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On a criterion of smoothness

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RENDICONTI

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Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *On a criterion of smoothness.* Nota di ALEXANDRU BREZULEANU, presentata (*) dal Socio B. SEGRE.

SUNTO. — In quest'articolo si dimostra – per il caso generale – là reciproca di un importante criterio di levigatezza (smoothness) congetturato da Grothendieck in [2], 9.6. Si danno anche alcune generalizzazioni parziali del suddetto criterio (2.1, 2.4, 3.2).

0. The rings considered in this note are commutative and unitary. We use the notations and terminology from: EGA (§§ 19–20) or IL (chap. IX and XI), [2] (pp. 95–110) and [3] (2.3, 3.1, 3.2).

Let $A \xrightarrow{u} B$ be a morphism of rings and M a B–module; u gives the homological function (in M) $T_i(B/A, M)$, $i = 0, 1, 2$ so that $T_0(B/A, M) = \Omega_{B/A} \otimes_B M$, where $\Omega_{B/A}$ is the module of A–differentials in B. If $A \xrightarrow{u} B \xrightarrow{v} C$ are morphisms of rings, then for any C–module M there is an exact sequence, $T_2(B/A, M) \to T_2(C/A, M) \to T_2(C/B, M) \to$ $\to T_1(B/A, M) \to T_1(C/A, M) \to T_1(C/B, M) \to T_0(B/A, M) \xrightarrow{v_{C/B/A} \otimes M}$ $T_0(C/A, M) \xrightarrow{u_{C/B/A} \otimes M} T_0(C/B, M) \to 0$ (see [3] 2.3).

1.0. Let $A \xrightarrow{u} B \xrightarrow{v} C = B/\mathfrak{c}$ be morphisms of rings, where $\mathfrak{c}$ is an ideal in B and v is the canonical surjection. We consider the following conditions:

a′) $T_2(C/B, C) = 0$;

c) The canonical morphism

$$N_{C/A} \xrightarrow{f} \mathfrak{c}/\mathfrak{c}^2$$

is injective (see [2] 9.2);

c′) $T_1(B/A, C) = 0$;

b) $T_0(B/A, C)$ is a projective c–module.

(*) Nella seduta del 15 novembre 1969.

1.1. LEMMA i). *If the ring* B *is noetherian and* $\mathfrak{c}$ *is generated by a* B*–regular sequence, then* $T_2(C/B, C) = 0$.

ii) *If the condition* c′) *is satisfied, then also* c) *is satisfied.*
If a′) *and* c) *are satisfied, then also* c′) *is satisfied.*

Proof. i) It results from [3] 3.2.1.

ii) It results from the exact sequence associated to $A \rightarrow B \rightarrow C$ and C ([3], 2.3):

$$\begin{array}{ccccccc} T_2(C/B, C) & \rightarrow & T_1(B/A, C) & \rightarrow & T_1(C/A, C) & \xrightarrow{f} & T_1(C/B, C) \\ & & & & \| & & \| \\ & & & & N_{C/A} & \xrightarrow{\quad f \quad} & \mathfrak{c}/\mathfrak{c}^2 \end{array}$$

(The equalities are given by [2], 9.2 and [3], 3.1.2).

1.2. Let $A \xrightarrow{u'} A' \xrightarrow{v'} A'/\mathfrak{b} = B$ be morphisms of rings, where $\mathfrak{b} = \ker v'$ and $v'u' = u$; let $\mathfrak{a}$ be an ideal of A' so that $\mathfrak{c} \subset v'(\mathfrak{a}) = \mathfrak{m}$.

LEMMA. *If the conditions* b) *and* c′) *are satisfied, then the morphism*

$$\delta_{B/A'/A} \otimes_B C : \mathfrak{b}/\mathfrak{b}^2 \otimes_B C \longrightarrow \Omega_{A'/A} \otimes_{A'} C$$

is invertible to the left.

Proof. $A \rightarrow A' \rightarrow B$ and C give the exact sequence ([3], 2.3)

$$\begin{array}{ccccccccc} T_1(B/A, C) & \rightarrow & T_1(B/A', C) & \xrightarrow{\qquad\qquad} & T_0(A'/A, C) & \rightarrow & T_0(B/A, C) & \rightarrow & 0 \\ \| & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mathfrak{b}/\mathfrak{b}^2 \otimes_B C & \xrightarrow{\delta_{B/A'/A} \otimes_B C} & \Omega_{A'/A} \otimes_{A'} C & \rightarrow & \Omega_{B/A} \otimes_B C & \rightarrow & 0 \end{array}$$

(the first equality is c′); the second is given by [3], 3.1.2.) Cf. b), $\Omega_{B/A} \otimes_B C$ is projective, hence the sequence splits.

