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## Uniform Structures from Abstract Spaces

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SUNTO. — Partendo da topologie e spazi astratti assegnati, se ne deducono uniformità (generalizzate od estese) nel senso di Weil coll'uso di opportuni insiemi di funzioni.

## INTRODUCTION.

Fréchet [2] and Appert [1] studied abstract spaces, and their work involved generalized uniform structures. In his extended topology P. C. Hammer considers abstract spaces using set-valued set-functions as the primitive notion. This approach has also been used by Z. P. Mamuzic [4]. In this paper the problem of obtaining generalized or extended uniformities from given topologies and abstract spaces is considered.

Briefly recall that a *Weil uniformity* [7] for a set M is a non-empty family $\Phi$ of subsets of $M \times M$ satisfying the following:

(*a*) $U \in \Phi$ implies $U \supseteq \Delta = \{(p, p) \mid p \in M\}$,
(*b*) $U \in \Phi$ implies $U^{-1} \in \Phi$,
(*c*) $U \in \Phi$ implies there exists $V \in \Phi$ such that $V \circ V \subseteq U$,
(*d*) $U, V \in \Phi$ implies $U \cap V \in \Phi$,
(*e*) $U \in \Phi$ and $U \subseteq V$ imply $V \in \Phi$. (This property is called « ancestral »).

The uniformity is separated if $\cap \Phi = \Delta$.

In regard to set-functions, the terminology used will be that of Hammer [3]. The empty set will be denoted by N. The term *Fréchet space* will denote an ordered pair $(M, g)$ in which $g$ is an expansive function from $2^M$ into $2^M$.

## UNIFORMITIES FROM ABSTRACT SPACES.

It is well-known that a topological space $(M, T)$ is uniformizable if and only if it is a completely regular space. Pervin has shown [5] that any topological space is quasi-uniformizable where a quasi-uniformity on a set M is collection of subsets of $M \times M$ satisfying conditions (*a*), (*c*), (*d*), and (*e*) given in the introduction above.

For spaces more general than those $(M, t)$ with Kuratowski closure function $t$, we want to know how the properties of the set-valued set-function determine the properties of the $\Phi$ obtained, in order to see what kind of uniform-like structure corresponds to the various abstract spaces. For this purpose let $(M, f)$ be a generalized topology in the sense of Mamuzic [4]

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and consider using $f$ in a way analogous to an interior function, so that for $A \subseteq M$, $fA$ is treated as the open set in Pervin's construction [5]. For each $A \subseteq M$ we define $U_A = (fA \times fA) \cup (cfA \times M)$. Then $S = \{U_A \mid A \subseteq M\}$, $B = \{V \mid V$ is a finite intersection of sets in $S\}$ and $\Phi = \{U \subseteq M \times M \mid U$ contains some $V \in B\}$.

THEOREM 1. *Let* $(M, f)$ *be a generalized topology and suppose* $\Phi$ *is defined from it as above. Then*:

(i) $\Phi$ *is ancestral and* $U \in \Phi$ *implies* $\Delta \subseteq U$,

(ii) $\Phi$ *is closed under intersection.*

(iii) $\Phi$ *has property* (*c*) *but need not be symmetric.*

*Proof*: (i) By the manner of its definition, $U \in \Phi$ and $V \supseteq U$ implies V contains some element of B, so $V \in \Phi$. For any $p \in M$ either $p \in fA$ or $p \in cfA$ for each set $A \subseteq M$. If $p \in fA$, then $(p, p) \in fA \times fA \subseteq U_A$, and if $p \in cfA$ then $(p, p) \in cfA \times M \subseteq U_A$. Therefore each $U_A$ must contain $\Delta$ and each member of $\Phi$ contains $\Delta$.

(ii) If $U, V \in \Phi$ then each one contains a finite intersection of the $U_A$'s, hence $U \cap V$ must also.

