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Kurt Mahler

On formal power series as integrals of algebraic differential equations

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Matematica. — *On formal power series as integrals of algebraic differential equations.* Nota di KURT MAHLER, presentata (*) dal Socio B. SEGRE.

In memory of my dear friend Jan Popken.

RIASSUNTO. — Si stabilisce l'esistenza di due costanti reali positive γ_1, γ_2 siffatte che, per una qualsiasi serie formale di potenze $\sum_0^\infty f_h z^h$ a coefficienti f_h complessi che sia soluzione di una qualche equazione differenziale algebrica, debba risultare $|f_h| \leq \gamma_1 (h!)^{\gamma_2}$ per $h = 0, 1, 2, \cdots$.

The following result will be proved. Let

$$f = \sum_{h=0}^{\infty} f_h z^h$$

be a formal power series with complex coefficients which satisfies any algebraic differential equation. Then two positive constants γ_1 and γ_2 exist such that

$$|f_h| \leq \gamma_1 (h!)^{\gamma_2} \qquad \text{for all } h.$$

This estimate is the best possible. For if n is any positive integer, the series

$$\sum_{h=0}^{\infty} (h!)^n z^h$$

is known to satisfy a linear differential equation with coefficients that are polynomials in z.

1. Denote by K an arbitrary subfield of the complex number field C, and by K* the ring of all formal power series

$$f = \sum_{h=0}^{\infty} f_h z^h \quad , \quad g = \sum_{h=0}^{\infty} g_h z^h , \quad \text{etc.},$$

with coefficients $f_h, g_h, \cdots$ in K. Here sum and product are as usual defined by

$$f + g = \sum_{h=0}^{\infty} (f_h + g_h) z^h \quad , \quad fg = \sum_{h=0}^{\infty} \left(\sum_{k=0}^{h} f_k g_{h-k} \right) z^h ,$$

and the elements a of K are identified with the special series

$$a = a + \sum_{h=1}^{\infty} 0 \cdot z^h$$

and play the role of constants.

(*) Nella seduta del 20 febbraio 1971.

Differentiation in K* is defined formally by

$$\frac{d^k f}{dz^k} = f^{(k)} = \sum_{h=k}^{\infty} h(h-1)\cdots(h-k+1) f_h z^{h-k},$$

a notation used also for $k = 0$ when

$$f^{(0)} = f.$$

In particular,

$$\frac{df}{dz} = 0 \quad \text{if and only if} \quad f = a \in K.$$

The usual rules for the derivatives of sum, difference, and product hold also in K*.

An important mapping from K* into K is defined by the formal substitution $z = 0$. For this substitution we use the notation

$$f(0) = f\big|_{z=0} = f_0.$$

More generally

$$f^{(k)}(0) = f^{(k)}\big|_{z=0} = k!\, f_k.$$

2. This paper is concerned with power series

$$f = \sum_{h=0}^{\infty} f_h z^h$$

in K* which satisfy any algebraic differential equation

(F) $$F((w)) = F(z; w, w', \cdots, w^{(m)}) = 0.$$

Here $F(z; w_0, w_1, \cdots, w_m) \not\equiv 0$ denotes a polynomial in the indeterminates $z, w_0, w_1, \cdots, w_m$ with coefficients in some extension field of K. By a well known method from linear algebra f can then be shown to satisfy also an algebraic differential equation (F) with coefficients in K; only this case will therefore from now on be considered.

Put

$$F_\mu(z; w_0, w_1, \cdots, w_m) = \frac{\partial}{\partial w_\mu} F(z; w_0, w_1, \cdots, w_m) \qquad (\mu = 0, 1, \cdots, m),$$

and

$$F_{(\mu)}((w)) = F_\mu(z; w, w', \cdots, w^{(m)}) \qquad (\mu = 0, 1, \cdots, m),$$

where w denotes an indeterminate element of K*. There is no loss of generality in assuming that both

(1) $$F_{(m)}((w)) \not\equiv 0$$

and

(2) $$F_{(m)}((f)) \neq 0.$$

The integer $m \geq 0$ is thus the exact order of the differential equation (F); when $m = 0$, (F) becomes an algebraic equation for f, a case which need not be excluded.

3. The differential operator $F((w))$ has the explicit form

$$F((w)) = \sum_{(\varkappa)} p_{(\varkappa)}(z)\, w^{(\varkappa_1)} \cdots w^{(\varkappa_N)} . \tag{3}$$

Here the summation

$$\sum_{(\varkappa)}$$

extends over all ordered systems

$$(\varkappa) = (\varkappa_1, \cdots, \varkappa_N)$$

of integers for which

$$0 \leq \varkappa_1 \leq m, \cdots, 0 \leq \varkappa_N \leq m \quad ; \quad \varkappa_1 \leq \varkappa_2 \leq \cdots \leq \varkappa_N \quad ; \quad 0 \leq N \leq n , \tag{4}$$

where n is a fixed positive integer, and the $p_{(\varkappa)}(z)$ are polynomials in $K[z]$. The integer N may vary with the system $(\varkappa)$, and there is just one improper system $(\varkappa)$ denoted by (ω) for which $N = 0$. The term in (3) corresponding to $(\varkappa) = (\omega)$ has the form

$$p_{(\omega)}(z)$$

and thus has no factors $w^{(j)}$, but is a polynomial in z alone.

