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## On a class of finite groups

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**Algebra.** — *On a class of finite groups.* Nota di ANTONIO MACHÍ, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si studiano i gruppi finiti tali che due loro sottogruppi qualsiansi dello stesso ordine risultano coniugati. Si dimostra che: nel caso nilpotente, tali gruppi sono ciclici; nel caso risolubile, sotto opportune condizioni per un 2–sottogruppo di Sylow, detti gruppi ammettono un quoziente isomorfo al gruppo alterno $A_4$ su quattro lettere; nel caso generale, sotto le stesse condizioni per un 2–sottogruppo di Sylow, tali gruppi hanno un subquoziente isomorfo ad $A_4$.

## INTRODUCTION.

In this paper we study the structure of finite groups G with the following property:

*If* H *and* K *are two subgroups of* G *and* $|H| = |K|$, *then* $H \sim K$. i.e., any two subgroups of the same order are conjugate. We will write $G \in (C)$ if G is a finite group with the above property. Under the stronger condition that two subgroups of the same order be conjugate in their union, these groups have been classified by G. Pazderski [1]. In that paper, the author proves that G′—the derived group of G—is cyclic, that the quotient $G/G'$ is also cyclic, and that the orders of G′ and $G/G'$ are relatively prime.

It is clear that $A_4$—the alternating group on four letters—belongs to (C). Our main result will be that, under suitable conditions on the 2–Sylow subgroups of G, the group $A_4$ is involved in G. This will be proved by an application of Lemma 1.2.3. of the Hall-Higman paper.

In section 1, we show that nilpotent groups belonging to (C) are cyclic. In section 2, Sylow subgroups of groups $G \in (C)$ are studied. In section 3, we consider solvable groups $G \in (C)$ and show that if a 2–Sylow subgroup of G is quaternion (of order 8) or elementary abelian of order 4 (Klein group), then G contains a normal subgroup H such that $G/H \cong A_4$. In section 4, dropping the hypothesis of solvability, we show that $A_4$ is involved in G.

We observe that, if $G \in (C)$ and $N \trianglelefteq G$, then $G/N \in (C)$. In fact, if $|K_1/N| = |K_2/N|$, then $|K_1| = |K_2|$ and therefore $K_1 \sim K_2$. Let $K_1^x = K_2$ for some $x \in G$. Then $(K_1/N)^{xN} = K_2/N$.

## 1.—THE NILPOTENT CASE.

PROPOSITION 1. *If* $G \in (C)$ *and* G *is nilpotent, then* G *is cyclic.*

*Proof.* As we observed at the end of the Introduction, if $G \in (C)$ every quotient of G belongs to (C). Let G′ be the derived group of G. Then $G/G' \in (C)$ and is abelian. Since then every subgroup of $G/G'$ is normal,

(*) Nella seduta del 14 febbraio 1970.

$G/G'$, belonging to (C), has at most one subgroup of order $n$ for every positive integer $n$. It is well known that such groups are cyclic. Since G is nilpotent, $\Phi(G) \geq G'$—where $\Phi(G)$ is the Frattini subgroup of G. Thus $G/\Phi(G)$ is cyclic, and this implies G cyclic, q.e.d.

## 2.—The Sylow subgroups.

If $G \in (C)$ and $P \in \mathrm{Syl}_p(G)$, P is a $p$-group in which any two subgroups of the same order are isomorphic (being conjugate in G). Such $p$-groups have been determined by R. Armstrong [2], and are of the following types:

i) *cyclic*;

ii) *elementary abelian*;

iii) *non abelian of order $p^3$, with exponent $p$ if $p$ is odd and quaternion if $p = 2$.*

The following Proposition shows that if G is solvable the non abelian case for $p$ odd cannot happen.

