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Analisi matematica. — *On a class of integro–differential equations.* Nota I di MEHMET NAMIK OĞUZTÖRELI (*) (**), presentata (***) dal Socio M. PICONE.

RIASSUNTO. — In questo lavoro si studia la soluzione di un'equazione integro–differenziale a derivate parziali del second'ordine dipendente da due parametri.

In the present article we investigate the solution of an integro-partial differential equation of the second order depending on two parameters.

1. INTRODUCTION.

In this paper we consider the integro-differential equation

$$(1.1)\quad \lambda \frac{\partial^2 u(x,y)}{\partial x^2} + (\lambda - 1)\, u(x,y) = g_0(x,y) + \mu \iint_R K(x,y;\xi,\eta)\, u(\xi,\eta)\, d\xi\, d\eta$$

subject to the boundary conditions

$$(1.2)\qquad u(0,y) = u(1,y) = 0 \qquad \text{for} \quad 0 \leq y \leq 1,$$

where $R = \{(x,y) \mid 0 \leq x, y \leq 1\}$, λ and μ are real parameters, $g_0(x,y)$ and $K(x,y;\xi,\eta)$ are given real-valued functions which are continuous on R and $R \times R$ respectively, and $u(x,y)$ is the unknown function. In the following we investigate continuous solutions of the boundary value problem (1.1)–(1.2) for arbitrary λ and for sufficiently small μ.

Before dealing with the general case, we consider the following special cases obtained from (1.1) by putting $\lambda = 0$, $\lambda = 1$ and $\mu = 0$.

For $\lambda = 0$ Eq. (1.1) reduces to the following Fredholm integral equation:

$$(1.3)\qquad u(x,y) + \mu \iint_R K(x,y;\xi,\eta)\, u(\xi,\eta)\, d\xi\, d\eta = g_0(x,y).$$

Clearly, Eq. (1.3) does not, in general, have a solution which satisfies the boundary conditions (1.2).

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Further, for $\mu = 0$ we have the differential equation

$$\lambda \frac{\partial^2 u(x,y)}{\partial x^2} + (\lambda - 1) u(x,y) = g_0(x,y). \tag{1.4}$$

Solving Eq. (1.4) subject to the boundary conditions (1.2) is equivalent to solving the following Fredholm integral equation:

$$u(x,y) = \frac{1}{\lambda} f_0(x,y) + \frac{\lambda - 1}{\lambda} \int_0^1 G(x,\sigma) u(\sigma,y) d\sigma, \tag{1.5}$$

where

$$G(x,\sigma) = \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{\sin k\pi x \sin k\pi\sigma}{k^2} = \begin{cases} x(1-\sigma) & \text{for } 0 \le x \le \sigma \le 1, \\ \sigma(1-x) & \text{for } 0 \le \sigma \le x \le 1, \end{cases} \tag{1.6}$$

and

$$f_0(x,y) = -\int_0^1 G(x,\sigma) g_0(\sigma,y) d\sigma. \tag{1.7}$$

It is well known that the eigenvalues and eigenfunctions of the symmetric kernel $G(x,\sigma)$ are $k^2\pi^2$ and $\Phi_k(x) = \sqrt{2} \sin k\pi x$, respectively. Thus, if $\frac{\lambda - 1}{\lambda} \neq k^2\pi^2$, i.e., if

$$\lambda \neq \lambda_k = \frac{1}{1 - k^2\pi^2} \qquad (k = 1, 2, 3, \cdots) \tag{1.8}$$

then Eq. (1.4) has a unique solution which is given by the formula

$$u(x,y) = \frac{1}{\lambda} f_0(x,y) + \frac{\lambda - 1}{\lambda^2} \int_0^1 \Gamma\left(x, \sigma; \frac{\lambda - 1}{\lambda}\right) f_0(\sigma,y) d\sigma, \tag{1.9}$$

where $\Gamma(x,\sigma;\nu)$ is the resolvent of the kernel $G(x,\sigma)$:

$$\Gamma(x,\sigma;\nu) = 2 \sum_{k=1}^{\infty} \frac{\sin k\pi x \sin k\pi\sigma}{k^2\pi^2 - \nu}. \tag{1.10}$$

The solution can also be given in the form

$$u(x,y) = \frac{1}{\lambda} f_0(x,y) + \frac{\lambda - 1}{\lambda^2} \sqrt{2} \sum_{k=1}^{\infty} \frac{f_{0,k}(y) \sin k\pi x}{k^2\pi^2 - \frac{\lambda - 1}{\lambda}}, \tag{1.11}$$

with

$$f_{0,k}(y) = \sqrt{2} \int_0^1 f_0(x,y) \sin k\pi x \, dx, \qquad k = 1, 2, 3, \cdots. \tag{1.12}$$

Further, if $\lambda = \lambda_{k_0}$ for some index k_0, then Eq. (1.4) has a solution, not uniquely determined, if and only if $f_{0,k_0}(y) = 0$ for $0 \le y \le 1$. In this case the solution is given by the formula

$$(1.13) \qquad u(x, y) = (1 - k_0^2 \pi^2) f_0(x, y) + (1 - k_0^2 \pi^2) k_0^2 \sqrt{2} \sum_{\substack{k=1 \\ k \neq k_0}}^{\infty} \frac{f_{0,k}(y) \sin k\pi x}{k^2 - k_0^2} + \Phi(y) \sin k_0 \pi x ,$$

where $\Phi(y)$ is an arbitrary function continuous in the interval $0 \leq y \leq 1$.

