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Some Remarks on a S. C. Chu and R. D. Moyer's theorem

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Analisi funzionale. — *Some Remarks on a S. C. Chu and R. D. Moyer's theorem* (*). Nota di ALESSIO VOLČIČ, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Si generalizza la definizione di trasformazione non ciclica introdotta dai due Autori citati nel titolo e si caratterizzano (Teorema 2) le trasformazioni non cicliche che operano su un insieme totalmente ordinato, completo e denso. Si fanno poi alcune considerazioni sul classico metodo di approssimazione di Newton, che si può dedurre dal Teorema 1.

In this paper we study a particular class of continuous transformations of a closed and bounded interval of the real line into itself (1) referring to the problem of the effective construction of the fixed points of such transformations.

It is well known that if f is a continuous transformation of a Hausdorff topological space S into itself and if, for a certain $x \in S$ the sequence $\{f^h(x)\}$ (2) (which is called Picard sequence) converges to x_0, then $f(x_0) = x_0$. In the mathematical literature the sufficient conditions (either on S or on f) for the convergence of the Picard sequences are numberless. Recently, in a seminar held by U. Barbuti and S. Guerra at the Mathematical Istitute of Trieste, some new results in this direction were presented (see [1]). In particular, it was observed that sometimes (3) it is sufficient to suppose that the transformation f has no involutory couple of points, i.e. couples x_1, x_2 such that $f(x_1) = x_2 \neq x_1 = f(x_2)$ and it was conjectured that conditions of this type are sufficient when the topological space is ordered and the order is (in some way) consistent with the topology. This conjecture is proved to be true and Theorem 2 extends the theorem stated above due to S. C. Chu and R. D. Moyer and which contains also previous results of W. A. Coppel, J. E. Maxfield and W. J. Mourant:

THEOREM.—*If f is a continuous transformation of the closed interval $[a, b]$ into itself, then the following properties are equivalent*:

I a – *for each* $x \in [a, b]$ *such that* $f(x) \neq x$ *we have* $f^2(x) \neq x$;

I b – *for each* $x \in [a, b]$ *such that* $f(x) > x$ *we have* $f^2(x) > x$ *and for each* $x \in [a, b]$ *such that* $f(x) < x$ *we have* $f^2(x) < x$;

II – *If* G *is any non empty closed subset of* $[a, b]$, *mapped into itself by f, then f has a fixed point in* G;

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(1) Transformations already studied in [4], [8] and [3].

(2) We define $f^0(x) = x$ and $f^{h+1}(x) = f(f^h(x))$.

(3) See [2], [3], [4] and [8].

III a – *for each* $x \in [a, b]$ *such that* $f(x) \neq x$, *we have* $f^h(x) \neq x$ *for every* $h > 1$;

III b – *for each* $x \in [a, b]$ *such that* $f(x) > x$, *we have* $f^h(x) > x$ *for every* $h > 1$ *and for each* $x \in [a, b]$ *such that* $f(x) < x$, *we have* $f^h(x) < x$ *for every* $h > 1$;

IV – $\{f^h(x)\}$ *is a convergent sequence for every* $x \in [a, b]$.

This Theorem, and in particular condition I a, suggests the following definition (see [3]):

Definition.—A continuous transformation f of a topological space S into itself is said to be cyclic if there exists a point x such that $f(x) \neq x$ and $f^2(x) = x$. It is said non cyclic in the opposite case.

In this paper we obtain in section 1 a new characterization of cyclic transformations operating on a closed and bounded interval of the real line. In section 2 the Theorem 1 and Chu and Moyer's Theorem are generalized to the case of continuous transformations operating on totally ordered, complete and dense (4) sets. In section 3, finally, we use the results of the former sections to reobtain the classical Newton approximation method.

1. For the cyclic transformations we can prove the following theorem:

THEOREM 1.—*If* f *is a continuous transformation of* $[a, b]$ *into itself, then the two following properties are equivalent*:

A) f *is cyclic*

B) *there exists some couple of points* x_1, x_2 *such that*

$$f(x_1) \geq x_2 > x_1 \geq f(x_2) . \tag{1}$$

$A \Rightarrow B$. If f is cyclic, there exists x, which is not fixed for f, such that $f^2(x) = x$. Let be, for example $f(x) > x$. Then the couple $x_1 = x$ and $x_2 = f(x)$ satisfies (1).

$B \Rightarrow A$. If for the couple x_1, x_2 the two signs of equality hold in (1), then the couple is involutory and f is cyclic. Let be then

$$f(x_1) > x_2 . \tag{2}$$

If $f^2(x_2) \geq x_2$ or $f^2(x_1) \leq x_1$, f is cyclic by property I b of Chu and Moyer's Theorem. Let be then

$$f^2(x_2) < x_2 \tag{3}$$

and

$$f^2(x_1) > x_1 . \tag{4}$$

(4) An ordered set is said to be complete, iff each subset has the least upper bound. It is said to be dense, if for each couple $x, y \in$ S, $x < y$ there exists a z such that $x < z < y$.

