculus of variations to generalize the classical comparison and oscillation theorems. The purpose of this note is to point out how a technique used by M. Picone to study a single Sturm-Liouville equation can be generalized to systems and thereby provide a direct method for analyzing such problems.

We shall consider real column vectors U and V which are, respectively, solutions of the selfadjoint systems

$$(AU')' + CU = 0 \tag{1}$$

$$(GV')' + HV = 0 \tag{2}$$

on an interval $x_1 \leq x \leq x_2$. At the end points we prescribe boundary conditions for U of the form

$$U'(x_i) + (-1)^i S_i U(x_i) = 0 \quad ; \quad i = 1, 2. \tag{3}$$

Here $A(x)$ and $G(x)$ are to be $n \times n$ real, symmetric, positive definite matrices of class C^2 for $x_1 \leq x \leq x_2$ and C, H, S_1, S_2, are to be $n \times n$ real symmetric matrices, continuous for $x_1 \leq x \leq x_2$. The notation "$S_i = +\infty$" will be used to denote the boundary condition $U(x_i) = 0$.

Following Morse [3], we call two solutions V_1 and V_2 of (2) *mutually conjugate* if

$$V_1^* GV_2' = V_2^* GV_1'. \tag{4}$$

Furthermore a $n \times n$ matrix W is called a *conjugate system* for (2) if its columns are mutually conjugate solutions of (2) which are linearly independent on $[x_1, x_2]$. The determinant of a conjugate system W will be denoted by $|W|$.

If $|W| \neq 0$ in $[x_1, x_2]$, then a direct calculation using (4) and the relation

$$(W^{-1})' = -W^{-1} W' W^{-1}$$

(*) Nella seduta del 19 aprile 1969.

establishes the following identity:

$$\frac{d}{dx}(U^* AU' - U^* GW' W^{-1} U) =$$
$$U^* (AU')' - U^* (GW')' W^{-1} U + U'^* (A - G) U'$$
$$+ (U'^* - U^* W^{-1*} W'^*) G (U' - W' W^{-1} U).$$

If U and W satisfy (1) and (2) respectively, then an integration yields the following generalization of the classical identity of M. Picone:

$$[U^* AU' - U^* GW' W^{-1} U]_{x_1}^{x_2} =$$

$$\int_{x_1}^{x_2} [U^* (H - C) U + U'^* (A - G) U'] dx \tag{5}$$

$$+ \int_{x_1}^{x_2} (U' - W' W^{-1} U)^* G (U' - W' W^{-1} U) dx$$

The following comparison theorem follows readily from this identity.

THEOREM 1. *Suppose* U (x) *is a nontrivial solution of* (1) (3) *and that* W (x) *is a conjugate system for* (2). *If*

(i) $A - G$ *is positive semi-definite for* $x_1 \leq x \leq x_2$;
(ii) $H - C$ *is positive semi-definite for* $x_1 \leq x \leq x_2$;
(iii) $-S_1 + GW' W^{-1}$ *is positive semi-definite at* $x = x_1$;
$S_2 + GW' W^{-1}$ *is positive semi-definite at* $x = x_2$,

but at least one of the above is positive definite, then $|W| = 0$ *for some* x *in* $[x_1, x_2]$.

Proof. Suppose $|W| \neq 0$ in $[x_1, x_2]$ so that (5) is valid. Using (3) and the positive definiteness of G, it follows from (5) that

$$[-U^* (S_2 + GW' W^{-1}) U] (x_2) + [U^* (S_1 - GW' W^{-1}) U] (x_1)$$
$$\geq \int_{x_1}^{x_2} [U^* (H - C) U + U'^* (A - G) U'] dx.$$

Our hypotheses assure that the left side of this inequality is nonpositive while the right side is nonnegative and preclude both sides being zero. This contradiction shows that $|W| = 0$ for some x in $[x_1, x_2]$.

Remarks.

1) If U $(x_i) = 0$, then "$S_i = +\infty$" and we need not impose any boundary condition on W at x_i.
2) If W satisfies boundary conditions of the form

$$W'(x_i) + (-1)^i T_i W(x_i) = 0 \quad ; \quad i = 1, 2,$$

then condition (iii) can be replaced by (iii') $S_i - T_i$ is positive semi-definite for $i = 1, 2$.

3) Conditions (i) and (ii) can be replaced by the weaker integral inequalities

$$(i') \qquad \int_{x_1}^{x_2} U'^{*} (A - G) U'^{*} \, dx \geq 0 ;$$

$$(ii') \qquad \int_{x_1}^{x_2} U^{*} (H - C) U^{*} \, dx \geq 0 .$$

Oscillation theorems for conjugate systems of (2) now follow readily by comparing (2) with a scalar equation of the form (1). Specifically, if I denotes the identity matrix and

$$A(x) = a(x) I$$

$$C(x) = c(x) I$$

then any scalar solution of

$$(6) \qquad (au')' + cu = 0$$

generates a vector valued solution of (1), namely a vector $U(x)$ all of whose components are identical with $u(x)$. Applying Theorem 1, we obtain the following oscillation criterion for (2).

THEOREM 2. *Suppose* $(au')' + cu = 0$ *is oscillatory at a singular point* $x = b$ *(possibly* $b = \infty$*). If*

(i) $a(x) I - G$ *is positive semi-definite in some neighborhood of* b;

(ii) $H - c(x) I$ *is positive semi-definite in some neighborhood of* b, *then* (2) *is oscillatory at* b *in the sense that for any conjugate system* W *of* (2), $|W| = 0$ *in every neighborhood of* b.

REFERENCES.

[1] G. A. BLISS and I. J. SCHOENBERG, *On separation, comparison and oscillation theorems for selfadjoint systems of linear second order differential equations*, « Amer. J. of Math. », *53* 781–800 (1931).

[2] M. MORSE, *The Calculus of Variations in the Large*, « Amer. Math. Soc. Colloquium Publications », Vol. 18, 1934.

[3] M. MORSE, *A generalization of the Sturm Separation and Comparison Theorems in n-space*, « Math. Ann. », *103*, 52–69 (1930).