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On the Zeros of an Entire Function and its Second Derivative

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Analisi matematica. — *On the Zeros of an Entire Function and its Second Derivative.* Nota (*) di CHUNG-CHUN YANG, presentata dal Socio G. SANSONE.

RIASSUNTO. — Il presente studio riguarda il comportamento di una funzione intera f della variabile complessa $z = x + y$ priva di zeri. Si dimostra che quando tutti gli zeri di f'' sono di molteplicità $m\,(m \geqq 3)$, allora f ha la forma $f = e^{az+b}$, oppure la forma $f = e^{p(z)}$, indicando $p(z)$ un polinomio.

As a consequence of a quite general result of Clunie's [1] that if f is an entire function, and $ff^{(l)} \neq 0$ for some $l \geq 2$, then f must have the form $f = e^{az+b}$. The method used to attack the above problem, based on the Nevanlinna fundamental theorems for meromorphic functions was delicate but rather complicated. The purpose of this paper is to show that for $l = 2$, by altering the approach, we are able to obtain and improve the above result. It is assumed that the reader has some familiarity with the standard notations which are associated with the Nevanlinna theory.

Our main result can be stated as follows:

THEOREM. *Let f be an entire function, $f \neq 0$ and that $f''(z)$ can be expressed as $f''(z) \equiv [H(z)]^m$ for some non-constant entire function* H, *m an integer ≥ 3. Then when m is even $f = e^{az+b}$ where a, b are constants; while when m is odd, $f = e^{p(z)}$, where $p(z)$ is a polynomial.*

As a immediate consequence, we have the following:

COROLLARY. *Let f be an entire function, if $ff'' \neq 0$, then $f = e^{az+b}$.*

Proof of the Theorem. By hypothesis we have

$$f(z) = e^{g(z)} \tag{1}$$

where g is an entire function.

Then

$$f''(z) = \{[g'(z)]^2 + g''(z)\}\, e^{g(z)}\,. \tag{2}$$

Setting $g'(z) \equiv h(z)$, we get

$$\begin{aligned}[g'(z)]^2 + g''(z) &= [h(z)]^2 + h'(z) \\ &= -h^2 \left\{ \left(\frac{1}{h}\right)' - 1 \right\}.\end{aligned} \tag{3}$$

(*) Pervenuta all'Accademia il 15 luglio 1970.

According to the hypothesis we have

$$[g'(z)]^2 + g''(z) = [H_0(z)]^m \tag{4}$$
$$= [h(z)]^2 + h'(z),$$

where $H_0(z)$ is an entire function.

By a result of Milloux's [3] we see that

$$2\,T(r, h) \geq T(r, h^2 + h') = T(r, H_0^m) \tag{5}$$
$$= m T(r, H_0)$$

for all r except a set of r of finite measure. Thus

$$T(r, h) \geq \frac{m}{2}\, T(r, H_0) \tag{6}$$

for all r except a set of r of finite measure.

Now we apply an inequality of Hayman's [2] to $1/h$ and obtain

$$T\left(r, \frac{1}{h}\right) < \left(2 + \frac{1}{1}\right) N(r, h) + \left(2 + \frac{2}{1}\right) \bar{N}\left(r, \frac{1}{\left(\frac{1}{h}\right)' - 1}\right) \tag{7}$$
$$+ o\{T(r, h)\}$$

for all r except a set of r of finite measure.

Hence, by h is entire

$$T\left(r, \frac{1}{h}\right) < 4\, \bar{N}\left(r, \frac{1}{\left(\frac{1}{h}\right)' - 1}\right) + o\{T(r, h)\} \tag{8}$$

for all r except a set of r of finite measure.

But

$$\bar{N}\left(r, \frac{1}{\left(\frac{1}{h}\right)' - 1}\right) \leq \bar{N}\left(r, \frac{1}{h^2 + h'}\right) \tag{9}$$
$$= \bar{N}\left(r, \frac{1}{H_0^m}\right).$$

Hence

$$\bar{N}\left(r, \frac{1}{\left(\frac{1}{h}\right)' - 1}\right) \leq \frac{1}{m}\, T(r, H_0) + o\{T(r, h)\}. \tag{10}$$

Thus if h is not a polynomial, then this and (5) yield

$$\frac{m}{2}\, T(r, H_0) \leq T\left(r, \frac{1}{h}\right) + o(1) \leq \frac{4}{m}\, T(r, H_0) + o\{T(r, h)\}, \tag{11}$$

for all r except a set of r of finite measure.

Consequently

$$T(r, H_0) \leq \frac{8}{m^2} T(r, H_0) + o\{T(r, h)\}, \tag{12}$$

for all r except a set of r of finite measure.

This is possible only when $m^2 < 8$ that is $m = 1$ or 2.

Hence we reach the conclusion that if $m \geq 3$ then h has to be a polynomial.

We then have two cases.

Case (i) m is even, then from (4) by a degree argument that h has to be a constant, it follows from this that $f = e^{az+b}$ where a, b are constants.

Case (ii) m is odd, then we do not have the above conclusion, since one can exhibit a polynomial $h(z)$ such that equation (4) holds. Hence, in this case $f = e^{\int h(z)\,dz} = e^{p(z)}$, where $p(z)$ is a polynomial. The theorem is thus proved.

Remark: We have left the case $m = 2$ from our theorem which our argument fails to apply. Therefore, the following question raised by Professor Hellerstein is not without interest.

Conjecture: Let f be an entire function, $f \neq 0$. Suppose that $f'' = h^2$ for some non-costant entire function h, then f must have the form $f = e^{az+b}$ where a, b are constants.

REFERENCES

[1] J. CLUNIE, *On integral and meromorphic functions*, « J. London Math. Soc. », *37*, 12–17 (1962).

[2] W. K. HAYMAN, *Meromorphic functions, Oxford Mathematical Monographs*, Clarendon Press, Oxford 1964, p. 60.

[3] H. MILLOUX, *Les fonctions meromorphes et leurs derivées*, Paris 1940, p. 18.