4. An explicit expression for the successive derivatives

$$F^{(h)}((w)) = \left(\frac{d}{dz}\right)^h F((w)) \qquad (h = 1, 2, 3, \cdots)$$

of $F((w))$ can be obtained by means of the following simple lemma.

LEMMA 1: *Let* $h \geq 1$ *and* $N \geq 0$ *be arbitrary integers, and let*

$$w_0, w_1, \cdots, w_N$$

be any $N+1$ *elements of* K^*. *Then*

$$\left(\frac{d}{dz}\right)^h (w_0 w_1 \cdots w_N) = h! \sum_{\lambda_0, \lambda_1, \cdots, \lambda_N} \frac{w_0^{(\lambda_0)}}{\lambda_0 !} \frac{w_1^{(\lambda_1)}}{\lambda_1 !} \cdots \frac{w_N^{(\lambda_N)}}{\lambda_N !} , \tag{5}$$

where the summation extends over all ordered systems of $N+1$ *integers* $\lambda_0, \lambda_1, \cdots, \lambda_N$ *satisfying*

$$\lambda_0 \geq 0, \lambda_1 \geq 0, \cdots, \lambda_N \geq 0 \quad ; \quad \lambda_0 + \lambda_1 + \cdots + \lambda_N = h .$$

Proof. The assertion is evident when $h = 1$. Assume it has already been established for some $h \geq 1$. We now show that then it holds also for $h + 1$ and hence is always true.

On differentiating (5),

$$\left(\frac{d}{dz}\right)^{h+1} (w_0 w_1 \cdots w_N) = h! \sum_{\lambda_0, \lambda_1, \cdots, \lambda_N} \sum_{\nu=0}^{N} \frac{w_0^{(\lambda_0)}}{\lambda_0!} \cdots \frac{w^{(\lambda_\nu+1)}}{\lambda_\nu!} \cdots \frac{w_N^{(\lambda_N)}}{\lambda_N!} =$$

$$= h! \sum_{\mu_0, \mu_1, \cdots, \mu_N} \left(\sum_{\nu=0}^{N}\right) \frac{w_0^{(\mu_0)}}{\mu_0!} \frac{w_1^{(\mu_1)}}{\mu_1!} \cdots \frac{w_N^{(\mu_N)}}{\mu_N!},$$

where the new summation extends over all ordered systems of $N+1$ integers $\mu_0, \mu_1, \cdots, \mu_N$ satisfying

$$\mu_0 \geq 0, \mu_1 \geq 0, \cdots, \mu_N \geq 0 \quad ; \quad \mu_1 + \mu_0 + \cdots + \mu_N = h + 1 .$$

Since thus

$$h! \sum_{\nu=1}^{N} \mu_\nu = (h + 1)!,$$

the assertion follows.

5. Apply Lemma 1 to all the separate terms

$$p_{(\varkappa)}(z)\, w^{(\varkappa_1)} \cdots w^{(\varkappa_N)}$$

in the formula (3) for $F((w))$. It follows then that

$$F^{(h)}((w)) = h! \sum_{(\varkappa)} \sum_{[\lambda]} \frac{p_{(\varkappa)}^{(\lambda_0)}(z)}{\lambda_0!} \frac{w^{(\varkappa_1+\lambda_1)}}{\lambda_1!} \cdots \frac{w^{(\varkappa_N+\lambda_N)}}{\lambda_N!}. \tag{6}$$

Here the inner summation

$$\sum_{[\lambda]}$$

extends over all ordered systems $[\lambda] = [\lambda_0, \lambda_1, \cdots, \lambda_N]$ of $N+1$ integers for which

$$\lambda_0 \geq 0, \lambda_1 \geq 0, \cdots, \lambda_N \geq 0 \quad ; \quad \lambda_0 + \lambda_1 + \cdots + \lambda_N = h, \tag{7}$$

and N denotes the same integer as for the system $(\varkappa)$. There is exactly one term

$$p_{(\omega)}^{(h)}(z)$$

in the development (6) for which $N = 0$. This term does not involve w, and it vanishes as soon as h exceeds the degree of the polynomial $p_{(\omega)}(z)$.

6. From its definition, $F^{(h)}((w))$ is a polynomial in z and $w, w', \cdots, w^{(h+m)}$. We next show that, for sufficiently large k, $F^{(h)}((w))$ is linear in the derivative $w^{(k)}$.