1.3. LEMMA. *Let* $h : M \rightarrow N$ *be a morphism of* B*–modules, let* M *be separated in the* $\mathfrak{m}$*–adic topology and* N *be a projective* B*–module. If*

$$h_1 = h \otimes B/\mathfrak{m} : M/\mathfrak{m}M \rightarrow N/\mathfrak{m}N$$

is invertible to the left, then h *is also invertible to the left.*

Proof. Let $g' : N/\mathfrak{m}N \rightarrow M/\mathfrak{m}M$ be such that $g'h_1 = 1$. N being projective, there is $g : N \rightarrow M$ so that $N \xrightarrow{g} M \rightarrow M/\mathfrak{m}M = N \rightarrow N/\mathfrak{m}N \rightarrow M/\mathfrak{m}M$. Obviously $g_1 = g \otimes_B B/\mathfrak{m}$ is equal to g': it follows that $(gh)_1 = 1$, hence $(gh)_n = (gh) \otimes_B B/\mathfrak{m}^n$ is equal to 1 (see the proof of EGA, 19.1.10, i), or IL, XI, 2.2.1). Let $x \in M$; it results that $x - (gh)(x) \in \mathfrak{m}^n M$ for any $n \geq 1$, hence $gh = 1$.

2.1. THEOREM. *Let* A, A′, B, C *be as in* 1.0, 1.2 *and suppose that:*

– A′ *is a formally smooth* A*–algebra (for the discrete topologies)*

– *or* $\mathfrak{b}/\mathfrak{b}^2$ *is* $\mathfrak{m}$*–separated or* B *is a local ring and* $\mathfrak{b}/\mathfrak{b}^2$ *is a* B*–module of finite type.*

Then B *is a formally smooth* A*–algebra (for the discrete topologies) if and only if the conditions* b) *and* c′) *are satisfied.*

Proof. " If part ". Without any hypothesis about A' and $\mathfrak{b}/\mathfrak{b}^2$, from [3] 3.1.3 (or [2], 9.5.7) it results that $T_1(B/A, C) = 0$ and that $\Omega_{B/A}$ is a projective B–module.

" Only if part ". Since A' is a formally smooth A–algebra, $\Omega_{A'/A}$ is a projective A'–module (EGA, 20.4.9); hence $\Omega_{A'/A} \otimes_{A'} B$ is a projective B–module. From 1.2, it follows that $\delta_{B/A'/A} \otimes_B C$ is invertible to the left; then $\delta_{B/A'/A}$ is also invertible to the left (if $\mathfrak{b}/\mathfrak{b}^2$ is $\mathfrak{m}$–separated by 1.3; else $\delta_{B/A'/A} \otimes_B K$ is invertible to the left, where K is the residue field of B, and apply EGA, 19.1.12, b) $\Rightarrow$ a), or IL, XI, 3.1). Then B is a formally smooth A–algebra (EGA, 20.5.12, or IL, XI, 2.13). Q.E.D. Consequently $\Omega_{B/A}$ is a projective B–module.

2.2. COROLLARY. *In the hypotheses of* 2.1, *let the condition* a′) *be satisfied. Then* B *is a formally smooth* A*–algebra (for the discrete topologies) if and only if the condition* b) *and* c) *are satisfied. (Since* a′) *and* c) *imply* c′) *and* c′) *implies* c), *by* 1.1).

2.3. " *Theorem* ([2] 9.6). *Let* $A \to B \to C$ *be local homomorphisms of local noetherian rings, with* A *and* C *regular,* $B \to C$ *surjective thus* $C \simeq B/\mathfrak{c}$, $\mathfrak{c}$ *an ideal of* B, *and* B *a localisation of an* A*–algebra of finite type. Then* B *is a formally smooth* A*–algebra if and only if the following conditions are satisfied:*

a) B *is regular, i.e. the ideal* $\mathfrak{c}$ *is a regular ideal*

b) $\Omega_{B/A} \otimes_B C$ *is a projective* C*–module*

c) *The characteristic homomorphism*

$$N_{C/A} \longrightarrow \mathfrak{c}/\mathfrak{c}^2$$

is injective ".

