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On a Conjecture of Severi

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Geometria. — *On a Conjecture of Severi.* Nota di LEONARD ROTH † presentata (*) dal Socio B. SEGRE.

SUNTO. — Questo è un frammento lasciato fra le sue carte dal compianto matematico inglese, assai legato alla Scuola geometrica italiana, il quale perì tragicamente nei pressi di Pittsburgh il 28 novembre 1968. In esso viene analizzata una congettura di Severi relativa alla razionalità di certe varietà algebriche.

1. In his expositions of the theory of rational equivalence Severi has repeatedly (1932 and again [6] vol. III, p. 306 etc.) conjectured (1) that, on an algebraic variety, an irreducible algebraic series Σ of sets of n points such that any two sets lie in a g^1_n of sets, is necessarily a rational series (of equivalence). We are thus led to the conjecture: any algebraic variety V_r $(r \geq 2)$ is unirational provided any two of its points lie on some rational curve of V_r.

2. Even in the case $r = 2$, this conjecture—which is then certainly true—is by no means simple to justify, except in the one case where the curves in question form a net. The existence, on a surface V_2, of an ∞^2 algebraic system of rational curves implies that $p_g = p_i = 0$: hence (Castelnuovo-Enriques) V_2 in either rational or scrollar; but in the latter case, V_2 contains a unique pencil of rational curves and no ∞^2 system of rational curves. But the demonstration rests on the theory of surfaces for which $p_g = 0$, $p_a = -1$.

An alternative proof, again not simple, is obtained by appealing to the $n > 2\pi - 2$ theorem [2]. This does not extend to varieties of dimension $r > 2$.

3. B. Segre [5] has shown that, if Severi's conjecture were confirmed in case $r = 3$, it would follow that the general V^4_3 of S_4 is unirational. Segre has given an example of a non singular V^4_3 which is unirational, but this is a special type. Moreover, Segre's proof of the unirationality does not make use of the known ∞^4 system of rational curves on the V^4_3.

4. We may compare Severi's conjecture, in the case $r = 3$, with a known result for threefolds. Let V_3 be a non-singular variety containing a *net* $|A|$ of rational surfaces, the general member of which is non-singular.

(*) Nella seduta del 15 novembre 1969.

(1) However, in 1955, Severi [7] proposed the following wider definition: two sets G_n, G'_n of Σ are rationally equivalent provided there exists a third set H_m, say, such that $G_n + H_m$, $G'_n + H_m$ belong to a rational series on the given variety. With this definition in mind we see that the conjecture loses some of its force.

Then Enriques has shown that, provided the characteristic curves of $|A|$ do *not* belong to a congruence of rational or elliptic curves, then V_3 must be unirational. In the two exceptional cases just mentioned, Enriques' demonstration may fail. Yet through any two points of V_3 there pass an *infinity* of rational curves, lying on a surface A.

That V_3 may actually prove to be non-unirational in the exceptional cases has been conjectured by Fano [3]. According to him, the general V_3^n of S_4 $(n > 4)$ whose only singularity is an ordinary $(n - 2)$–fold line l is presumably not unirational. On any non-singular model of V_3^n we have a net $|A|$ corresponding to prime sections of V_3^n through l; the characteristic curves of $|A|$ are rational curves forming a congruence—they are images of plane sections of V_3^n through l. Thus, if Fano's conjectures were confirmed, the possibility of an exception to Enriques' theorem would be established and Severi's conjecture for $r = 3$ would be disproved.

5. With regard to the case $r > 3$, we observe that there is no known extension of Enriques' theorem to manifolds of higher dimension.

Let us first see what can be said of a variety V_r $(r \geq 3)$ which contains an irreducible algebraic ∞^{2r-2} system of rational curves C of which a *finite number* ν (≥ 1) pass through two general points of V_r.

If just one curve C *passes through two general points of* V_r, *then* V_r *is birational.*

For the curves C issuing from a general point P of V_r have in general a *simple* point there (though they may possess singularities elsewhere). Thus the subsystem of such curves is in $(1, 1)$ correspondence with the first neighbourhood of P, and hence is birational. We therefore have on V_r a birational congruence of rational curves with a unisecant V_{r-1}, namely the first neighbourhood of P. Whence the result.

Next, *if* ν (> 1) *curves* C *pass through two general points of* V_r, *while those curves which pass through a general point of* V_r *form a unirational subsystem, then* V_r *is unirational.*

For, as before, those curves C issuing from P have in general a simple point there, while the first neighbourhood of P is in effect a birational plurisecant V_{r-1} to the subsystem in question [4].

6. Neither of these results is however strictly analogous to the known general result for $r = 2$, which contemplates *birational* systems. What Castelnuovo–Enriques have done is to arrive by a roundabout route at a *rational pencil* of *rational curves*, which do not belong to the given birational system of rational curves.

It is possible that the results just obtained are as much as Severi requires for his theory of rational equivalence, so that the conjecture in its most general form is superfluous. The same remark applies to the next criterion which we shall give.

7. After considering the case of a g_n^1 it is natural to pass on to a g_n^2. Assuming that any two sets of our series are contained in a unique g_n^2, we have, on the associated V_r (assumed non-singular), a family of rational surfaces F such that just one F passes through two general points P, Q of V_r. Each F is generated by ∞^1 rational curves C which in general have P and Q as *simple* base points.

Supposing first that $r = 3$, then we have a net $|F|$ of surfaces, without base points, if we assume that the curves C have no fixed points. Hence the general member of $|F|$ is non-singular.

Now, by Enriques' theorem [1], provided the characteristic curves of $|F|$ do not belong to a congruence of rational or elliptic curves, it follows that V_3 must be unirational. [It is essential to the proof that the general member of $|F|$ should be non-singular].

Secondly, suppose that $r > 3$: then, with hypotheses similar to the above, we have on V_r a system $\{F\}$ of rational surfaces, the general member of which is non-singular, such that just one F passes through two general points of V_r. Then, as in the proof of Enriques' theorem, we see that, if $\{F\}$ does not belong to a congruence of rational or elliptic curves, and if the subsystem passing through a given point of V_r is unirational, then V_r is unirational.

In conclusion, then

If a non-singular variety V_r *contains a system of rational surfaces such that*

(i) *the general member is non-singular*;

(ii) *just one surface passes through two general points of* V_r;

(iii) *the system having an assigned base point is unirational*;

(iv) *the system does not belong to a congruence of rational or elliptic curves*; *then* V_r *is unirational.*

8. There is next the possibility that the subsystem of $\{F\}$ having an assigned base points is such that $\nu > 1$ members pass through a second assigned point.

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