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Curvature collineations in Fins1er space

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Geometria differenziale. — *Curvature collineations in Finsler space.* Nota[(*)] di BAIJ NATH PRASAD, presentata dal Socio E. BOMPIANI.

RIASSUNTO. — Estensione agli spazi di Finsler della nozione di « collineazione di curvatura » di spazi Riemanniani e proprietà relative.

INTRODUCTION

In a recent paper Katzin *et al.* [1] formulated that a Riemannian space V_n is said to admit a symmetry called a curvature collineation provided there exists a vector v^i such that $\mathfrak{L}_v R^m_{ijk} = 0$, where R^m_{ijk} is the Riemann curvature tensor [3] and $\mathfrak{L}_v$ denotes the Lie derivative [4]. In [2] they also studied the curvature collineation in a conformally flat space C_n. In the present paper I wish to continue the analysis of curvature collineation in Finsler space F_n. The relation between curvature collineation and other symmetries admitted by an affinely connected space is discussed.

1. *Fundamental formulae.* Let F_n be an n–dimensional Finsler space equipped with the line element $(x^i, \dot{x}^i)$ and the positively homogeneous metric function $F(x, \dot{x})$ of degree one in $\dot{x}^i$. The entities [5]

$$g_{ij}(x, \dot{x}) = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x}) \quad (1) \tag{1.1}$$

form the covariant components of the metric tensor of F_n.

The Cartan covariant derivative of a vector field $X^i(x, \dot{x})$ (depending on the element of support $\dot{x}^i$) with respect to x^j is given by (Cartan [6])

$$X^i_{|j}(x, \dot{x}) = \partial_j X^i - \dot{\partial}_m X^i G^m_j + X^m \Gamma^{*i}_{mj} \tag{1.2}$$

where Γ^{*i}_{mj} are the Cartan connection coefficients and the symbol

$$G^m_j(x, \dot{x}) = \dot{\partial}_j G^m(x, \dot{x}) \tag{1.3}$$

has its usual meaning given in [5] (page 71).

We have ([5] Ch. III)

$$2G^i = \Gamma^{*i}_{jk} \dot{x}^j \dot{x}^k \tag{1.4 a}$$

$$G^k_{ij} = \Gamma^{*k}_{ij} + C^k_{ij|l} \dot{x}^l \tag{1.4 b}$$

where

$$G^k_{ij} \overset{\text{def.}}{=} \dot{\partial}_j G^k_i(x, \dot{x}) \qquad \text{and} \qquad C^k_{ij} = g^{kh} C_{ihj} = \frac{1}{2} g^{kh} \dot{\partial}_j g_{ih}.$$

If the Finsler space F_n is affinely connected space then $C^k_{ij|l} = 0$ and we have $G^k_{ij} = \Gamma^{*k}_{ij}$.

(*) Pervenuta all'Accademia il 10 settembre 1970.

(1) $\partial_j X^i = \dfrac{\partial X^i}{\partial x^j}$ and $\dot{\partial}_j X^i = \dfrac{\partial X^i}{\partial \dot{x}^j}$.

We have the following commutation formulae involving the Lie derivative of any tensor T^i_{jk} and the connection Γ^{*i}_{kj}

$$(\mathfrak{L}_v T^i_{jk})_{|l} - \mathfrak{L}_v (T^i_{jk|l}) = - T^a_{jk} (\mathfrak{L}_v \Gamma^{*i}_{al}) + T^i_{ak} (\mathfrak{L}_v \Gamma^{*a}_{jl}) + + T^i_{ja} (\mathfrak{L}_v \Gamma^{*a}_{kl}) + (\dot\partial_a T^i_{jk}) (\mathfrak{L}_v \Gamma^a_{bl}) \dot x^b , \tag{1.5 a}$$

$$\dot\partial_l (\mathfrak{L}_v T^i_{jk}) - \mathfrak{L}_v (\dot\partial_l T^i_{jk}) = 0 , \tag{1.5 b}$$