Proof. "Only if part". From a) follows a′), by 1.1 i). Let $\mathfrak{p}$ be a prime ideal in $A[X_1, \cdots, X_n]$, $A' = A[X_1, \cdots, X_n]_{\mathfrak{p}}$ and $\mathfrak{b}$ an ideal in A' so that $B = A'/\mathfrak{b}$. A' is a formally smooth A–algebra (EGA, 19.3). Now apply 2.2.

Here A can be arbitrary, but B must be noetherian and essentially of finite presentation over A (i.e. there is A' as above with $\mathfrak{b}$ an ideal of finite type and $B = A'/\mathfrak{b}$).

We also give an alternative proof of the " if part. ". The conditions b) and c) result as in the " if part " of 2.1. Let K be the residue field of B. Let $B' = A[X_1, \cdots, X_m]_{\mathfrak{q}}$ where $\mathfrak{q}$ is a prime ideal, and $\mathfrak{u}$ and ideal of B' such that $K = B'/\mathfrak{u}$. B' is a regular ring (since A is regular), hence $\mathfrak{u}$ is generated by a B–regular sequence (since K is regular); from [3], 3.2.2. it follows that $T_2(K/A, K) = 0$. $A \to B \to K$ and K give the exact sequence ([3], 2.3)

$$T_2(K/A, K) \to T_2(K/B, K) \to T_1(B/A, K).$$

But $T_1(B/A, K) = 0$ ([3], 3.1.3. or [2], 9.5.7), i.e. $T_2(K/B, K) = 0$; hence B is a regular ring ([3], 3.2.1). Q.E.D.

For the $\mathfrak{m}$–adic topology, the theorem 2.1 has the following analogous form.

2.4. PROPOSITION. *Let* A , A′, $\mathfrak{a}$, B , $\mathfrak{m}$ *and* C *be as in* 1.2, *and suppose that*:

– A′ (*with the* $\mathfrak{a}$*–adic topology*) *is a formally smooth* A*–algebra*;
– *the topology of* $\mathfrak{b}/\mathfrak{b}^2$ *induced by* $\mathfrak{b}$ *is equal to the* $\mathfrak{m}$*–adic topology.*

If conditions b) *and* c′) *are satisfied, then* B (*with the* $\mathfrak{m}$*–adic topology*) *is a formally smooth* A*–algebra. Consequently* $\Omega_{B/A}$ *is a formally projective* B*–module.*

Proof. By EGA, 20.4.9, $\Omega_{A'/A}$ is a formally projective A–module and its topology is $\mathfrak{a}$–adic (EGA, 20.4.5), hence $\Omega_{A'/A} \otimes_{A'} B$ is a formally projective B–module (IL, IX, 1.19) and its topology is $\mathfrak{m}$–adic. By 1.2, $\delta_{B/A'/A} \otimes_B C$, hence also $\delta_{B/A'/A} \otimes_B K$ (where $K = B/\mathfrak{m}$), is invertible to the left; then $\delta_{B/A'/A}$ is formally invertible to the left (EGA, 19.1.9). From EGA 22.6.1 it follows that B (with the $\mathfrak{m}$–adic topology) is a formally smooth A–algebra. The last statement follows from EGA 20.4.9. Q.E.D.

REMARK. Let B (with the $\mathfrak{m}$–adic topology) be a formally smooth A–algebra. Then $\Omega_{B/A}$ is a formally projective B–module.

If the $\mathfrak{m}/\mathfrak{c}$–adic topology of C is discrete, or if C is noetherian and $\Omega_{B/A} \otimes_B C$ is a C–module of finite type, then $\Omega_{B/A} \otimes_B C$ is a projective C–module (by IL, XI, 2.5.1).

Hence, under the hypotheses of the proposition, less b) and under the above hypothes, the condition b) is satisfied if and only if B is a formally smooth A–algebra.

From now on all the topologies are discrete.

2.5. Let $Z \xrightarrow{i} Y \xrightarrow{h} X$ be morphisms of schemes, where i is a closed immersion of Ideal $\mathcal{I}$. Let h be decomposed in $Y \xrightarrow{i'} X' \xrightarrow{h'} X$ with i' a closed immersion of Ideal $\mathcal{J}$. Let z be a point of Z, $y = i(z)$, $x' = i'(y)$ and $x = h(y)$.

By [3], 2.2.4, the T_i commute with localisation, hence 2.1 and 2.2 can be written in terms of $\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'} \to \mathcal{O}_{Y,y} \to \mathcal{O}_{Z,z}$, $T_2(Y/X, \mathcal{O}_Z)_z$, $i = 0, 1$, $T_2(Z/Y, \mathcal{O}_Z)_z$, $T_1(Z/X, \mathcal{O}_Z)_z \to (\mathcal{I}/\mathcal{I}^2)_z$ and give the criteria for h to be locally formally smooth in y (see [2], 9.5.8).

