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On quasibounded mappings and nonlinear functional equations

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Analisi funzionale. — *On quasibounded mappings and nonlinear functional equations.* Nota di ALFONSO VIGNOLI (*), presentata (**) dal Socio S. SANSONE.

RIASSUNTO. — Si dimostra il seguente teorema: « Sia $T : X \to X$ una applicazione quasilimitata e addensante (vedi Introduction) di uno spazio di Banach in se stesso. Sia $T = \limsup_{x} T(x)/x < 1$, allora l'equazione $y = x - T(x)$ ammette almeno una soluzione per ogni $y \in X$ ». Come corollari si ottengono alcuni risultati già noti.

1. INTRODUCTION

Let $T : X \to Y$ be a continuous mapping from a Banach space X into a Banach space Y. If the number

$$|T| = \limsup_{\|x\| \to \infty} \frac{\|T(x)\|}{\|x\|},$$

is finite then the mapping T is said to be *quasibounded* and $|T|$ is called the *quasinorm* of T. (See A. Granas [1]). Clearly a continuous linear mapping $L : X \to Y$ is quasibounded and $|L| = \|L\|$, where $\|L\|$ is the norm of L.

In [1] Granas proved the following theorem

THEOREM A. *Let* $T : X \to X$ *be a quasibounded completely continuous mapping from a Banach space* X *into itself. Let* $|T| < 1$, *then the equation* $y = x - T(x)$ *has at least a solution for each* $y \in X$.

The aim of the present paper is to give an extension of Theorem A to a more general class of mappings. As corollaries we give some known results obtained previously by other authors.

For this purpose we will use the following terminology.

Let $A \subset X$ be a bounded set of a metric space (X, d). By $\alpha(A)$ we denote the infimum of all $\varepsilon > 0$ such that A admits a finite covering consisting of subsets with diameter less than ε. (See C. Kuratowski [2]). Clearly $\alpha(A) = 0$ if and only if A is precompact.

Let $T : X \to X$ be a continuous mapping of a metric space (X, d) into itself. If for any bounded set $A \subset X$ such that $\alpha(A) > 0$ we have

$$\alpha(T(A)) < \alpha(A),$$

then the mapping T is said to be *densifying* (see [3]).

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Note that contractive mappings (i.e. $d(T(x), T(y)) \leq kd(x, y)$, $0 \leq k < 1$, for all $x, y \in X$) and completely continuous mappings are densifying; also sums of contractive and completely continuous mappings defined on Banach spaces are densifying.

The following theorem regarding densifying mappings was proved in [4].

THEOREM B. *Let* $T : Q \to Q$ *be a densifying mapping defined on a bounded convex and closed subset* Q *of a Banach space* X. *Then* T *has at least one fixed point* (1).

2. SOLVABILITY OF FUNCTIONAL EQUATIONS

The main result of this paper is the following

THEOREM 1. *Let* $T : X \to X$ *be a quasibounded densifying mapping from a Banach space* X *into itself. Let* $|T| < 1$, *then the equation* $y = x - T(x)$ *has at least one solution for each* $y \in X$.

Proof. Given any $y^{\wedge} \in X$ we have to prove that $y^{\wedge} = x^{\wedge} - T(x^{\wedge})$ for some $x^{\wedge} \in X$. Let $G = y^{\wedge} - T$. It is readily seen that G is a densifying mapping. Consider the following family of balls with center in $y^{\wedge}$

$$Q(k) = \{x \in X : \| x - y^{\wedge} \| \leq k\} \quad , \quad k = 1, 2, \cdots$$

We want to show that for some integer $k' > 0$ the mapping G maps $Q(k')$ into itself. Assume the contrary. Then for any positive integer k there exists an element x_k such that $\| x_k - y^{\wedge} \| \leq k$ and

$$\| G(x_k) - y^{\wedge} \| > k.$$

But

$$\| G(x_k) - y^{\wedge} \| = \| T(x_k) \| ,$$

hence

$$\frac{\| T(x_k) \|}{\| x_k \|} > \frac{k}{\| x_k \|} ;$$

on the other hand

$$\| x_k \| \leq \| y^{\wedge} \| + k ,$$

from which it follows

$$1 > |T| = \limsup_{\| x_k \| \to \infty} \frac{\| T(x_k) \|}{\| x_k \|} \geq \limsup_{k \to \infty} \frac{k}{\| x_k \|} \geq \limsup_{k \to \infty} \frac{k}{\| y^{\wedge} \| + k} = 1 .$$

(1) We would like to point out that a similar theorem was proved by Sadovskij [5]. But he uses a different definition of the number α, so strictly speaking the class of densifying mappings introduced in [3] does coincide with the class of mappings introduced by Sadovskij. Also, should be remarked that the proof of our theorem is completely different from the one given by Sadovskij.