Finally, for $\lambda = 1$ Eq. (1.1) reduces to the following simple integro-differential equation:

$$(1.14) \qquad \frac{\partial^2 u(x, y)}{\partial x^2} = g_0(x, y) + \mu \iint_R K(x, y; \xi, \eta) u(\xi, \eta) \, d\xi \, d\eta .$$

We can easily show that solving Eq. (1.14) subject to the boundary conditions (1.2) is equivalent to solving the Fredholm integral equation:

$$(1.15) \qquad u(x, y) = f_0(x, y) + \mu \iint_R H(x, y; \xi, \eta) u(\xi, \eta) \, d\xi \, d\eta ,$$

where

$$(1.16) \qquad H(x, y; \xi, \eta) = \int_0^1 G(x, \sigma) K(\sigma, y; \xi, \eta) \, d\sigma.$$

The kernel $H(x, y; \xi, \eta)$ is not symmetric, in general. By the theory of integral equations Eq. (1.15) has a unique solution for sufficiently small μ.

Clearly, the numbers λ_k defined by Eq. (1.8) lie in the interval $\left[\frac{1}{1 - \pi^2}, 0\right)$ and form a monotonic increasing sequence approaching zero:

$$(1.17) \qquad \frac{1}{1 - \pi^2} < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots < 0 \quad , \quad \lim_{k \to \infty} \lambda_k = 0 .$$

In the following we always assume $\lambda \neq 0$.

2. Solution for small μ.

In this section we deal with the boundary value problem (1.1)–(1.2) for sufficiently small μ. For this purpose we seek a solution of the form

$$(2.1) \qquad \begin{cases} u(x, y) = \sum_{n=0} \mu^n u_n(x, y) , \\ u_n(0, y) = u_n(1, y) = 0 \quad \text{for } 0 \leq y \leq 1 , \qquad n = 0, 1, 2, \cdots . \end{cases}$$

Substituting (2.1) into (1.1) and comparing the coefficients of μ^n on both sides, we find

$$(2.2) \qquad \begin{cases} \lambda \frac{\partial^2 u_0(x, y)}{\partial x^2} + (\lambda - 1) u_0(x, y) = g_0(x, y) , \\ \lambda \frac{\partial^2 u_n(x, y)}{\partial x^2} + (\lambda - 1) u_n(x, y) = \iint_R K(x, y; \xi, \eta) u_{n-1}(\xi, \eta) \, d\xi \, d\eta, \\ \qquad n = 1, 2, 3, \cdots . \end{cases}$$

Note that the first equation in (2.2) is exactly the same as Eq. (1.4) whose solution is discussed in § 1. Further, we can easily verify that each $u_n(x,y)$, for $n \geq 1$, satisfies the Fredholm integral equation

$$u_n(x,y) = -\frac{1}{\lambda}\iint_R H(x,y;\xi,\eta)\,u_{n-1}(\xi,\eta)\,d\xi\,d\eta + \frac{\lambda-1}{\lambda}\int_0^1 G(x,\sigma)\,u_n(\sigma,y)\,d\sigma, \tag{2.3}$$

where $H(x,y;\xi,\eta)$ is given by Eq. (1.15). Thus, if $\lambda \neq \lambda_k$ then Eq. (2.3) has a unique solution given by the formula:

$$u_n(x,y) = -\frac{1}{\lambda}\iint_R \Omega\left(x,y;\xi,\eta;\frac{\lambda-1}{\lambda}\right)u_{n-1}(\xi,\eta)\,d\xi\,d\eta, \tag{2.4}$$

where

$$\Omega(x,y;\xi,\eta;\nu) = H(x,y;\xi,\eta) + \nu\int_0^1 \Gamma(x,\sigma;\nu)\,H(\sigma,y;\xi,\eta)\,d\sigma. \tag{2.5}$$

If $\lambda = \lambda_k$ for some $k = k_0$, then Eq. (2.3) has a solution, not uniquely determined, if and only if

$$\int_0^1\int_0^1\int_0^1 H(x,y;\xi,\eta)\,u_{n-1}(\xi,\eta)\sin k_0\pi x\,d\xi\,d\eta\,dx = 0 \tag{2.6}$$

holds for $0 \leq y \leq 1$. In this case the solution is of the form

$$u_n(x,y) = (1-k_0^2\pi^2)\left\{f_n(x,y) + k_0^2\sqrt{2}\sum_{\substack{k=1\\k\neq k_0}}\frac{f_{n,k}(y)\sin k\pi x}{k^2-k_0^2}\right\} + \Phi_n(y)\sin k_0\pi x \tag{2.7}$$

where $\Phi_n(y)$ is an arbitrary continuous function in the interval $0 \leq y \leq 1$, and

$$\begin{cases} f_0(x,y) = -\displaystyle\int_0^1 G(x,\sigma)\,g_0(\sigma,y)\,d\sigma, \\ f_n(x,y) = -\displaystyle\iint_R H(x,y;\xi,\eta)\,u_{n-1}(\xi,\eta)\,d\xi\,d\eta, \quad n = 1,2,3,\cdots \\ f_{n,k}(y) = \sqrt{2}\displaystyle\int_0^1 f_n(x,y)\sin k\pi x\,dx, \quad k = 1,2,3,\cdots, \quad n = 0,1,2,\cdots. \end{cases} \tag{2.8}$$

