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Proximities and Abstract Spaces

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Topologia. — *Proximities and Abstract Spaces*. Nota di GEORGE C. GASTL, presentata [(*)] dal Socio B. SEGRE.

RIASSUNTO. — Si studiano le connessioni che intercedono fra varie relazioni di prossimità in un insieme M e varie topologie generalizzate od estese inerenti ad M.

INTRODUCTION.

This paper is concerned with various proximity relations and their associated set-functions.

A *proximity space* consists of a set M and a binary relation P on subsets of M such that the following conditions are satisfied.

P. 1. For all $A \subseteq M$, $(A, N) \notin P$.

P. 2. If $(A, B) \in P$, then $(B, A) \in P$.

P. 3. If $(A \cup B, C) \in P$, then $(A, C) \in P$ or $(B, C) \in P$.

P. 4. $(\{x\}, \{y\}) \in P$ iff $x = y$.

If in addition P also satisfies

P. 5. If $(A, B) \notin P$, then there exist $C, D \subseteq M$ such that $C \cup D = M$ and $(A, C) \notin P$, $(B, D) \notin P$

then P is a *separated proximity*.

In regard to abstract spaces, suppose k is a function from 2^M into 2^M. Then (M, k) will be called a *Fréchet space* if g is expansive, an *Appert space* if g is a closure function, and a *Čech space* if g is enlarging and additive.

SPACES FROM PROXIMITY RELATIONS.

A topology on M corresponding to a proximity P on M can be obtained by defining the set-valued set-function $k : 2^M \to 2^M$ by $kA = \{q \mid (\{q\}, A) \in P\}$; and if P satisfies all five given conditions, then it is well-known that k is a Kuratowski closure function on M and (M, k) is a completely regular T_1–space. The relationships between the conditions on P and the properties of k will be studied. First the term ancestral, as applied to binary relations among sets, is defined.

DEFINITION 1. Let R be a binary relation on subsets of M. If $(A, B) \in R$ and $A \subseteq C$ imply $(C, B) \in R$, then R is *left ancestral*. If $(A, B) \in R$ and $B \subseteq C$ imply $(A, C) \in R$, then R is *right ancestral*. If R has both of these properties it is *ancestral*.

(*) Nella seduta del 19 aprile 1969.

THEOREM 1. *Let* P *be a binary relation on subsets of* M *and define the function* $k : 2^M \to 2^M$ *by* $kA = \{q \mid (\{q\}, A) \in P\}$.

(a) *If* P *is right ancestral then* k *is isotonic.*
(b) *If* P *has property* P.1 *then* $kN = N$.
(c) *If* P *is right ancestral and* $(\{q\}, \{q\}) \in P$ *for each* $q \in M$, *then* k *is enlarging.*
(d) *If* P *is right ancestral and satisfies*: $(C, A \cup B) \in P$ *implies* $(C, A) \in P$ *or* $(C, B) \in P$, *then* k *is additive.*
(e) *If* P *is ancestral and satisfies*: $(A, B) \notin P$ *implies there exist* C, D *disjoint for which* $(A, cC) \notin P$ *and* $(cD, B) \notin P$, *and* $(\{q\}, \{q\}) \in P$ *for each* $q \in M$, *then* k *is idempotent.*

Proof: (a) Suppose $A \subseteq B$ and $q \in kA$. Then $(\{q\}, A) \in P$, and if P is right ancestral $(\{q\}, B) \in P$, hence $q \in kB$.

(b) If $q \in kN$, then $(\{q\}, N) \in P$, so property P. 1 requires that $kN = N$.

(c) Let $A \subseteq M$ and $q \in A$. If $(\{q\}, \{q\}) \in P$ for each $q \in M$, and P is right ancestral, then $(\{q\}, A) \in P$ and hence $q \in kA$. This is true for each $q \in A$, so $A \subseteq kA$.

(d) If P is right ancestral then k is isotonic, so $k(A \cup B) \supseteq kA \cup kB$. Then it must be shown $kA \cup kB \supseteq k(A \cup B)$. Let $q \in k(A \cup B)$. Then $(\{q\}, A \cup B) \in P$, and if this implies either $(\{q\}, A) \in P$ or $(\{q\}, B) \in P$ then $q \in kA$ or $q \in kB$ whence $q \in kA \cup kB$.