PROPOSITION 2. *Let $G \in (C)$ and G solvable. If $p$ is an odd prime, then a $p$-Sylow subgroup of G is abelian.*

*Proof.* Let $P \in \mathrm{Syl}_p(G)$, $H \lhd G$, $P \not\leq H$. Let $|P \cap H| = p^a$. The subgroup $P \cap H$ is unique of its order in P, since if $|K| = |P \cap H|$ and $K \leq P$, then

$$K = (P \cap H)^x < H^x = H\,, \qquad \text{for some} \quad x \in G\,,$$

and therefore $K \leq H$, so that $K = P \cap H$. Thus if $a \geq 1$, P is cyclic [3], unless $P \leq H$. If $P = H$, $P \lhd G$. Suppose, in this case, P not abelian. Then $1 < Z(P) < P$, so that $Z(P)$ is unique of its order in P: if not, let $K < P$ with $|K| = |Z(P)|$, $K \neq Z(P)$. Then $Z(P)^x = K$, for some $x \in G$, and since $P \lhd G$, conjugation by $x$ induces an automorphism of P moving $Z(P)$, which is impossible. Thus P is cyclic, a contradiction. Therefore P is abelian. Suppose now $H = G'$, so that $G/H$ is abelian. By above, if P is not cyclic, $P \leq G'$. Let

$$G > G' > G'' > \cdots > G^{(n)} = 1$$

be the derived series of G. Since $P \neq 1$, there exists an $i \geq 1$ such that $P \not\leq G^{(i)}$. Consider $P \cap G^{(i)}$. We have the following possibilities:

*a)* $|P \cap G^{(i)}| = 1$. If $P < G^{(i-1)}$, P is abelian, since $G^{(i-1)}/G^{(i)}$ is abelian. If $P \cap G^{(j)} = 1$, with $2 \leq j \leq i - 1$, G is abelian since $G'/G''$ is abelian: if $P \cap G^{(j)} \neq 1$, P is cyclic if $P \not\leq G^{(i-1)}$, and abelian in the other case.

*b)* $|P \cap G^{(i)}| = p^a$, $a \geq 1$. In this case P is cyclic. q.e.d.

Thus, in case G solvable, we have the following possibilities for a $p$–Sylow subgroup of $G \in (C)$:

i) *cyclic*;
ii) *elementary abelian*;
iii) *quaternion of order* 8.

Let now P be any $p$–Sylow subgroup of a solvable group $G \in (C)$, and consider $P \cap G'$ with $P \not\leq G'$. Let $|P \cap G'| = p^a$. If $a = 0$, P is cyclic, being isomorphic to a subgroup of $G/G'$. If $a = 1$, since $P \cap G'$ is unique of order $p$ in P, we have $P \cap G' \leq Z(P)$, and since $PG'/G' \cong P/P \cap G'$ is cyclic, we have P abelian. Thus P cannot be quaternion, and so is cyclic. If $a > 1$, P is cyclic. We have proved:

*If* G *is a solvable group belonging to* (C) *and* $P \in Syl_p(G)$ *is not cyclic, then* $P \leq G'$.

## 3.—THE SOLVABLE CASE.

Theorem 1 below is a straightforward application of the following result of Zassenhaus:

LEMMA (Zassenhaus [4]). *Let* G *be a solvable finite group. Suppose that for some integer* $s > 1$, $2^{s+1} \nmid G$ *and that* G *has an element of order* $2^{s-1}$. *Then* G *has a normal subgroup* H *such that a* 2*–Sylow subgroup of* H *is cyclic and either* $G/H \cong Z_2$, *or* $G/H \cong A_4$ *or* $G/H \cong S_4$.

THEOREM 1. *Let* $G \in (C)$ *and solvable. Let a* 2*–Sylow subgroup of* G *be quaternion of order* 8 *or elementary abelian of order* 4. *Then* G *contains a normal subgroup* H *such that* $G/H \cong A_4$.

*Proof.* Clearly, G satisfies the hypothesis of the Lemma. Suppose $G/H \cong Z_2$; it follows:

i) if a 2–Sylow subgroup is quaternion, Q, then $Q/Q \cap H \cong QH/H \cong Z_2$, which implies $|Q \cap H| = 4$. Since $H \lhd G$, the subgroup $Q \cap H$ of Q cannot be conjugate to another subgroup of order 4 of Q;
ii) if a 2–Sylow subgroup is elementary abelian of order 4, V, then $V/V \cap H \cong VH/H \cong Z_2$, which implies $|V \cap H| = 2$; the subgroup $V \cap H$ of V cannot be conjugate to another subgroup of order 2 of V.

Thus, $G/H \not\cong Z_2$. Observe that $S_4 \notin (C)$, since a 2–Sylow subgroup of $S_4$ is dihedral of order 8. But if $G \in (C)$, every quotient does. It follows that G has a normal subgroup H such that $G/H \cong A_4$. q.e.d.