(iii) We want to prove that for each $A \subseteq M$, $U_A \circ U_A \subseteq U_A$. Let $(x, z) \in U_A \circ U_A$. Then for some $y$ we have $(x, y) \in U_A$ and $(y, z) \in U_A$. If $y \in fA$ then $z \in fA$ and $(x, z) \in U_A$. If $y \in cfA$, then $x \in cfA$ and also $(x, z) \in U_A$. Therefore $U_A \circ U_A \subseteq U_A$ for each $A \subseteq M$. Now if $U \in \Phi$, U contains some $U_{A_1} \cap U_{A_2} \cap \cdots \cap U_{A_n} = V$, and $V \in \Phi$. But $V \circ V \subseteq V$ because $U_{A_i} \circ U_{A_i} \subseteq U_{A_i}$ for $i = 1, 2, \cdots, n$, so this is a $V \in \Phi$ which satisfies $V \circ V \subseteq U$. To show that symmetry is not to be expected in $\Phi$, let $(M, f)$ be a $T_1$–topology with $f$ the Kuratowski closure and M at least countably infinite. Then for $p \in M$, $fp = p$, and we have $U_{\{p\}} = \{(p, p)\} \cup \cup \{(q, x) \mid x \in M, q \neq p\} \in \Phi$. But $U_{\{p\}}^{-1} = \{(p, p)\} \cup \{(x, q) \mid x \in M, q \neq p\}$, and therefore $V = U_{\{p\}} \cap U_{\{p\}}^{-1} = \{(p, p)\} \cup (c\{p\} \times c\{p\})$. If $U_{\{p\}}^{-1} \in \Phi$ then $V \in \Phi$ and V would have to contain some set $U_A = (fA \times fA) \cup (cfA \times M)$. But $\{q\} \times M$ is not contained in V for any $q$, so $U_A \subseteq V$ implies $cfA = N$ and $fA = M$. This means $U_A = M \times M$. Thus V cannot be in $\Phi$ which means $U_{\{p\}}^{-1} \notin \Phi$, and $\Phi$ is not symmetric.

Therefore the $\Phi$ defined in such a way is a quasi-uniformity regardless of the properties of $f$.

THEOREM 2. *Let* $(M, f)$ *be a generalized topology. If* $\Phi$ *is constructed from it as above, and* $r : 2^M \to 2^M$ *is defined by* $rA = \{p \mid U[p] \subseteq A$ *for some* $U \in \Phi\}$, *then*:

(i) If $f$ is shrinking, $f \subseteq r$,

(ii) If $f$ is an interior function, $f = r$.

*Proof*: (i) Let $A \subseteq M$ and $p \in fA$. Then $U_A = (fA \times fA) \cup (cfA \times M) \in \Phi$ and $U_A[p] = fA \subseteq A$ because $f$ is a shrinking function. Thus $p \in rA$ and $f \subseteq r$.

(ii) Although the definition given for $\Phi$ seems to use every $A \subseteq M$, if $f$ is a topological interior function then $fA$ is an open set G, so in reality, in this case, only the open sets are used, and the $\Phi$ constructed is the same as that of Pervin. Since the quasi-uniform topology from $\Phi$ is the topology we had originally, the interior function $r$ must be the same as $f$.

Therefore at least in these two cases we know the relation between $f$ and the topology obtained from $\Phi$. Without such restrictions on $f$ such a relationship between $f$ and $r$ does not necessarily hold.

We will now consider a second method of constructing a collection $\Phi$ of subsets of $M \times M$ starting with an extended topology. Let $(M, f)$ be an extended topology and for each $A \subseteq M$ define $V_A = \{(a, b) \mid b \notin A$, or else $b \in A$ and $a \in fA\} = (M \times cA) \cup (fA \times A)$. Note that $V_N = M \times M$ and $V_M = fM \times M$. Let $\Phi = \{V_A \mid A \subseteq M\}$ and consider the properties of $\Phi$.

THEOREM 3. *Let* $(M, f)$ *be an extended topology and* $\Phi$ *a nonempty family of subsets of* $M \times M$ *defined as above. Then:*

(i) *If* $f$ *is enlarging,* $\Phi$ *satisfies property (a) for Weil uniformities,*
(ii) *If* $f$ *is isotonic, then the function* $t$ *defined by* $tA = \{p \mid \{p\} \times A$ *intersects every* $V \in \Phi\}$ *is the same as* $f$, *provided* $fN = N$.

*Proof*: (i) If $fA \supseteq A$ for each $A \subseteq M$, then for a particular $A_0$ look at $V_{A_0} = (M \times cA_0) \cup (fA_0 \times A_0)$. If $p \in A_0$, $p \in fA_0$ and thus $(p, p) \in fA_0 \times A_0 \subseteq V_{A_0}$. If $p \in cA_0$, then $(p, p) \in M \times cA_0$. Hence $(p, p) \in V_{A_0}$ for every $p \in M$; i.e., $\Delta \subseteq V_{A_0}$.