(e) If P is right ancestral and contains $(\{q\}, \{q\})$ for all $q \in M$, then from (c) k is enlarging; i.e., $k(kA) \supseteq kA$ for each $A \subseteq M$. Then only $k^2 \subseteq k$ is needed. Let $q \in k^2 A = k(kA)$. Then $(\{q\}, kA) \in P$. Suppose $q \notin kA$. Then $(\{q\}, A) \notin P$. By assumption there are sets C and D such that $C \cap D = N$, $(\{q\}, cC) \notin P$, and $(cD, A) \notin P$. If $A \cap cD \neq N$, then there is some $s \in A \cap cD$ and $(\{s\}, \{s\}) \in P$, hence $(cD, A) \in P$ which is a contradiction. Thus $A \subseteq D$. Also if $s \in kA \cap cD$, then $(\{s\}, A) \in P$ and then $(cD, A) \in P$ which is not true. Hence $kA \subseteq D$. Since $q \in k^2 A$, $(\{q\}, kA) \in P$, and because $kA \subseteq D \subseteq cC$ the right ancestral property yields $(\{q\}, cC) \in P$ which is a contradiction. Therefore $k^2 \subseteq k$, and k is idempotent.

Therefore k is a Kuratowski closure function when P satisfies the properties:

(i) For all $A \subseteq M$, $(A, N) \notin P$
(ii) P is ancestral
(iii) For each $q \in M$, $(\{q\}, \{q\}) \in P$
(iv) When $(C, A \cup B) \in P$, then $(C, A) \in P$ or $(C, B) \in P$
(v) When $(A, B) \notin P$, then there exist C and D disjoint such that $(A, cC) \notin P$ and $(cD, B) \notin P$.

The symmetry property P. 2 is not necessary, and P. 3 is replaced by the same property on the right. Also it may be true that $(\{x\}, \{y\}) \in P$ even when $x \neq y$.

Relations on subsets of M which are weaker than a proximity have been studied by Mattson and Pervin. Pervin [5] has defined what he calls a *quasi-proximity* as a relation between subsets of M which has the four properties

1) For all $A \subseteq M$, $(A, N) \notin P$
2) For each $q \in M$, $(\{q\}, \{q\}) \in P$
3) $(C, A \cup B) \in P$ iff $(C, A) \in P$ or $(C, B) \in P$
4) If $(A, B) \notin P$ then there exist two disjoint sets U and V such that $(A, cU) \notin P$ and $(cV, B) \notin P$.

Clearly a quasi-proximity plus the symmetry condition is a proximity, not necessarily separated. These conditions used by Pervin are equivalent to (i), (iii), (iv), (v), plus the right ancestral property for P. If the set-valued set-function k is defined as it has been above, then it is not a Kuratowski closure function when P is only a quasi-proximity. E. F. Steiner [6] has given an example showing this.

In order to assure that k is a Kuratowski closure it is necessary to include the one condition which appears in (i)—(v) above and is not required of a quasi-proximity, and that is that P is left ancestral. Steiner has added the following condition: $(A \cup B, C) \in P$ iff $(A, C) \in P$ or $(B, C) \in P$. Certainly this is sufficient when added to the quasi-proximity requirements to make k a Kuratowski closure, but it is not necessary because it includes the "right hereditary" property $(A \cup B, C) \in P \Rightarrow (A, C) \in P$ or $(B, C) \in P$, which is not used.

The above results on obtaining the set-function k by using a relation P can be summarized in terms of abstract spaces as follows:

THEOREM 2. *Let* P *be a relation on the subsets of* M *and* $k: 2^M \to 2^M$ *be given by* $kA = \{q \mid (\{q\}, A) \in P\}$.

(a) (M, k) *is an isotonic space if* P *is right ancestral.*
(b) (M, k) *is a Fréchet space if* P *is right ancestral and if* $(\{q\}, \{q\}) \in P$ *for each* $q \in M$; *i.e., if* P *is Mattson's generalized quasi-proximity.*
(c) (M, k) *is an Appert space if* P *has conditions* (ii), (iii), *and* (v).
(d) (M, k) *is a Čech space if* P *has conditions* (iii) *and* (iv) *and is right ancestral.*

Mattson [4] has studied a weaker form of proximity, called a *generalized quasi-proximity*. He required that P have property (iii) and the right ancestral property, hence k for this case is expansive and (M, k) is a Fréchet space. By adding the symmetry requirement to the two given for a generalized quasi-proximity, Mattson obtained a *generalized proximity* for M and proved that this type of proximity is the complement in $2^M \times 2^M$ of a *Wallace separation* [7]. Other similar forms of weaker proximity relations have been considered by Leader [2] and Lodato [3], and these are complements in $2^M \times 2^M$ of weak topological separations [4].