The above recursive process works if we can choose the arbitrary functions $\Phi_0(y), \Phi_1(y), \cdots, \Phi_n(y), \cdots$ in such a manner that the orthogonality condition (2.6) will be satisfied for any n.

To establish the convergence of the series (2.1) we first introduce the notation

$$\|w\| = \max_{R} |w(x, y)| \tag{2.9}$$

for any $w = w(x, y)$ defined on R, and

$$\left\{\begin{aligned} H &= \left\| \iint_R |H(x,y;\xi,\eta)|\, d\xi d\eta \right\| , \quad K = \left\| \iint_R |K(x,y;\xi,\eta)|\, d\xi\, d\eta \right\| , \\ \Gamma_\nu &= \left\| \int_0^1 |\Gamma(x,\sigma;\nu)|\, d\sigma \right\| , \quad \Omega_\nu = \left\| \iint_R |\Omega(x,y;\xi,\eta;\nu)|\, d\xi\, d\eta \right\| , \end{aligned}\right. \tag{2.10}$$

where $\nu = \frac{\lambda - 1}{\lambda}$. Note that $\|f_0\| < \frac{1}{8}\|g_0\|$, $H < \frac{1}{8} K$ and $\Omega_\nu \leq (1 + |\nu| \Gamma_\nu) H$ by Eqs. (1.7), (1.15), (2.5) and (2.10). Further, by virtue of Eqs. (1.9) and (2.4), we may write

$$\left\{\begin{aligned} &\|u_0\| \leq \frac{1 + |\nu| \Gamma_\nu}{|\lambda|} \|f_0\| \\ &\|u_n\| \leq \frac{\Omega_\nu}{|\lambda|} \|u_{n-1}\| \leq \left(\frac{\Omega_\nu}{|\lambda|}\right)^n \frac{1 + |\nu| \Gamma_\nu}{|\lambda|} \|f_0\| , \qquad n = 1, 2, 3, \cdots \end{aligned}\right. \tag{2.11}$$

and, by Eqs. (1.4) and (2.2),

$$\left\{\begin{aligned} &\left\| \frac{\partial^2 u_0}{\partial x^2} \right\| < \frac{8 + |\nu| + \nu^2 \Gamma_\nu}{8|\lambda|} \|g_0\| , \\ &\left\| \frac{\partial^2 u_n}{\partial x^2} \right\| < \frac{(1 + |\nu| \Gamma_\nu)(K + |\nu| \Omega_\nu)}{8\lambda^2} \left(\frac{\Omega_\nu}{|\lambda|}\right)^{n-1} \|g_0\| , \qquad n = 1, 2, 3, \cdots . \end{aligned}\right. \tag{2.12}$$

Clearly, the series

$$\sum_{n=0}^{\infty} \frac{1 + |\nu| \Gamma_\nu}{|\lambda|} \|f_0\| \left(\frac{\Omega_\nu}{|\lambda|}\right)^n |\mu|^n \tag{2.13}$$

and

$$\frac{8 + |\nu| + \nu^2 \Gamma_\nu}{8|\lambda|} \|g_0\| + \sum_{n=1}^{\infty} \frac{(1 + |\nu| \Gamma_\nu)(K + |\nu| \Omega_\nu)}{8\lambda^2} \|g_0\| \left(\frac{\Omega_\nu}{|\lambda|}\right)^{n-1} |\mu|^n \tag{2.14}$$

dominate the series $\sum_{n=0}^{\infty} \mu^n u_n(x, y)$ and $\sum_{n=0}^{\infty} \mu^n \frac{\partial^2 u_n(x, y)}{\partial x^2}$ respectively, and both converge for

$$|\mu| < \frac{|\lambda|}{\Omega_\nu} . \tag{2.15}$$

The uniform and absolute convergence of the series $\sum_{n=0}^{\infty} \mu^n \frac{\partial u_n(x, y)}{\partial x}$ can be established similarly. Thus, the series (2.1) converges absolutely and uniformly on the square R, and can be twice differentiated term by term with respect to x for μ satisfying the inequality (2.15), where $\lambda \neq \lambda_k$. Hence, assuming that $\lambda \neq 0$, we have the following

THEOREM 1. *For* $\lambda \neq \lambda_k$ *and for* μ *satisfying the inequality* (2.15), *there exists a unique solution* $u(x, y)$, *given by Eq.* (2.1), *of the boundary value problem* (1.1)-(1.2).

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