(ii) From the definition of the function $t$, we have $p \in tA$ iff $\{p\} \times A$ intersects every $V \in \Phi$ and this means iff $\{p\} \times A$ intersects every $V_B$, $B \subseteq M$. If $A \cap cB \neq N$, then clearly $\{p\} \times A$ intersects $M \times cB \subseteq V_B$, hence $\{p\} \times A$ intersects every $V_B$ iff it intersects every $V_B$ for which $A \subseteq B$. If $\{p\} \times A \cap V_B \neq N$ for all $B \supseteq A$, then $\{p\} \times A \cap V_A \neq N$ and thus $\{p\} \times A \cap fA \times A \neq N$. Then $p \in fA$. Conversely if $p \in fA$ and $A \subseteq B$ then isotonicity gives $p \in fB$. Then, assuming $A \neq N$, $(\{p\} \times A) \cap (fB \times B) \neq N$ so $\{p\} \times A \cap V_B \neq N$ for all $B \supseteq A$. This proves $p \in tA$ iff $\{p\} \times A$ intersects every $V_B$ with $B \supseteq A$, which is true iff $p \in fA$; under the assumptions $f$ isotonic and $A \neq N$. Hence we know that when $f$ is isotonic, $fA = tA$ for all $A \neq N$. Clearly $tN = N$, so $t = f$ is possible only if $fN = N$. Otherwise $t$ and $f$ agree everywhere except at N.

COROLLARY 1. *If* $f$ *is a contractive function, then* $f = t$.

*Proof*: When $f$ is contractive it is isotonic, hence from part (ii) in the above theorem we know $fA = tA$ for all $A \neq N$. Since $f$ is shrinking, $fN = N$; therefore $f = t$.

Because the given $\Phi$ is not ancestral, the function $r$ which was defined by $rA = \{p \mid$ there exists $V \in \Phi$ for which $V[p] = A\}$ need not be isotonic and hence its dual $crc$ also may not be isotonic. The following example illustrates this situation.

*Example 1.* Let $M = \{a, b, c, d, e, f, g, h, k, m, n\}$, and define

$$V_1 = \{(a, b), (a, e), (b, m), (b, c), (b, g)\},$$
$$V_2 = \{(d, e), (g, b), (d, b), (k, m), (k, c), (k, g)\}, \quad \text{and}$$
$$V_3 = \{(m, d), (n, k), (g, d), (k, a)\}.$$

For $\Phi = \{V_1, V_2, V_3\}$ we have $r(\{b, e\}) = \{a, d\}$ but $r(\{a, b, e\}) = N$ and in fact $r(M) = N$. Thus $r$ is not isotonic for this $\Phi$.

When $\Phi$ is not ancestral, the function *crc* may not be the same as $t$, but when the $\Phi$ is obtained as described above using $(M, f)$ we have the following information about *crc*.

THEOREM 4. *Let* $(M, f)$ *be an isotonic space and construct* $\Phi = \{V_A \mid A \subseteq M\}$ where $V_A = (M \times cA) \cup (fA \times A)$. *Then the function* $r$ *as defined previously has its dual* $crc = t$.

*Proof*: From the definition of $r$ we know $p \in crcA$ iff for each $V_B \in \Phi$, $\{y \mid (p, y) \in V_B\} \neq cA$. Since $V_N = M \times M$, $crcN = N = tN$. Let $A \neq N$. Then $p \notin fA$ implies $\{y \mid (p, y) \in V_A\} = cA$, which means $p \notin crcA$ and $crcA \subseteq fA$. If $B \neq A$, then $cB \neq cA$ and $\{y \mid (p, y) \in V_B\} = M$ or $cB$, neither of which is $cA$. Thus $p \in crcA$ iff $\{y \mid (p, y) \in V_A\} \neq cA$ which is true iff $p \in fA$, provided $A \neq N$. But when $f$ is isotonic, $fA = tA$ for all $A \neq N$.

This theorem shows that in this situation, even though $\Phi$ is not ancestral, the function *crc* is isotonic because it agrees with $t$ everywhere, and they both agree with $f$ on every set except perhaps at N. Therefore, using *crc* it is not possible to get a function which comes any closer to $f$ than the function $t$ does.