$$(\mathfrak{L}_v \Gamma^{*i}_{hk})_{|j} - (\mathfrak{L}_v \Gamma^{*i}_{jh})_{|k} = \mathfrak{L}_v K^i_{hkj} + (\mathfrak{L}_v \Gamma^{*l}_{jb}) \dot x^b (\dot\partial_l \Gamma^{*i}_{hk}) - - (\mathfrak{L}_v \Gamma^{*l}_{kb}) \dot x^b (\dot\partial_l \Gamma^{*i}_{jh}) , \tag{1.6 a}$$

$$\dot\partial_j (\mathfrak{L}_v \Gamma^{*i}_{hk}) - \mathfrak{L}_v (\dot\partial_j \Gamma^{*i}_{hk}) = 0 , \tag{1.6 b}$$

where K^i_{hkj} is the curvature tensor defined by

$$K^i_{hkj} = (\partial_j \Gamma^{*i}_{hk} - \dot\partial_l \Gamma^{*i}_{hk} G^l_j) - (\partial_k \Gamma^{*i}_{hj} - \dot\partial_l \Gamma^{*i}_{hj} G^l_k) + \Gamma^{*i}_{mj} \Gamma^{*m}_{hk} - \Gamma^{*i}_{mk} \Gamma^{*m}_{hj} . \tag{1.7}$$

From the equations (1.4 a) and (1.6 b) we deduce

$$\dot\partial_j (\mathfrak{L}_v G^i) - \mathfrak{L}_v (\dot\partial_j G^i) = 0 , \tag{1.8 a}$$

$$\dot\partial_j (\mathfrak{L}_v G^i_k) - \mathfrak{L}_v (\dot\partial_j G^i_k) = 0 , \tag{1.8 b}$$

and

$$\dot\partial_j (\mathfrak{L}_v G^i_{hk}) - \mathfrak{L}_v (\dot\partial_j G^i_{hk}) = 0 . \tag{1.8 c}$$

We quote the following definitions for reference in the later articles of this paper.

Motion. (Rund [5]). A F_n is said to admit a motion provided there exists a vector v^i such that

$$H_{ij} = \mathfrak{L}_v g_{ij} = v_{j|i} + v_{i|j} + 2\, C_{ijk}\, v^h_{|k}\, \dot x^k = 0 . \tag{1.9}$$

Affine motion (Yano [4]). A F_n is said to admit an affine motion provided there exists a vector v^i such that

$$\mathfrak{L}_v G^i_{hk} = 0 . \tag{1.10}$$

Homothetic motion. (Hiramatu [7]). A F_n is said to admit a homothetic motion if there exists a vector v^i such that

$$\mathfrak{L}_v g_{ij} = H_{ij} = 2\sigma g_{ij} \tag{1.11}$$

holds with σ a non zero constant.

Applying the formula (1.5 a) to the fundamental tensor g_{jk} and noting that $g_{jk|l} = 0$, we have

$$H_{jk|l} = g_{ak} (\mathfrak{L}_v \Gamma^{*a}_{jl}) + g_{ja} (\mathfrak{L}_v \Gamma^{*a}_{kl}) + 2\, C_{jka} (\mathfrak{L}_v \Gamma^{*a}_{bl}) \dot x^b$$

from which we deduce

$$\mathfrak{L}_v \Gamma^{*i}_{hk} + \frac{1}{2} g^{im} (H_{hm|k} + H_{mk|h} - H_{hk|m}) - C^i_{hl} (\mathfrak{L}_v \Gamma^{*l}_{bk}) \dot x^b - - C^i_{kl} (\mathfrak{L}_v \Gamma^{*l}_{bh}) \dot x^b + g^{im} C_{hkl} (\mathfrak{L}_v \Gamma^{*l}_{bm}) \dot x^b . \tag{1.12}$$

We, therefore, observe that the following relation holds

$$(1.13)\qquad (\mathcal{L}_v \Gamma^{*i}_{hk})\, \dot{x}^h \dot{x}^k = \frac{1}{2} g^{im} (H_{hm|k} + H_{mk|h} - H_{hk|m})\, \dot{x}^h \dot{x}^k .$$