This contradiction shows that for some $k' > 0$, $G : Q(k') \to Q(k')$. Hence by Theorem B the mapping G has at least one fixed point, say $x^\wedge \in Q(k')$. Then $G(x^\wedge) = x^\wedge = y^\wedge + T(x^\wedge)$, i.e. $y^\wedge = x^\wedge - T(x^\wedge)$, which completes the proof.

From Theorem 1 follows

COROLLARY 1. *Let* $T : X \to X$ *be a quasibounded densifying mapping from a Banach space* X *into itself. Let the quasinorm of* T *satisfy the inequality* $|\lambda|\,|T| < 1$, *where* λ *is a real number such that* $|\lambda| \leq 1$. *Then the equation* $y = x - \lambda T(x)$ *has at least one solution for each* $y \in X$.

COROLLARY 2. *Let* $T : X \to X$ *be a densifying mapping from a Banach space* X *into itself. Let* $\|T(x)\| = o(\|x\|)$ *as* $\|x\| \to \infty$. *Let* λ *be a real number such that* $|\lambda| \leq 1$. *Then the equation* $y = x - \lambda T(x)$ *has at least one solution for each* $y \in X$.

In the above corollaries the condition $|\lambda| \leq 1$ is required in order that the mapping λT be densifying.

If the mapping T is assumed to be completely continuous both corollaries can be proved without the assumption $|\lambda| \leq 1$. (See Granas [1] for Corollary 1 and Dubrovskij [6] for Corollary 2).

Since a sum of a contractive mapping (or more generally a densifying mapping) and a completely continuous one is a densifying mapping, from Theorem 1 we get the following.

COROLLARY 3. *Let* $F : X \to X$ *be a quasibounded densifying mapping from a Banach space* X *into itself with quasinorm* $|F| \leq k$, $0 \leq k < 1$, *and let* $G : X \to X$ *be completely continuous with quasinorm* $|G| < 1 - k$. *Then the equation* $y = x - F(x) - G(x)$ *has at least one solution for each* $y \in X$.

Remark. In particular if in Corollary 3 the mapping F is assumed to be contractive with constant k, then F is densifying and satisfies the condition $|F| \leq k$. Indeed

$$\frac{\|F(x)\|}{\|x\|} \leq \frac{\|F(x) - F(0)\|}{\|x\|} + \frac{\|F(0)\|}{\|x\|} \leq k + \frac{\|F(0)\|}{\|x\|} \quad , \quad \forall x \in X,$$

hence $|F| \leq k$. Hence as a particular case of Corollary 3 (when F is a contraction with constant k) we obtain a result of Nashed and Wong [7].

COROLLARY 4. *Let* $F : X \to X$ *be a quasibounded densifying mapping from a Banach space* X *into itself with quasinorm* $|F| \leq k$, $0 \leq k < 1$, *and let the mapping* $G : X \to X$ *be quasibounded and completely continuous. Let* λ *be a real number such that* $|\lambda|\,|G| < 1 - k$. *Then the equation* $y = x - F(x) - \lambda G(x)$ *has at least one solution for each* $y \in X$.

Nussbaum [8] showed that a strict semicontraction is densifying, hence all our results hold for that class of mappings. We recall that a mapping $T : X \to X$ is said to be *strict semicontractive* (see Browder [9]) if there exists a continuous mapping $S : X \times X \to X$ such that $T(x) = S(x, x)$ for all $x \in X$,

$$\|S(x, z) - S(y, z)\| \leq k\|x - y\| \quad , \quad 0 \leq k < 1; \qquad x, y, z \in X,$$

and the mapping $x \to S(\cdot, x)$ is completely continuous from X to the space of mappings from X into itself with the uniform metric.

We also remark that an interesting surjectivity theorem involving quasibounded P–compact mappings was given by Petryshyn [10].

A geometric application of Theorem 1 is given in [11].

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