Let j be any integer in the interval

$$0 \leq j \leq \left[\frac{h-1}{2}\right],$$

and define k in terms of h by

$$k = h + m - j .$$

Further denote by

$$F^{(h,k)}((w)) \cdot w^{(k)}$$

the sum of all terms on the right-hand side of (6) which have at least one factor $w^{(k)}$, and denote by

$$F^{(h,k)}_{(\varkappa)}((w)) \cdot w^{(k)}$$

the sum of all those contributions to $F^{(h,k)}((w))\, w^{(k)}$ which are obtained from the h–th derivative

$$(8) \qquad \left(\frac{d}{dz}\right)^h (p_{(\varkappa)}(z) w^{(\varkappa_1)} \cdots w^{(\varkappa_N)}) = h! \sum_{[\lambda]} \frac{p^{(\lambda_0)}_{(\varkappa)}(z)}{\lambda_0!} \frac{w^{(\varkappa_1+\lambda_1)}}{\lambda_1!} \cdots \frac{w^{(\varkappa_N+\lambda_N)}}{\lambda_N!}$$

of the term

$$p_{(\varkappa)}(z) w^{(\varkappa_1)} \cdots w^{(\varkappa_N)}, \qquad = t_{(\varkappa)} \text{ say,}$$

in the representation (3) of $F((w))$. It is then clear that

$$(9) \qquad F^{(h,k)}((w)) = \sum_{(\varkappa)} F^{(h,k)}_{(\varkappa)}((w)),$$

and that further

$$F^{(h,k)}_{(\omega)}((w)) = 0 .$$

Hence there are non-zero contributions only from those terms $t_{(\varkappa)}$ for which

$$(\varkappa) \neq (\omega) \quad \text{and therefore} \quad 1 \leq N \leq n .$$

7. Let now ν be any element of the set $\{1, 2, \cdots, N\}$, and let ν' be any element of this set which is distinct from ν. It is obvious that the binomial coefficient

$$\binom{h}{k - \varkappa_\nu}$$

vanishes if either

$$k - \varkappa_\nu < 0 \quad \text{or} \quad k - \varkappa_\nu > h .$$

It suffices therefore to consider those suffixes ν for which

$$0 \leq k - \varkappa_\nu \leq h .$$

Such suffixes will be said to be *admissible*.

To every admissible suffix ν there exist systems $[\lambda] = [\lambda_0, \lambda_1, \cdots, \lambda_N]$ of $N+1$ integers satisfying both

$$(7) \qquad \lambda_0 \geq 0, \lambda_1 \geq 0, \cdots, \lambda_N \geq 0 \; ; \; \lambda_0 + \lambda_1 + \cdots + \lambda_N = h$$

and

(10) $$\varkappa_\nu + \lambda_\nu = k\,.$$

Hence, by the definitions of j and k,

$$\lambda_\nu = k - \varkappa_\nu = (h-j) + (m-\varkappa_\nu) \geq h - j > \frac{h}{2},$$

and therefore

$$\lambda_{\nu'} < \frac{h}{2} \quad , \quad \varkappa_{\nu'} + \lambda_{\nu'} < \frac{h}{2} + m = h + m - \frac{h}{2} \leq h + m - j = k\,.$$

It follows that the corresponding term

$$h!\,\frac{p_{(\varkappa)}^{(\lambda_0)}(z)}{\lambda_0\,!}\;\frac{w^{(\varkappa_1+\lambda_1)}}{\lambda_1!}\cdots\frac{w^{(\varkappa_N+\lambda_N)}}{\lambda_N!}\,, \qquad = T_{(\varkappa),[\lambda]} \text{ say,}$$

on the right-hand side of (8) has one and only one factor $w^{(k)}$. Hence the contribution from $T_{(\varkappa),[\lambda]}$ to $F_{(\varkappa)}^{(h,k)}((w))$ is equal to

$$\frac{\partial T_{(\varkappa),[\lambda]}}{\partial w^{(k)}} = \frac{h!}{\lambda_\nu!}\;\frac{p_{(\varkappa)}^{(\lambda_0)}(z)}{\lambda_0\,!}\;\prod_{\nu'}\frac{w^{(\varkappa_{\nu'}+\lambda_{\nu'})}}{\lambda_{\nu'}\,!}\,.$$

On the other hand, by Lemma 1, also

$$\left(\frac{d}{dz}\right)^{h-k+\varkappa_\nu}\left(p_{(\varkappa)}(z)\prod_{\nu'} w^{(\varkappa_{\nu'})}\right) = (h-k+\varkappa_\nu)!\sum_{[\lambda]}{}'\;\frac{p_{(\varkappa)}^{(\lambda_0)}(z)}{\lambda_0!}\;\prod_{\nu'}\frac{w^{(\varkappa_{\nu'}+\lambda_{\nu'})}}{\lambda_{\nu'}\,!}$$

where the summation $\sum\limits_{[\lambda]}{}'$ is extended only over those systems $[\lambda]$ which have both properties (7) and (10). Therefore

$$\sum_{[\lambda]}{}'\;\frac{\partial T_{(\varkappa),[\lambda]}}{\partial w^{(k)}} = \binom{h}{k-\varkappa_\nu}\left(\frac{d}{dz}\right)^{h-k+\varkappa_\nu}\left(p_{(\varkappa)}(z)\prod_{\nu'} w^{(\varkappa_{\nu'})}\right) =$$