Proximity Relations from Spaces.

Consider the converse problem of obtaining a binary relation on subsets of M from a given set-valued set-function. In the case of a topological space when k is a Kuratowski closure, the separation $(A \cap kB) \cup (kA \cap B) \neq N$ is a familiar one, and it suggests the "closeness" relation $(A, B) \in P$ iff $(A \cap kB) \cup (kA \cap B) \neq N$. This relation would certainly be symmetric regardless of the properties of k. Similarly if $(A, B) \in P$ provided $kA \cup kB \neq N$, this would be symmetric by the manner of definition. A definition which does not require symmetry will be used, so the resulting relation P will not have all properties of a proximity.

THEOREM 3. *Assume* $k : 2^M \to 2^M$ *and* $(A, B) \in P$ *iff* $A \cap kB \neq N$.

(a) P *is left ancestral by definition.*

(b) *If* k *is isotonic, then* P *is right ancestral.*

(c) *If* $kN = N$, *then* $(A, N) \notin P$ *for each* $A \subseteq M$.

(d) *If* k *is enlarging, then* $(\{q\}, \{q\}) \in P$ *for each* $q \in M$.

(e) *If* k *is additive, then* P *has the property*: $(C, A \cup B) \in P$ *implies either* $(C, A) \in P$ *or* $(C, B) \in P$.

(f) *If* k *is idempotent, then* P *has the property*: *if* $(A, B) \notin P$ *then there exist* C, D *disjoint such that* $(A, cC) \notin P$ *and* $(cD, B) \notin P$.

Proof: (a) By the definition of P, if $(A, B) \in P$ then $A \cap kB \neq N$, hence if $C \supseteq A$, $C \cap kB \neq N$ and $(C, B) \in P$.

(b) If k is isotonic, then $B \subseteq C$ implies $kB \subseteq kC$; hence $(A, B) \in P$ and $B \subseteq C$ imply $A \cap kB \neq N$ and $A \cap kC \neq N$ which means $(A, C) \in P$.

(c) If $kN = N$, then $A \cap kN = N$ for each $A \subseteq M$ and $(A, N) \notin P$.

(d) Suppose k is enlarging. Then $q \in A$ implies $q \in kA$ which means $(\{q\}, A) \in P$. Thus $(\{q\}, \{q\}) \in P$ for each $q \in M$.

(e) When $k(A \cup B) = kA \cup kB$, k is isotonic, hence P is right ancestral by (a). Also $k(A \cup B) \subseteq kA \cup kB$, whence $(C, B \cup A) \in P$ implies $C \cap k(B \cup A) \neq N$ and consequently either $C \cap kB \neq N$ or $C \cap kA \neq N$. This means $(C, B) \in P$ or else $(C, A) \in P$.

(f) Suppose k is idempotent and $(A, B) \notin P$. Then $A \cap kB = N$. Choose $C = ckB$ and $D = kB$. Then $(A, kB) \notin P$ because $A \cap k(kB) = A \cap kB = N$. Also $(ckB, B) \notin P$ because $ckB \cap kB = N$. Thus C and D are disjoint and $(A, cC) = (A, kB) \notin P$ and $(cD, B) = (ckB, B) \notin P$. Also in this case $C \cup D = M$.

Defining P in the given way from a function k means that an isotonic space (M, k) determines an ancestral relation P. A Fréchet space determines a generalized quasi-proximity (Mattson) with the additional left ancestral property. Mattson has proved the function $k'A = \{q \mid (\{q\}, A) \in P\}$ corresponding to this constructed generalized quasi-promixity is equal to the k of he Fréchet space. If (M, k) is an Appert space then the resulting P has properties (ii), (iii), and (v) given above after Theorem 1. If (M, k) is a topological space, then P has all properties (i) through (v) and is a quasi-proximity on M, but is not necessarily symmetric.