Since the Lie derivative of $\dot{x}^h$ vanishes, we have from (1.4 a) and (1.13)

$$(1.14)\qquad 2\, \mathcal{L}_v G^i = \frac{1}{2} g^{im} (H_{hm|k} + H_{mk|h} - H_{hk|m})\, \dot{x}^h \dot{x}^k .$$

Suppose that F_n admits a motion ($H_{ij} = 0$). Using the equations (1.14), (1.8 a), (1.8 b), and (1.8 c) we get $\mathcal{L}_v G^i_{jk} = 0$. Hence we have

THEOREM (1.1). *In a* F_n *every motion is an affine motion.* *From* (1.13) *and* (1.11) *we have for a homothetic motion*

$$(\mathcal{L}_v \Gamma^{*i}_{hk})\, \dot{x}^h \dot{x}^k = 0 \qquad \text{i.e.} \quad \mathcal{L}_v G^i_{jh} = 0 .$$

Hence

THEOREM (1.2). *Every homothetic motion is an affine motion.*

2. *Curvature collineations.* The infinitesimal transformation

$$(2.1)\qquad \bar{x}^i = x^i + v^i(x)\, \delta t$$

where δt is a positive infinitesimal, defines a curvature collineation provided that the space F_n admits a vector field $v^i(x)$ such that

$$(2.2)\qquad \mathcal{L}_v K^i_{hkj} = 0 .$$

Substituting the values of $\mathcal{L}_v \Gamma^{*i}_{hk}$ and $\mathcal{L}_v \Gamma^{*i}_{jh}$ from (1.12) in the equation (1.6 a) and using the fact that the covariant derivative of $\dot{x}^i$ vanishes, we have, for a curvature collineation,

$$\begin{aligned}
&(H_{ha|k} + H_{ak|h} - H_{hk|a})_{|j} - (H_{ha|j} + H_{aj|h} - H_{hj|a})_{|k} - 2\, C_{hal|j} (\mathcal{L}_v \Gamma^{*l}_{bk})\, \dot{x}^b - \\
&\quad - 2\, C_{hal} (\mathcal{L}_v \Gamma^{*l}_{bk})_{|j}\, \dot{x}^b - 2\, C_{hal|j} (\mathcal{L}_v \Gamma^{*l}_{bh})\, \dot{x}^b - 2\, C_{kal} (\mathcal{L}_v \Gamma^{*l}_{bh})_{|j}\, \dot{x}^b + \\
&\quad + 2\, C_{hkl|j} (\mathcal{L}_v \Gamma^{*l}_{ba})\, \dot{x}^b + 2\, C_{hkl} (\mathcal{L}_v \Gamma^{*l}_{ba})_{|j}\, \dot{x}^b + 2\, C_{hal|k} (\mathcal{L}_v \Gamma^{*l}_{bj})\, \dot{x}^b + \\
&\quad + 2\, C_{hal} (\mathcal{L}_v \Gamma^{*l}_{bj})_{|k}\, \dot{x}^b + 2\, C_{jal|k} (\mathcal{L}_v \Gamma^{*l}_{bh})\, \dot{x}^b + 2\, C_{jal} (\mathcal{L}_v \Gamma^{*l}_{bh})_{|k}\, \dot{x}^b - \\
&\quad - 2\, C_{hjl|k} (\mathcal{L}_v \Gamma^{*l}_{ab})\, \dot{x}^b - 2\, C_{hjl} (\mathcal{L}_v \Gamma^{*l}_{ab})_{|k}\, \dot{x}^b - (\mathcal{L}_v \Gamma^{*l}_{jb})\, \dot{x}^b (\dot{\partial}_l \Gamma^{*i}_{hk})\, 2\, g_{ia} + \\
&\quad + (\mathcal{L}_v \Gamma^{*l}_{kb})\, \dot{x}^b (\dot{\partial}_l \Gamma^{*i}_{jk})\, 2\, g_{ia} = 0 ,
\end{aligned}$$

where we have multiplied by $2\, g_{ia}$ in the simplification and used the fact that $g_{ia|j} = 0$.