$$= \binom{h}{k-\varkappa_\nu}\left(\frac{d}{dz}\right)^{h-k+\varkappa_\nu}\frac{\partial}{\partial w^{(\varkappa_\nu)}}\left(p_{(\varkappa)}(z)\,w^{(\varkappa_1)}\cdots w^{(\varkappa_N)}\right),$$

whence, on summing over $\nu = 1, 2, \cdots, N$,

$$F_{(\varkappa)}^{(h,k)}((w)) = \sum_{\nu=1}^{N}\binom{h}{k-\varkappa_\nu}\left(\frac{d}{dz}\right)^{h-k+\varkappa_\nu}\frac{\partial}{\partial w^{(\varkappa_\nu)}}\left(p_{(\varkappa)}(z)\,w^{(\varkappa_1)}\cdots w^{(\varkappa_N)}\right).$$

Here the terms belonging to non-admissible suffixes ν vanish on account of the factor $\binom{h}{k-\varkappa_\nu} = 0$.

The formula so obtained may also be written as

$$F_{(\varkappa)}^{(h,k)}((w)) = \sum_{\mu=0}^{m}\binom{h}{k-\mu}\left(\frac{d}{dz}\right)^{h-k+\mu}\frac{\partial}{\partial w^{(\mu)}}\left(p_{(\varkappa)}(z)\,w^{(\varkappa_1)}\cdots w^{(\varkappa_N)}\right),$$

because

$$\frac{\partial}{\partial w^{(\mu)}}\left(p_{(\varkappa)}(z)\, w^{(\varkappa_1)}\cdots w^{(\varkappa_N)}\right) = \sum_{\nu} \frac{\partial}{\partial w^{(\varkappa_\nu)}}\left(p_{(\varkappa)}(z)\, w^{(\varkappa_1)}\cdots w^{(\varkappa_N)}\right)$$

where ν in $\sum_{\nu}$ runs over all suffixes $1, 2, \cdots, N$ which satisfy $\varkappa_\nu = \mu$.

Finally, by (3) and (9),

$$\text{(11)} \qquad F^{(h,k)}((w)) = \sum_{\mu=0}^{m} \binom{h}{k-\mu} \left(\frac{d}{dz}\right)^{h-k+\mu} F_{(\mu)}((w))$$

where, as in § 2, $F_{(\mu)}((w))$ denotes the partial derivative of $F((w))$ with respect to $w^{(\mu)}$. This formula is due to A. Hurwitz (1889) and S. Kakeya (1915).

8. The basic identities (6) and (11) hold for all elements w of K^*. We apply them now to the integral f of (F). We so firstly obtain the equations

$$\text{(12)} \qquad F^{(h)}((f)) = h! \sum_{(\varkappa)} \sum_{[\lambda]} \frac{p_{(\varkappa)}^{(\lambda_0)}(z)}{\lambda_0!} \frac{f^{(\varkappa_1+\lambda_1)}}{\lambda_1!} \cdots \frac{f^{(\varkappa_N+\lambda_N)}}{\lambda_N!} = 0 \quad (h=1,2,3,\cdots),$$

and secondly, for all $h = 1, 2, 3, \cdots$ and all $j = 0, 1, \cdots, \left[\frac{h-1}{2}\right]$, find that

$$\text{(13)} \qquad F^{(h,k)}((f)) = \sum_{\mu=0}^{m} \binom{h}{k-\mu} \left(\frac{d}{dz}\right)^{h-k+\mu} F_{(\mu)}((f)),$$

a formula which gives the coefficients of $f^{(k)} = f^{(h+m-j)}$ in (12).

In (12) and (13) we finally put $z = 0$. Since

$$p_{(\varkappa)}^{(\lambda_0)}(z)\Big|_{z=0} = p_{(\varkappa)}^{(\lambda_0)}(0) \quad \text{and} \quad f^{(h)}\Big|_{z=0} = h!\, f_h,$$

this leads to the equations

$$\text{(14)} \qquad h! \sum_{(\varkappa)} \sum_{[\lambda]} \frac{p_{(\varkappa)}^{(\lambda_0)}(0)}{\lambda_0!} \frac{(\varkappa_1+\lambda_1)!}{\lambda_1!} \cdots \frac{(\varkappa_N+\lambda_N)!}{\lambda_N!} f_{\varkappa_1+\lambda_1} \cdots f_{\varkappa_N+\lambda_N} = 0$$

$$(h = 1, 2, 3, \cdots).$$

Furthermore, the coefficient of $k!\, f_k$ on the left hand side is given by

$$F^{(h,k)}((f))\Big|_{z=0} = \sum_{\mu=0}^{m} \binom{h}{k-\mu} \left(\frac{d}{dz}\right)^{h-k+\mu} F_{(\mu)}((f))\Big|_{z=0}.$$