Suppose we changed the method of constructing the family $\Phi$ from a given $(M, f)$ in order to try to make the resulting function *crc* coincide with $f$; i.e., in order to have $crcN = fN$. This we already have if $fN = N$, so assume $fN \neq N$.

THEOREM 5. *If* $(M, f)$ *is an isotonic space and* $fN \neq N$, *and* $\Phi$ *is any family of subsets of* $M \times M$, *then the function given by* $crcA = \{p \mid$ *for each* $V \in \Phi, V[p] \neq cA\}$ *does not agree with* $f$ *everywhere.*

*Proof*: Assume that *crc* agrees with $f$. Then *crc* is isotonic, and $p \in crcM$ iff for each $V \in \Phi$, $V[p] \neq N$ hence $\{p\} \times M$ intersects every $V \in \Phi$. Also $p \in crcN$ iff for each $V \in \Phi$, $\{p\} \times M$ is not a subset of V. Then $p \notin crcN$ iff there exists some $V_0 \in \Phi$ such that $\{p\} \times M \subseteq V_0$. For any $A \subseteq M$ we have $crcN \subseteq crcA \subseteq crcM$. Let $p \in crcN$. Then $\{p\} \times M$ is not a subset of any $V \in \Phi$, but $p \in crcM$ so $\{p\} \times M$ intersects every $V \in \Phi$. Let $U \in \Phi$. Then $\{p\} \times M \not\subseteq U$, but $U[p] \neq N$. Let $A = U[p]$. We know $A \neq N$ and $A \neq M$. Consider $cA$. Since $U[p] = A = c(cA)$, $p \notin crc(cA) = crA$. This is a contradiction since $p \in crcN \subseteq crc(cA)$.

Therefore, no matter how the family $\Phi$ is constructed, the function *crc* could not be the same as the isotonic function $f$ if $fN \neq N$. It could never be closer to $f$ than it is for the given construction of $\Phi$ for which $f = crc$ for all sets except N. Given a space $(M, f)$ then, it is not possible to have a uniformity $\Phi$ on M for which $f$ is the associated function if $f$ is isotonic and $fN \neq N$.

The construction of $\Phi$ can be changed slightly to make $\Phi$ ancestral. For a given $(M, f)$ let $V_A = (M \times cA) \cup (fA \times A)$ as before but define $\Phi = \{U \subseteq M \times M \mid \text{for some } A \subseteq M,\ U \supseteq V_A\}$. This does not interfere with the functions $t$ or $r$ because the small elements of $\Phi$ determine the function values. Suppose the given space $(M, f)$ is a Fréchet space with $fN = N$. Then $\Phi$, under the new definition, is ancestral, and it has property $(a)$ of Weil uniformities because $f$ is enlarging. Due to the fact that $f$ is isotonic and $fN = N$ we know that $t = f$. These results are summarized in the following.

THEOREM 6. *Let* $(M, f)$ *be a given space and* $\Phi = \{U \subseteq M \times M \mid U \supseteq V_A$ *for some* $A \subseteq M\}$. (i) *If* $f$ *is isotonic and* $fN = N$, *then* $\Phi$ *is a generalized uniformity (i.e. is ancestral) for which* $t = f$. (ii) *If* $f$ *is expansive and* $fN = N$, *then* $\Phi$ *is an extended uniformity* [6] *for which* $t = f$.

The following example is a case in which $f$ is isotonic.

*Example 2*. Let M be the set of positive integers and suppose $f: 2^M \to 2^M$ is defined by $fA = \{z \mid z = a_1 \cdot a_2 \text{ for } a_1, a_2 \in A\}$. Clearly $f$ is isotonic and $fN = N$. For $A \subseteq M$ we have $V_A = (M \times cA) \cup (fA \times A)$, and $\Phi$ is defined as $\{U \subseteq M \times M \mid U \supseteq V_A \text{ for some } A \subseteq M\}$. Then $p \in tA$ iff $p \in fA$ which is true iff $p$ can be factored in A. Notice that if $1 \in A$ then $A \subseteq fA$, but $f$ is not an enlarging function. The dual function $r = ctc$ and $p \in rA$ means $p \notin t(cA)$ and hence $p$ cannot be factored in $cA$.

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