From (2) and (3) we infer that there is at least one point y between $f(x_2)$ and x_1, such that $f(y) = x_2$. Let y_0 be the largest of all those y's. We observe that f has no fixed point in the interval $[y_0, x_1]$, as for each $t \in [y_0, x_1]$ the disequalities $f(y) \geq x_2 > t$ hold. More, we have that

$$f^2(y_0) = f(f(y_0)) = f(x_2) < y_0 . \tag{5}$$

By (4) and (5) we see that the continuous transformation $f^2(x)$ has a fixed point in the interval $[y_0, x_1]$, which does not contain any fixed point of f, so f is cyclic.

2. The purpose of this section is to generalize Theorem 1 and Chu and Moyer's Theorem. We observe that some of the properties listed in the Chu and Moyer's Theorem (precisely I a, II, III a and IV) keep their meaning, as well, for the continuous transformations operating on arbitrary topological space. But these properties are not in general equivalent. For instance, I a and III a are not equivalent even for continuous transformations of a compact of R^2 into itself (see [1], p. 79). We can easily see that the following implications hold: IV $\Rightarrow$ II $\Rightarrow$ III $a \Rightarrow$ I a.

In order to keep the meaning of properties I b and III b and property B of Theorem 1, it seems natural to make our considerations in a totally ordered set $(S, <)$ (5). In S we can introduce in a natural way a topology, which will be called " order topology ", assigning the following family of open sets as subbasis:

$$\{t : t < x\} \quad , \quad \{t : t > x\} \quad ; \quad x \in S .$$

It will be useful to recall the following propositions:

PROPOSITION 1.—*If* S *is a totally ordered set, the three following properties are equivalent* (6):

(a) S *is connected in the order-topology;*
(b) S *is dense and every upper bounded set has a least upper bound;*
(c) S *is dense and every lower bounded set has a greatest lower bound.*

PROPOSITION 2.—*If* S *is a totally ordered set, such that each its upper bounded subset has a least upper bound, the three following properties are equivalent* (7):

(a) S *is compact in the order-topology*;
(b) S *has the first and the last point*;
(c) S *is complete.*

(5) It is understood that $x < y$ excludes $y < x$. We shall write $x \leq y$ if $x < y$ or $x = y$.

(6) For a Theorem of this type, see [5] page 359, or [7] page 58 problem I d.

(7) See [7] page 162, problem C. This proposition follows also from the Proposition 1 and Theorem 20 B 12 page 264 [5].

PROPOSITION 3.—*If* S *is complete, every monotone sequence is convergent in the order-topology.*

The simple proof is left to the reader.

PROPOSITION 4.—*If* S *is connected and compact, f and g are two continuous transformations of* S *into itself, such that $f(a) < g(a)$, $f(b) > g(b)$ (where a and b are, respectively, the first and the last point of* S), *then there exists a point $x_0 \in S$, such that $f(x_0) = g(x_0)$.*

The proof is straightforward. If we suppose the contrary, the two sets

$$A = \{t = g(t) \leq f(t)\} \quad , \quad B = \{t = f(t) \leq g(t)\}$$

would be closed, non empty and disjoint, such that $A \cup B = S$ and therefore it would not be connected.

If S is a totally ordered set, connected and compact in the order-topology, it has by Proposition 2 the first point a and the last point b. Let us use the symbol $[x, y]$ for the set (which will be still called " interval ") of points z such that $x \leq z \leq y$: we can so write $[a . b]$ instead of S.

By this convention and by Propositions 1, 2 and 4 the Theorem 1 can be extended (without any change neither in the statement nor in the proof) to the case of a continuous transformation f operating on $[a, b] = S$.

In the same way the Chu and Moyer Theorem can be extended using this time also the Proposition 3. We can state this Theorem:

THEOREM 2.—*If* S *is a totally ordered set, connected and compact in the order-topology and f is a continuous transformation of* S *into itself, the following properties are equivalent*:

I a – *for each $x \in S$ such that $f(x) \neq x$, we have $f^2(x) \neq x$;*

I b – *for each $x \in S$ such that $f(x) > x$, we have $f^2(x) > x$ and for each $x \in S$ such that $f(x) < x$, we have $f^2(x) < x$;*

II – *if* G *is any non empty closed subset of* S, *mapped into itself by f, then f has a fixed point in* G;

III a – *for each $x \in S$ such that $f(x) \neq x$, we have $f^h(x) \neq x$ for every $h > 1$;*

III b – *for each $x \in S$ such that $f(x) > x$, we have $f^h(x) > x$ for every $h > 1$ and for each $x \in S$ such that $f(x) < x$, we have $f^h(x) < x$ for every $h > 1$;*

IV – $\{f^h(x)\}$ *is a convergent sequence for every $x \in S$;*

V – *there exists no couple of points x_1, x_2 such that $f(x_1) \geq x_2 > x_1 \geq f(x_2)$.*

The proof of this Theorem is omitted. Only a few modifications are to be made to the proof thought out by Chu and Moyer (see [3]) of the equivalence, in this generalized case, of properties I a, I b, II, III a, III b and IV. The two properties I b and V are proved to be equivalent as in Theorem 1.