Here the expressions $F_{(\mu)}((f))$ are elements of K^*, hence have the explicit form

$$F_{(\mu)}((f)) = \sum_{h=0}^{\infty} F_{\mu,h}\, z^h \qquad (\mu = 0, 1, \cdots, m)$$

with certain coefficients $F_{\mu,h}$ in K. Thus

$$\left(\frac{d}{dz}\right)^{h-k+\mu} F_{(\mu)}((f))\Big|_{z=0} = (h-k+\mu)!\, F_{\mu,h-k+\mu} \qquad \text{for all } h \text{ and } \mu.$$

Therefore further

$$F^{(h,k)}((f))\Big|_{z=0} = \sum_{\mu=0}^{m} \binom{h}{k-\mu} (h-k+\mu)\,!\; F_{\mu,h-k+\mu}\,,$$

whence, on replacing μ by $m-\mu$ and remembering that $k = h+m-j$,

(15) $$F^{(h,k)}((f))\Big|_{z=0} = \sum_{\mu=0}^{m} \binom{h}{j-\mu} (j-\mu)\,!\; F_{m-\mu,j-\mu}\,.$$

All these expressions are polynomials in h with coefficients in K. We can easily prove that they do not all vanish identically. For by hypothesis,

(2) $$F_{(m)}((f)) \neq 0\,.$$

Hence the coefficients $F_{m,h}$ do not all vanish, and so there exists an integer

$$t \geq 0\,,$$

such that

$$F_{m,0} = F_{m,1} = \cdots = F_{m,t-1} = 0\,, \qquad \text{but} \qquad F_{m,t} \neq 0\,.$$

Thus, on choosing $j = t$ and $k = h+m-t$,

$$F^{(h,k)}((f))\Big|_{z=0} = \binom{h}{t} t\,!\; F_{m,t} + \sum_{\mu=1}^{m} \binom{h}{t-\mu} (t-\mu)\,!\; F_{m-\mu,t-\mu}$$

is a polynomial in h of the exact degree t and certainly does not vanish identically.

It follows that there exists a smallest integer s satisfying

$$0 \leq s \leq t$$

such that the polynomial (15) vanishes identically in h for $j = 0, 1, \cdots, s-1$, but that the polynomial

$$F^{(h,k)}((f))\Big|_{z=0} = \sum_{\mu=0}^{m} \binom{h}{s-\mu} (s-\mu)\,!\; F_{m-\mu,s-\mu}\,, \quad \text{where} \quad k = h+m-s\,,$$

is not identically zero. On changing over from h to k, put

(16) $$a(k) = F^{(k-m+s,k)}((f))\Big|_{z=0} = \sum_{\mu=0}^{m} \binom{k-m+s}{s-\mu} (s-\mu)\,!\; F_{m-\mu,s-\mu}\,.$$

Then $a(k)$ is now a polynomial in k which likewise does not vanish identically.

With s, k, and $a(k)$ as just defined, we can now assert that for

$$h = k-m+s \geq 2s+1$$

and hence for

$$k \geq m+s+1\,,$$

the left-hand side of the equation (14) involves at most the coefficients

$$f_0, f_1, \cdots, f_k$$

of f, but is free of

$$f_{k+1}, f_{k+2}, \cdots, f_{k+s} = f_{h+m}.$$

Furthermore, on this left-hand side, $k! f_k$ has the exact factor $a(k)$.

9. The result just proved will enable us now to find both recursive equations and inequalities for the coefficients f_k of f.

Put, firstly,

$$\alpha(k) = \begin{cases} \dfrac{k!}{h!} \\ 1 \end{cases} \quad \text{and} \quad \beta(k) = \begin{cases} 1 & \text{if} \quad k \geq h, \\ \dfrac{h!}{k!} & \text{if} \quad k \leq h, \end{cases}$$

and, secondly,

$$A(k) = a(k)\,\alpha(k),$$

so that evidently all three expressions $\alpha(k)$, $\beta(k)$, and $A(k)$ are polynomials in k which do not vanish identically.

Thirdly, denote by

$$\varphi_k = \varphi_k(f_0, f_1, \cdots, f_{k-1})$$

the double sum

$$\varphi_k = -\beta(k) \sum_{(\varkappa)} \sum_{[\lambda]}{}^{*} \frac{p_{(\varkappa)}^{(\lambda_0)}(0)}{\lambda_0!} \frac{(\varkappa_1+\lambda_1)!}{\lambda_1!} \cdots \frac{(\varkappa_N+\lambda_N)!}{\lambda_N!} f_{\varkappa_1+\lambda_1} \cdots f_{\varkappa_N+\lambda_N}, \tag{17}$$

where the asterisk at $\sum_{(\varkappa)} \sum_{[\lambda]}^{*}$ signifies that all terms having one of the factors

$$f_k, f_{k+1}, \cdots, f_{k+s}$$

are to be omitted.