THEOREM 4. *Let* (M , k) *be a topology and construct a relation* P *by*: (A , B) $\in$ P *provided* A$\cap k$B $\neq$ N. *Then* P *has properties* (i) *through* (v) *given above, and the function t obtained from* P *by*: tA $= \{q \mid (\{q\}, \mathrm{A}) \in \mathrm{P}\}$, *is equal to k.*

Proof: From the results of Theorem 3, P has the given five properties when k has the properties of a Kuratowski closure, so $k = t$ must be proved. Let A $\subseteq$M and $q \in t$A. Then $(\{q\}, \mathrm{A}) \in \mathrm{P}$ which means $\{q\} \cap k\mathrm{A} \neq \mathrm{N}$; i.e., $q \in k$A. Thus $t \subseteq k$. If $q \in k$A, then $\{q\} \cap k\mathrm{A} \neq \mathrm{N}$, hence $(\{q\}, \mathrm{A}) \in \mathrm{P}$ and $q \in t$A. Therefore $t = k$.

The proof used only the definition of P in terms of k and the definition of t in terms of P, and was independent of the properties of k and P. Given any extended topology (M , k) the function t obtained in the given way must be identical with k, hence the construction (M , k) $\rightarrow$ (M , P) $\rightarrow$ (M , t) always produces the same extended topology as that given. The same procedures when beginning with a proximity (M , P) do not always yield the original (M , P) however.

THEOREM 5. *Let* P *be a relation on subsets of* M *and define* $k : 2^{\mathrm{M}} \rightarrow 2^{\mathrm{M}}$ *by* $k\mathrm{A} = \{q \mid (\{q\}, \mathrm{A}) \in \mathrm{P}\}$. *Then the relation* P′ *given by* (A , B) $\in$ P′ *provided* A$\cap k$B $\neq$ N, *satisfies* P′ $\subseteq$ P *if* P *is left ancestral.*

Proof: Suppose (A , B) $\in$ P′. Then A$\cap k$B $\neq$ N, hence there exists some $q \in \mathrm{A} \cap k\mathrm{B}$. For this q, $(\{q\}, \mathrm{B}) \in \mathrm{P}$ by the definition of k. Therefore if P is left ancestral, (A , B) $\in$ P and P′ $\subseteq$ P.

If one considers the original relation P and has (A , B) $\in$ P, this does not imply that there is some point $q \in$ A for which $(\{q\}, \mathrm{B}) \in \mathrm{P}$. If that were true, then $q \in k$B and hence (A , B) $\in$ P′. The following example is one in which P $\neq$ P′.

EXAMPLE 1. Let M be the real line E^1 and t the closure function of the usual topology. Define the relation P by (A , B) $\in$ P iff $t\mathrm{A} \cap t\mathrm{B} \neq \mathrm{N}$. Then the function k given by $k\mathrm{A} = \{q \mid (\{q\}, \mathrm{A}) \in \mathrm{P}\} = \{q \mid q \in t\mathrm{A}) = t\mathrm{A}$. The new relation P′ is then (A , B) $\in$ P′ provided $\mathrm{A} \cap k\mathrm{B} = \mathrm{A} \cap t\mathrm{B} \neq \mathrm{N}$. Thus P′ $\subseteq$ P and P′ $\neq$ P.

Steiner [6] has proved that, when the original relation P satisfies the condition: (A , B) $\in$ P iff $(\{a\}, \mathrm{B}) \in \mathrm{P}$ for some $a \in$ A, the construction produce P′ = P. The condition he gives is just the condition mentioned prior to Example 1 in addition to left ancestral. It is stronger than the "left hereditary" condition which was mentioned above as being required for the relation P in order to assure that k is a Kuratowski closure. But it is necessary to assure that the procedure (M , P) $\rightarrow$ (M , k) $\rightarrow$ (M , P′) will give P′ = P. Clearly, any P′ which is defined using k, as (A , B) $\in$ P′ iff A$\cap k$B $\neq$ N, has this property. Therefore when P and P′ are to be the same, P also has the property. The relation P is called *strongly left hereditary* provided $(A, B) \in P$ implies $(\{a\}, B) \in P$ for some $a \in A$. For each abstract space then there is the corresponding proximity relation.

THEOREM 6. *Let* (M, k) *be an extended topology and* P *a relation on subsets of* M.

(a) (M, k) *an isotonic space corresponds to an ancestral and strongly left hereditary relation* P.

(b) (M, k) *a Fréchet space corresponds to an ancestral and strongly left hereditary relation* P *which has condition* (iii) *as given above following Theorem* 1.