*Remark.* Under the weaker hypothesis that two subgroups of the same order are isomorphic, we can conclude that G has a normal subgroup H such that either $G/H \cong Z_2$ or $G/H \cong A_4$. The case $G/H \cong S_4$ is excluded because QH/H and VH/H cannot be dihedral of order 8.

## 4.—THE GENERAL CASE.

*Notation.* If G is a finite group and $p$ is a prime, by $O_p(G)$ we mean the maximal normal $p$–subgroup of G, and by $O_{p'}(G)$ the maximal normal subgroup of G whose order is not divisible by $p$.

*Definition.* A group G is said to be $p$–solvable if it has a normal series in which every quotient is either a $p$–group or a $p'$–group.

If G is $p$–solvable and $O_{p'}(G) = 1$, it is a consequence of Lemma 1.2.3. of the Hall-Higman paper [5] that $G/O_p(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(O_p(G)/\Phi(O_p(G)))$ [6].

*Definition.* A group H is said to be *involved* in a group G if G contains two subgroups $K_1$ and $K_2$ with $K_1 \lhd K_2$ such that $K_2/K_1 \cong H$. We can now prove:

THEOREM 2. *Let* $G \in (C)$. *If a 2–Sylow subgroup* P *of* G *is either quaternion or elementary abelian of order* 4, *then* $A_4$ *is involved in* G.

*Proof.*

i) $P = Q$, quaternion. Let $N = N_G(Q)$. Observe that $N > Q$: if $N = Q$, two subgroups of order 4 in Q, being conjugate in G and normal in Q are conjugate in $N = Q$ [7], a contradiction. Thus $N > Q$. Consider $C = C_G(Q)$. It is well known that $\mathrm{Aut}(Q) \cong S_4$; thus $N/C \lesssim S_4$. Since $|Q \cap C| = 2$, and a 2–Sylow subgroup of N/C is $QC/C \cong Q/Q\cap C \cong V$, Klein group, we have $N/C < S_4$ and $|N/C| = 4$ or 12. Suppose $|N/C| = 4$. Then, if $i, j$ are two elements of order 4 in Q—not belonging to the same subgroup of order 4—such that $i^x = j$, $x \in N$, we have

$$(iC)^{xC} = jC = iC$$

a contradiction. Thus $|N/C| = 12$, and since $N/C \lesssim S_4$, we have $N/C \cong A_4$.

ii) $P = V$, elementary abelian of order 4. As above, we have $N = N_G(V) > V$, since the subgroups of order 2 must be conjugate in N. In N we have the normal series

$$N \rhd V \rhd 1$$

with N/V odd and V a 2–group. Thus N is 2–solvable. Suppose $O_{2'}(N) = 1$. Then since $O_2(N) = V$, we have $\Phi(O_2(N)) = 1$, and, by [6], N/V isomorphic to a subgroup of $\mathrm{Aut}(V) \cong S_3$. Since N/V is odd, we have $|N/V| = 1$ or 3. In the first case, $N = V$, excluded. Thus, V splits N by a subgroup of order 3. Since the elements of order 3 must induce a non trivial conjugation, the splitting is a semidirect product, i.e. $N \cong A_4$.

Suppose now $O_{2'}(N) > 1$. Then $\bar{N} = N/O_{2'}(N)$ is such that $O_{2'}(\bar{N}) = 1$ and $O_2(\bar{N}) \cong V$. Thus $|\bar{N}/O_2(\bar{N})| = 1$ or 3. In the first case, $\bar{N} = O_2(\bar{N})$, i.e. $N/O_{2'}(N) \cong V$. But then N is the direct product of V and $O_{2'}(N)$, so that V is centralized by every element of N. Thus $|\bar{N}/O_2(\bar{N})| = 3$, so that $\bar{N} = N/O_{2'}(N) \cong A_4$. q.e.d.

*Remark.* The above method could also be used to prove Theorem 1 of section 3. In fact, G being solvable, a minimal normal subgroup of G is an elementary abelian $p$–group. Thus, for some $p$, $O_p(G) > 1$. If $O_{2'}(G) = 1$, then $O_2(G) > 1$, and in case a 2–Sylow subgroup of G be quaternion or elementary abelian of order 4 it is easily seen that $O_2(G)$ is the full Sylow subgroup of G. Thus, $\mathrm{Aut}(O_2(G)/\Phi(O_2(G))) \cong S_3$. By arguments similar to those of Theorem 2 the proof can be completed.

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