With this notation, we arrive at the basic recursive formula

$$A(k) f_k = \varphi_k(f_0, f_1, \cdots, f_{k-1}). \tag{18}$$

Here the polynomial $A(k)$ is not identically zero, and hence, if k_0 denotes any sufficiently large integer, then

$$A(k) \neq 0 \quad \text{if} \quad k \geq k_0. \tag{19}$$

Thus, whenever $k \geq k_0$, then the recursive formula (17) expresses f_k as a polynomial in $f_0, f_1, \cdots, f_{k-1}$. By means of this representation, we shall now establish an upper estimate for $|f_k|$.

10. For the moment, denote by $\varkappa_0, \varkappa_1, \cdots, \varkappa_N$ arbitrary non-negative integers, and put

$$w_\nu = \varkappa_\nu\,!\,(1-z)^{-(\varkappa_\nu+1)} \qquad (\nu = 0, 1, \cdots, N)$$

so that

$$w_\nu^{(\lambda_\nu)} = (\varkappa_\nu + \lambda_\nu)\,!\,(1-z)^{-(\varkappa_\nu+\lambda_\nu+1)} \qquad (\nu = 0, 1, \cdots, N)$$

and also

$$\frac{1}{h!}\left(\frac{d}{dz}\right)^h (w_0 w_1 \cdots w_N) =$$

$$= \varkappa_0!\,\varkappa_1!\cdots\varkappa_N!\binom{\varkappa_0+\varkappa_1+\cdots+\varkappa_N+h+N}{\varkappa_0+\varkappa_1+\cdots+\varkappa_N+N}(1-z)^{-(\varkappa_0+\varkappa_1+\cdots+\varkappa_N+h+N+1)}$$

On the other hand, by Lemma 1,

$$\frac{1}{h!}\left(\frac{d}{dz}\right)^h (w_0 w_1 \cdots w_N) =$$

$$= \sum_{\lambda_0,\lambda_1,\cdots,\lambda_N} \frac{(\varkappa_0+\lambda_0)!}{\lambda_0!}\cdots\frac{(\varkappa_N+\lambda_N)!}{\lambda_N!}(1-z)^{-(\varkappa_0+\varkappa_1+\cdots+\varkappa_N+h+N+1)}$$

where the summation again extends over all systems $[\lambda]$ with the properties (7). On comparing these two formulae, we obtain the identity

$$(20) \qquad \sum_{[\lambda]} \frac{(\varkappa_0+\lambda_0)!}{\lambda_0!}\,\frac{(\varkappa_1+\lambda_1)!}{\lambda_1!}\cdots\frac{(\varkappa_N+\lambda_N)!}{\lambda_N!} = \binom{\varkappa_0+\varkappa_1+\cdots+\varkappa_N+h+N}{\varkappa_0+\varkappa_1+\cdots+\varkappa_N+N}\varkappa_0!\,\varkappa_1!\cdots\varkappa_N!\,.$$

Here assume that

$$\varkappa_0 = 0 \;;\; 0 \le \varkappa_1 \le m, \cdots, 0 \le \varkappa_N \le m \;;\; 1 \le N \le n\,.$$

Then

$$0 \le \varkappa_0 + \varkappa_1 + \cdots + \varkappa_N + N \le (m+1)\,n\,,$$

and so the binomial coefficient

$$\binom{\varkappa_0+\varkappa_1+\cdots+\varkappa_N+h+N}{\varkappa_0+\varkappa_1+\cdots+\varkappa_N+N} \le \{h + (m+1)\,n\}^{(m+1)n}\,.$$

The identity (19) implies therefore for all systems $(\varkappa)$ as before that

$$(21) \qquad \sum_{[\lambda]} \frac{(\varkappa_1+\lambda_1)!}{\lambda_1!}\cdots\frac{(\varkappa_N+\lambda_N)!}{\lambda_N!} \le m^{mn}\{h+(m+1)\,n\}^{(m+1)n}\,.$$

11. The operator $F((w))$ depends on only finitely many polynomials $p_{(\varkappa)}(z)$, and these together have only finitely many coefficients

$$\frac{p_{(\varkappa)}^{(\lambda_0)}(0)}{\lambda_0!}\,.$$

The maximum

$$c_0 = \max_{(\varkappa),[\lambda]} \left| \frac{p_{(\varkappa)}^{(\lambda_0)}(0)}{\lambda_0!} \right|$$

of the absolute values of all these coefficients is then a finite positive constant which, naturally, does not depend on k.

On the right-hand side of the formula (17) for φ_k, the double sum $\sum\limits_{(\varkappa)} \sum\limits_{[\lambda]}^{*}$ is a subsum of the double sum $\sum\limits_{(\varkappa)} \sum\limits_{[\lambda]}$. It follows then from (17) that

$$|A(k)|\,|f_k| \leq |\beta(k)| \cdot c_0 \cdot m^{mn} \{k + (m+1)n - m + s\}^{(m+1)n} \cdot \tag{22}$$
$$\cdot \max_{(\varkappa),[\lambda]}{}^{*} |f_{\varkappa_1+\lambda_1} \cdots f_{\varkappa_N+\lambda_N}| ,$$

where max* is extended over all pairs of systems $(\varkappa)$, $[\lambda]$ for which

$$1 \leq N \leq n \; ; \; 0 \leq \varkappa_1 + \lambda_1 \leq k - 1 , \cdots , 0 \leq \varkappa_N + \lambda_N \leq k - 1 . \tag{23}$$