Interchanging the indices a and h in the above equation and adding the resulting equation to the above we get

$$\begin{aligned}
(2.3)\qquad &H_{ha|kj} - H_{ha|jk} - 2\, C_{hal|j} (\mathcal{L}_v \Gamma^{*l}_{bk})\, \dot{x}^b - 2\, C_{hal} (\mathcal{L}_v \Gamma^{*l}_{bk})_{|j}\, \dot{x}^b + \\
&+ 2\, C_{hal|k} (\mathcal{L}_v \Gamma^{*l}_{bj})\, \dot{x}^b + 2\, C_{hal} (\mathcal{L}_v \Gamma^{*l}_{bj})_{|k}\, \dot{x}^b - (\mathcal{L}_v \Gamma^{*l}_{jb})\, \dot{x}^b (g_{ia}\, \dot{\partial}_l \Gamma^{*i}_{hk} + g_{ih}\, \dot{\partial}_l \Gamma^{*i}_{ak}) + \\
&+ (\mathcal{L}_v \Gamma^{*l}_{kb})\, \dot{x}^b (g_{ia}\, \dot{\partial}_l \Gamma^{*i}_{jh} + g_{ih}\, \dot{\partial}_l \Gamma^{*i}_{ja}) = 0 .
\end{aligned}$$

Again the relation (Rund [5] page 81)

$$\dot{\partial}_l \Gamma^{*i}_{ak} = C^i_{kl|a} + C^i_{al|k} - g^{i\gamma} C_{akl|\gamma} - (C^i_{km} C^m_{al|\gamma} + C^i_{am} C^m_{kl|\gamma} - C^m_{ak} C^i_{ml|\gamma})\, \dot{x}^\gamma$$

yields

$$(2.4)\qquad (\dot{\partial}_l \Gamma^{*i}_{ak})\, g_{ih} + (\dot{\partial}_l \Gamma^{*i}_{hk})\, g_{ia} = 2\, (C_{ahl|k} - C_{ahm} C^m_{kl|\gamma}\, \dot{x}^\gamma) .$$

Multiplying the relation (2.3) by $\dot{x}^h$, summing with respect to h and using (2.4) and conditions $C_{hal}\,\dot{x}^h = 0$, $C_{hal|j}\,\dot{x}^h = 0$ we deduce

$$(2.5) \qquad (H_{hal|kj} - H_{hal|jk})\,\dot{x}^h = 0\,.$$

Hence we have the following.

THEOREM (2.1). *A necessary condition that a F_n admits a curvature collineation is that there exists a vector v^i such that the equation (2.5) holds.*

3. *Relation between curvature collineation and other symmetries.* The following lemma is obvious from the definition of the Lie derivative of a tensor.

LEMMA. *The operations of contraction and the Lie derivative are commutative.*

Using the above lemma and contracting the indices i and j in (2.2), we observe that every curvature collineation vector v^i satisfies

$$(3.1) \qquad \mathfrak{L}_v\, K_{hk} = 0$$

where $K_{hk} = K^i_{hki}$ is the Ricci tensor.

If a F_n admits a vector v^i such that (3.1) holds then we shall say that F_n admits a ' Ricci collineation '. Hence we have

THEOREM (3.1). *In a F_n every curvature collineation is a Ricci collineation.*

From the definition (1.10) of an affine motion and the equation (1.6 a) it follows that

THEOREM (3.2). *In an affinely connected space every affine motion is a curvature collineation.*

From Theorems (1.1) and (3.2) we obtain

THEOREM (3.3). *In an affinely connected space every motion is a curvature collineation.*

Also it follows immediately from the Theorems (1.2) and (3.2) that

THEOREM (3.4). *In an affinely connected space every homothetic motion is a curvature collineation.*

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