The Theorem stated above is a real generalization of Theorem 1 and Chu and Moyer's Theorem, as it is not true that every totally ordered set, connected and compact in the order topology is homeomorfic to a closed and bounded interval of the real line.

Example. Let Γ be the set of all the ordinal numbers less (and not equal) than Ω, the first ordinal having more than one countable set of predecessors and let be $S = (\Gamma \times [0, 1)) \cup \{(\Omega, 0)\}$. In S let us consider the lexicografical order: $(\alpha, x) < (\beta, y)$ if $\alpha < \beta$ or, if $\alpha = \beta$ and $x < y$; $\alpha, \beta \in \Gamma \cup \{\Omega\}$; $x, y \in [0, 1)$. In the order topology S is connected and compact (8), but it is not homeomorfic to an interval of the real line, as the point $(\Omega, 0)$ does not admit a countable base of neighbourhoods.

3. Now it is our purpose to see how the classical Newton approximation method can be deduced from Theorem 1.

THEOREM 3.—*Let f be a real, real valued function, with continuous derivative, defined on the closed interval* $I = [x_0 - \delta', x_0 + \delta'']$. *Let x_2 be the unique fixed point of f in* I *and* $|f'(x)| < 1$ *for every* $x \in I$. *Then, the sequence* $\{f^h(y)\}$ *converges to x_0, where y is the endpoint of* I *which is the nearest to* x_0.

We can see that by the hypothesis on the derivative of f, if $\delta = \min(\delta', \delta'')$, the interval $[x_0 - \delta, x_0 + \delta]$ is mapped into itself by f. In fact, we have

$$f(x) = \int_{x_0}^{x} f'(t)\, dt + f(x_0)$$

that is

$$|f(x) - f(x_0)| \leq \left| \int_{x_0}^{x} |f'(t)|\, dt \right| < |x - x_0| < \delta .$$

As we have $f'(x) > -1$ in I, it is $\dfrac{f(x') - f(x'')}{x' - x''} > -1$ for every couple of points in $[x_0 - \delta, x_0 + \delta]$ and from this we draw, supposed $x'' > x'$, that

$$f(x') - f(x'') < x'' - x'.$$

So there exists no couple of points x', x'' of $[x_0 - \delta, x_0 + \delta]$ such that $f(x') \geq x'' > x' \geq f(x'')$.

The restriction of f to the interval $[x_0 - \delta, x_0 + \delta]$ is then, by Theorem 1, non cyclic and therefore, by Chu and Moyer's Theorem, the sequence $\{f^h(x)\}$ converges for every $x \in [x_0 - \delta, x_0 + \delta]$; in particular for $x = y$. The limit point of such a sequence is a fixed point, so every Picard sequence, for $x \in [x_0 - \delta, x_0 + \delta]$, converges to x_0, which is the unique fixed point in I.

(8) See [6] page 55.

BIBLIOGRAPHY

[1] U. BARBUTI and S. GUERRA, *Sopra alcuni teoremi di convergenza nella teoria costruttiva del punto fisso*, « Rend. Ist. di Matem. Univ. Trieste », *2* (1), 59–84 (1970).

[2] D. CASTALDO, *Sulla convergenza delle approssimazioni successive relative al problema di Cauchy etc.*, « Rend. Acc. Sc. Fisiche e Matem. della Soc. di Scienze, Lettere ed Arti di Napoli », ser. 4ª, *35*, 734–744 (1968).

[3] S. C. CHU and R. D. MOYER, *On continuous functions, commuting functions and fixed points*, « Fund. Math. », *59*, 91–95 (1966).

[4] W. A. COPPEL, *The solution of equations by iteration*, « Proc. Cambridge Philos. Soc. », *51*, 41–43 (1955).

[5] E. ČECH. *Topological spaces*. Wiley & Sons 1966.

[6] J. G. HOCKING and G. S. YOUNG, *Topology*. Addison–Wesley 1961.

[7] J. L. KELLEY, *General topology*. Van Nostrand 1955.

[8] J. E. MAXFIELD and W. J. MOURANT, *Common fixed points of commuting continuous functions of the unit interval*, « Indagationes Math. », *27*, 668–670 (1965).