The estimate (22) can be slightly simplified. Let k_0 be the same constant as in (19). There exist then two further positive constants c_1 and c_2, both independent of k, such that

$$\left| \frac{\beta(k)}{A(k)} \right| \leq k^{c_1} \qquad \text{for} \quad k \geq k_0 ,$$

and hence also

$$\left| \frac{\beta(k)}{A(k)} \right| \cdot c_0 \cdot m^{mn} \{k + (m+1)n - m + s\}^{(m+1)n} \leq k^{c_2} \qquad \text{for} \quad k \geq k_0 . \tag{24}$$

Next, with any two systems $(\varkappa)$ and $[\lambda]$ we can associate a further ordered system of N integers $\{\nu\} = \{\nu_1, \cdots, \nu_N\}$ by putting

$$\nu_1 = \varkappa_1 + \lambda_1 , \cdots , \nu_N = \varkappa_N + \lambda_N . \tag{25}$$

Then, by (23),

$$1 \leq N \leq n \; ; \; 0 \leq \nu_1 \leq k - 1 , \cdots , 0 \leq \nu_1 \leq k - 1 . \tag{26}$$

Further, by the properties of $(\varkappa)$ and $[\lambda]$,

$$\nu_1 + \cdots + \nu_N = (\varkappa_1 + \cdots + \varkappa_N) + h = k + (\varkappa_1 + \cdots + \varkappa_N - m + s) ,$$

and hence there exists a further positive constant c_3 independent of k such that

$$\nu_1 + \cdots + \nu_N \leq k + c_3 . \tag{27}$$

It follows therefore finally from (22) and (24) that

$$|f_k| \leq k^{c_2} \cdot \max_{\{\nu\}} |f_{\nu_1} \cdots f_{\nu_N}| \qquad \text{for} \quad k \geq k_0 , \tag{28}$$

where the maximum is extended over all systems $\{\nu\}$ with the properties (26) and (27).

12. We may now, without loss of generality, assume that

(29) $$k_0 > c_3 + 1 .$$

Choose any k_0 positive numbers $u_0, u_1, \cdots, u_{k_0-1}$ such that

(30) $$0 < u_0 < u_1 < \cdots < u_{k_0-1}, \quad \text{and} \quad |f_k| \leq e^{u_k} \quad \text{for} \quad k = 0, 1, \cdots, k_0 - 1,$$

and then, for each suffix $k \geq k_0$, define recursively a number u_k by the equation

(31) $$u_k = c_2 \log k + \max_{\{\nu\}} (u_{\nu_1} + \cdots + u_{\nu_N}).$$

Here $\{\nu\}$ is to run again over all systems of integers with the properties (26) and (27). For use below, denote by S_k the set of all such systems $\{\nu\}$.

We assert that with this definition of u_k,

(32) $$|f_k| \leq e^{u_k} \qquad \text{for all suffixes} \quad k \geq 0 .$$

For this is certainly true for $k \leq k_0 - 1$, and it is for larger k a consequence of (28) and (31) because

$$|f_k| \leq \exp\left(c_2 \log k + \max (u_{\nu_1} + \cdots + u_{\nu_N})\right) = e^{u_k}.$$

Let now again $k \geq k_0$, hence, by (29),

$$k > c_3 + 1 .$$

The recursive formula (31) implies then that

(33) $$u_{k+1} - u_k = c_2 \log \frac{k+1}{k} + \max_{\{\nu'\} \in S_{k+1}} (u_{\nu'_1} + \cdots + u_{\nu'_{N'}}) - \max_{\{\nu\} \in S_k} (u_{\nu_1} + \cdots + u_{\nu_N}).$$

Here S_k evidently is a subset of S_{k+1}; the maximum over S_{k+1} is therefore not less than that over S_k, and so (33) implies that

(34) $$u_{k+1} - u_k \geq c_2 \log \frac{k+1}{k} > 0 \qquad \text{for} \quad k \geq k_0 .$$

Together with the first inequalities (30), this proves that the numbers u_k form a strictly increasing sequence of positive numbers.