COROLLARY. *Let* i *and* $h = h'i'$ *be as above and suppose that* h' *is locally formally smooth, that* $\mathcal{J}/\mathcal{J}^2$ *is locally of finite type* (*or that* $(\mathcal{J}/\mathcal{J}^2)_y$ *is* $\mathfrak{m}_{Y,y}$*–separate for any* $y \in Y$) *and that the topological spaces of* Z *and* Y *are the same. Then*:

i) h *is locally formally smooth if and only if* $T_0(Y/X, \mathcal{O}_Z)$ *is locally projective and* $T_1(Y/X, \mathcal{O}_Z) = 0$.

ii) *If moreover* $T_2(Z/Y, \mathcal{O}_Z) = 0$, h *is locally formally smooth if and only if* $T_0(Y/X, \mathcal{O}_Z)$ *is locally projective and the canonical morphism* $T_1(Z/X, \mathcal{O}_Z) \to \mathcal{I}/\mathcal{I}^2$ *is injective.*

3.1. Let $A \to B \to C = B/\mathfrak{c}$ be as in 1.0 and suppose that:

d) B is a laskerian [(1)] local ring, $\mathfrak{c}$ is an ideal of finite type and $\Omega_{B/A}$ is a B–module of finite presentation for any ideal $\mathfrak{d}$ in B, $\mathfrak{d} \otimes_B \Omega_{B/A}$ is $\mathfrak{c}$–separated.

LEMMA. *If the condition* c′) (*or* a′)), b) *and* d) *are satisfied, then* $\Omega_{B/A}$ *is a projective* B–*module.*

Proof. $A \to B$ and $0 \to \mathfrak{c} \to B \to C \to 0$ give the exact sequences ([3], 2.3).

$$(1)\quad T_1(B/A, \mathfrak{c}) \to T_1(B/A, B) \to T_1(B/A, C) \to \Omega_{B/A} \otimes_B \mathfrak{c} \to \Omega_{B/A} \to \Omega_{B/A} \otimes_B C \to 0$$

$$0 \longrightarrow \mathrm{Tor}_1^B(\Omega_{B/A}, C) \nearrow$$

But $\mathrm{Tor}_1^B(\Omega_{B/A}, C) = 0$, since $T_1(B/A, C) = 0$; from this and form b) and d) it results that $\Omega_{B/A}$ is a flat B–module (IL, IV, 6.12 ii) hence it is a projective B–module (IL, IV, 6).

3.2. PROPOSITION. *Let the condition* d) (*resp. and* a′)) *be satisfied.* B *is a formally smooth* A–*algebra if and only if* $T_1(B/A, \mathfrak{c}) = 0$ *and the conditions* c′) (*resp.* c)) *and* b) *are satisfied.*

Proof. The " if part " is true without d) and a′) (see [3], 3.1.3).

" Only if part ". By sequence (1), c′) and $T_1(B/A, \mathfrak{c}) = 0$ it results that $T_1(B/A, B) = 0$. $\Omega_{B/A}$ is a projective B–module (by 3.1); hence B is a formally smooth A–algebra ([2] 9.5.7).

3.3. Let $Z \xrightarrow{i} Y \xrightarrow{h} X$ be morphisms of schemes, where i is a closed immersion of Ideal $\mathfrak{J}$, X is a noetherian scheme and h is locally of finite type; let $z \in Z$ and $y = i(z)$, $x = h(g)$.

The proposition gives a criterion of smoothness in y in terms of the stalks of T_i, hence:

COROLLARY. *Let i and h be as above and suppose that the topological spaces of* Z *and* Y *are the same* (*resp. and* $T_2(Z/Y, \mathcal{O}_Z) = 0$). *Then h is smooth if and only if the following conditions are satisfied*

– $T_1(Y/X, \mathcal{O}_Z) = 0$ (*resp. the canonical morphism* $T_1(Z/X, \mathcal{O}_Z) \to \mathfrak{J}/\mathfrak{J}^2$ *is injective*).

– $T_1(Y/X, \mathfrak{J}) = 0$.

– $T_0(Y/X, \mathcal{O}_Z)$ *is locally projective.*

(1) A ring B is called laskerian if every ideal in B is a finite intersection of primary ideals (IL, III, 2.1).

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