(c) *An Appert space* (M, k) *corresponds to an ancestral and strongly left hereditary relation* P *satisfying conditions* (iii) *and* (v).

(d) *A topology* (M, k) *is equivalent to a quasi-proximity* P *which is left ancestral and strongly left hereditary.*

EXAMPLE 2. Let P be defined on subsets of M by $(A, B) \in P$ provided $A \cap B \neq N$. Then P is clearly a proximity relation and is separated. The corresponding function k is the identity function $kA = A$, so the space (M, k) is the discrete topology on M.

EXAMPLE 3. Let (M, k) be a compact Hausdorff space and let $(A, B) \in P$ iff $kA \cap kB \neq N$. Because k is a Kuratowski closure function $(A, N) \notin P$ for each $A \subseteq M$, and $(A \cup B, C) \in P$ implies $(A, C) \in P$ or $(B, C) \in P$. P is symmetric since $kA \cap kB \neq N$ is symmetric in A and B. The Hausdorff property ensures $(\{x\}, \{y\}) \in P$ iff $x = y$. If $(A, B) \notin P$, then $kA \cap kB = N$. Both kA and kB are compact because they are closed subsets of a compact space. Thus kA and kB are disjoint compact subsets of a Hausdorff space and have disjoint neighborhoods. Say $kA \subseteq U$ open and $kB \subseteq V$ open and $U \cap V = N$. Let $C = cU$ and $D = U$. Then $(A, C) \notin P$ because $kA \cap kC = kA \cap cU = N$, and $(B, D) \notin P$ because $kB \cap kD \subseteq kB \cap cV = N$. Thus P. 5 for a proximity is satisfied and (M, P) is a separated proximity space. The function $k'A = \{q \mid (\{q\}, A) \in P\} = \{q / q \in kA\} = kA$.

EXAMPLE 4. Let $M = E^2$, the Euclidean plane, and let $k : 2^M \to 2^M$ be $kA =$ the convex hull of A. Define P by: $(A, B) \in P$ iff $A \cap kB \neq N$. Clearly k is isotonic, enlarging, idempotent, and $kN = N$, hence from Theorem 3, P satisfies all of the conditions for a quasi-proximity except the right hereditary property. A set C may intersect the convex hull of $A \cup B$ but not intersect either kA or kB as k is not additive. P satisfies the two conditions for Mattson's generalized quasi-proximity. The function k' obtained from P is again the convex hull function. Notice that the condition given by Pervin for quasi-proximity which states $(A, B) \notin P$ implies there exist U and V disjoint such that $(A, cU) \notin P$ and $(cV, B) \notin P$, is not stronger than condition P. 5 given for a proximity. The P in this example satisfies the former because k is idempotent, but it does not satisfy P. 5. To illustrate this let E^2 be given a Cartesian coordinate system and let $A = \{p_1, p_2\}$ and $B = \{p_3\}$ where p_1 is the point $(0, 1)$ and p_2 is $(0, -1)$ while p_3 is $(0, 0)$.

Then clearly $kB = B$ and $A \cap kB = N$, so $(A, B) \notin P$. It is not possible to find C and D which satisfy P. 5. Since B should not intersect the convex hull of D, at least one of the points p_1 and p_2 must lie in $cD = C$. This would mean $A \cap C \subseteq A \cap kC \neq N$ contrary to the restriction that $(A, C) \notin P$.

REFERENCES.

[1] P. C. HAMMER, *Extended topology: set-valued set-functions*, « Nieuw Arch. Wisk. », *10*, 55–77 (1962).

[2] S. LEADER, *On clusters in proximity spaces*, « Fund. Math. », *47*, 205–213 (1959).

[3] M. LODATO, *Generalized proximity relations*, « Proc. Amer. Math. Soc. », *15*, 417–422 (1964).

[4] D. A. MATTSON, *Extended Topology: On Abstract Spaces*, Doctoral Dissertation. University of Wisconsin, Madison, Wisconsin 1965.

[5] W. J. PERVIN, *Quasi-proximities for topological spaces*, «Math. Ann. », *150*, 325–326 (1963).

[6]. E. F. STEINER, *The relation between quasi-proximities and topological spaces*, « Math. Ann. », *155*, 194–195 (1964).

[7] A. D. WALLACE, *Separation spaces*, « Ann. of Math. », *42*, 686–697 (1941).