13. Consider now any system $\{\pi\} = \{\pi_1, \cdots, \pi_{N^*}\}$ in S_{k+1} at which the maximum

$$\max_{\{\nu'\} \in S_{k+1}} (u_{\nu'_1} + \cdots + u_{\nu'_{N'}}) = u_{\pi_1} + \cdots + u_{\pi_{N^*}}$$

is attained. Since the numbers u_k are positive and strictly increasing, the suffixes $\pi_1, \cdots, \pi_{N^*}$ cannot all be zero; moreover, since

$$\pi_1 + \cdots + \pi_{N^*} \leq k + c_3 + 1 \qquad \text{and} \qquad k > c_3 + 1 ,$$

at most one of these suffixes can be as large as k. Denote by

$$\pi_{N^*} > 0$$

the largest of the suffixes $\pi_1, \cdots, \pi_{N^*}$, or one of them if several of these suffixes have the same maximum value. The other suffixes

$$\pi_1, \cdots, \pi_{N^*-1}$$

are then non-negative and less than k. Hence the system $\{\nu^0\} = \{\nu_1^0, \cdots, \nu_{N^0}^0\}$ defined by

$$N^0 = N^* \quad , \quad \nu_1^0 = \pi_1, \cdots, \nu_{N^0-1}^0 = \pi_{N^*-1}, \; \nu_{N^0}^0 = \pi_{N^*} - 1 \geq 0$$

belongs to the set S_k, and therefore

$$\max_{\{\nu\} \in S_k} (u_{\nu_1} + \cdots + u_{\nu_N}) \geq u_{\pi_1} + \cdots + u_{\pi_{N^*-1}} + u_{\pi_{N^*}-1} =$$

$$= \max_{\{\nu'\} \in S_{k+1}} (u_{\nu'_1} + \cdots + u_{\nu'_{N'}}) - (u_{\pi_{N^*}} - u_{\pi_{N^*}-1}) .$$

Here

$$u_{\pi_{N^*}} - u_{\pi_{N^*}-1} \leq \max_{\nu=0,1,\cdots,k-1} (u_{\nu+1} - u_\nu) .$$

Therefore, on combining the equation (33) with these two inequalities, it follows that

$$u_{k+1} - u_k \leq c_2 \log \frac{k+1}{k} + \max_{\nu=0,1,\cdots,k-1} (u_{\nu+1} - u_\nu) \qquad \text{for} \quad k \geq k_0 . \tag{35}$$

14. Finally put

$$v_k = u_{k+1} - u_k \quad \text{and} \quad c_4 = \max(v_0, v_1, \cdots, v_{k_0-1}) ,$$

so that c_4 is a further positive constant independent of k. Now, by (35),

$$v_k \leq c_2 \log \frac{k+1}{k} + \max_{\nu=0,1,\cdots,k-1} v_\nu \qquad \text{for} \quad k \geq k_0 ,$$

or equivalently,

$$v_k \leq c_2 \log \frac{k+1}{k} + \max(c_4, v_{k_0}, v_{k_0+1}, \cdots, v_{k-1}) \qquad \text{for} \quad k \geq k_0 .$$

This inequality implies that

$$v_k \leq c_2 \log \frac{k+1}{k_0} + c_4 \qquad \text{for} \quad k \geq k_0 . \tag{36}$$

For this assertion certainly is true if $k = k_0$. Assume then that $k > k_0$ and that the assertion has already been proved for all suffixes up to and including $k - 1$. Then

$$\max(c_4, v_{k_0}, v_{k_0+1}, \cdots, v_{k-1}) \leq c_2 \log \frac{k}{k_0} + c_4 ,$$

whence

$$v_k \leq c_2 \log \frac{k+1}{k} + c_2 \log \frac{k}{k_0} + c_4 = c_2 \log \frac{k+1}{k_0} + c_4 .$$

This proves that the estimate (36) holds also for the suffix k and therefore is always true.

On putting

$$c_5 = c_4 - c_2 \log k_0 ,$$

the inequality (36) shows that

$$u_{k+1} - u_k \leq c_2 \log (k + 1) + c_5 \qquad \text{for} \quad k \geq k_0 .$$

We apply this formula for the successive suffixes $k_0 , k_0 + 1 , \cdots , k - 1$, and add all the results. This leads to the estimate

$$u_k \leq u_{k_0} + c_2 \log (k!/k_0!) + c_5 (k - k_0) \qquad \text{for} \quad k \geq k_0 ,$$

which, by (32), is equivalent to

$$|f_k| \leq e^{u_{k_0} + c_5 (k-k_0)} (k!/k_0!)^{c_2} \qquad \text{for} \quad k \geq k_0 .$$

In this formula, $k!$ increases more rapidly than any exponential function of k. We arrive then finally at the following result where we have replaced the suffix k again by h.

THEOREM: *Let*

$$f = \sum_{h=0}^{\infty} f_h z^h$$

be a formal power series with real or complex coefficients which satisfies an algebraic differential equation. Then there exist two positive constants γ_1 *and* γ_2 *such that*

$$|f_h| \leq \gamma_1 (h!)^{\gamma_2} \qquad (h = 0 , 1 , 2 , \cdots) . \tag{37}$$

By way of example, one can easily show that if r is any positive integer, then

$$f = \sum_{h=0}^{\infty} (h!)^r z^h$$

satisfies a linear differential equation, with coefficients that are polynomials in z. It is thus in general not possible to improve on the estimate (37).

The theorem seems to be new. In §§ 12–14, its proof makes use of an idea by a young Canberra mathematician, Mr. A. N. Stokes. For the technique of applying the algebraic differential equation to the coefficients f_h I am of course greatly indebted to Popken's doctor thesis (1935).

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