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Boundedness Criteria for Solutions of some Second-order Differential Equations

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Equazioni differenziali. — *Boundedness Criteria for Solutions of some Second-order Differential Equations.* Nota di H. O. Tẹjumọla, presentata (*) dal Socio G. Sansone.

Riassunto. — Si dimostrano due teoremi di limitatezza delle soluzioni di una classe di equazioni differenziali non lineari del secondo ordine.

1. Much work has been done by previous authors on the problem of the boundedness of solutions of second-order differential equations (see, for example, [1]–[7]). The object of the present note is to give new criteria for solutions of second-order equations of the form

$$\ddot{x} + f(x, \dot{x})\,\dot{x} + g(x) = p(t, x, \dot{x}) \tag{1.1}$$

to be ultimately bounded. It will be assumed throughout what follows that the functions f, g and p, which depend only on the arguments displayed in (1.1), are continuous. The following results will be proved.

THEOREM 1. Let $\delta > 0$, $M > 0$ *be finite constants such that* $\delta_* = \delta - M - 1 > 0$ *and suppose that*

(i) *the function* $f(x, y)$ *is such that*

$$|y|\, f(x, y) \geq \delta \quad (|y| \geq 1) \;, \quad \max_{|y|\leq 1} |f(x, y)| = \gamma(x) \; ; \tag{1.2}$$

(ii) *the function* $g(x)$ *satisfies*

$$\lim_{|x|\to\infty} \int_0^x g(\xi)\, d\xi = +\infty \,, \tag{1.3}$$

$$\varliminf_{|x|\to\infty} \{g(x) \operatorname{sgn} x - 2\gamma(x)\} > 2M \; ; \tag{1.4}$$

(iii) *for all* t, x *and* y,

$$|p(t, x, y)| \leq M \,. \tag{1.5}$$

Then, there exists a constant D, $0 < D < \infty$, *whose magnitude depends only on the constants* δ *and* M *as well as on the functions* f, g *and* p *such that every solution* $x(t)$ *of* (1.1) *ultimately satisfies*

$$|x(t)| \leq D \quad , \quad |\dot{x}(t)| \leq D \,. \tag{1.6}$$

The restriction on $p(t, x, y)$ in (1.5) can be relaxed at the expense of that on $f(x, y)$ for $|y| \geq 1$. Indeed we have

(*) Nella seduta del 17 aprile 1971.

THEOREM 2. *Let* $\delta > 0$, $M > 0$ *be finite constants such that* $\delta_* \equiv \delta - M > 0$ *and suppose further that*

(i) $f(x, y)$ *is such that*

$$f(x, y) \geq \delta \quad (|y| \geq 1), \quad \max_{|y| \leq 1} |f(x, y)| = \gamma(x); \tag{1.7}$$

(ii) *the function* $g(x)$ *satisfies*

$$\lim_{|x| \to \infty} \int_0^x g(\xi)\, d\xi = +\infty, \tag{1.8}$$

$$\underline{\lim}_{|x| \to \infty} \{g(x) \operatorname{sgn} x - 2\gamma(x)\} > 2 M_* \tag{1.9}$$

where

$$M_* = \max [4(\delta + M)\delta_*^{-1}, M]; \tag{1.10}$$

(iii) *for all* t, x *and* y

$$|p(t, x, y)| \leq M y^2. \tag{1.11}$$

Then, there exists a constant D, $0 < D < \infty$, *whose magnitude depends only on the constants* M *and* δ *as well as on the functions* f, g *and* p *such that every solution* $x(t)$ *of* (1.1) *ultimately satisfies* (1.6).

Note that if the left hand side of (1.4) equals $+\infty$ then the condition (1.3) will be met. Thus conditions (1.3) and (1.4) allow for bounded, as well as unbounded functions, $g(x)$.

In view of the form of (1.2) one might be tempted to compare Theorem 1 with the boundedness theorems in [3] and [4]. The main point of our result here is that it concerns the ultimate boundedness property of solutions, which property includes the notion of the relative boundedness dealt with in [3] and [4]. Moreover, we do not require here any uniqueness conditions on f, g and p.

Theorem 2 extends the boundedness theorem given in [2] although, here, $|p(t, x, y)|$ is bounded whenever $|y|$ is.

2. *Proof of Theorem 1.* The system

$$\dot{x} = y, \quad \dot{y} = -f(x, y)\, y - g(x) + p(t, x, y) \tag{2.1}$$

is equivalent to (1.1). Let the continuous function $V = V(x, y)$ be defined by

$$V = V_1 + V_2 \tag{2.2}$$

where

$$2 V_1 = y^2 + 2 \int_0^x g(\xi)\, d\xi, \tag{2.3}$$

$$V_2 = \begin{bmatrix} y \operatorname{sgn} x, & \text{if } |y| \leq |x| \\ x \operatorname{sgn} y, & \text{if } |x| \leq |y| \end{bmatrix}. \tag{2.4}$$

We shall show that $V(x, y)$ satisfies

$$V(x, y) \to +\infty \quad \text{as} \quad x^2 + y^2 \to \infty \tag{2.5}$$

and that, for any solution $(x(t), y(t))$ of (2.1),

$$\dot{V}^+ = \operatorname*{Lim\ sup}_{h \to +0} \left\{ \frac{V(x(t+h), y(t+h)) - V(x(t), y(t))}{h} \right\}$$

exists and satisfies

$$\dot{V}^+ \leq -D_0 \quad \text{if} \quad x^2(t) + y^2(t) \geq D_1 \tag{2.6}$$

for some finite constants $D_0 > 0$, $D_1 > 0$. As will be clear from the Yoshizawa-type technique employed in [2; § 5] the two results (2.5) and (2.6) together imply, ultimately, that

$$x^2(t) + y^2(t) \leq D$$

which is (1.6).

To verify (2.5), note from (2.4) that

$$|V_2| \leq |y|$$

and thus, by (2.2) and (2.3),

$$2V \geq y^2 - 2|y| + 2\int_0^x g(\xi)\, d\xi .$$

In view of (1.3), the right hand side here tends to $+\infty$ as $x^2 + y^2 \to \infty$.

The existence of $\dot{V}^+$ for any solution $(x(t), y(t))$ of (2.1) follows from the fact that $V = V(x, y)$ is at least locally Lipschitzian in x and y.

We now turn to the verification of (2.6). Note from (2.2), (2.3), (2.4), and (2.1) that

$$\dot{V}^+ = \dot{V}_1 + \dot{V}_2^+ , \tag{2.7}$$

where

$$\dot{V}_1 = -f(x, y)\, y^2 + y p(t, x, y) , \tag{2.8}$$

$$\dot{V}_2^+ = \begin{cases} -g(x) \operatorname{sgn} x - f(x,y) y \operatorname{sgn} x + p(t,x,y) \operatorname{sgn} x, & \text{if } |y| \leq |x| \\ |y| & \text{if } |x| \leq |y| \end{cases} . \tag{2.9}$$

Thus

$$\dot{V}^+ \leq -g(x) \operatorname{sgn} x - f(x, y)\, y^2 + |f(x, y)|\, |y| + (|y| + 1)\, |p(t, x, y)| \tag{2.10}$$

if $|y| \leq |x|$ or

$$\dot{V}^+ \leq -f(x, y) y^2 + |y| (1 + |p(t, x, y)|) \tag{2.11}$$

if $|x| \leq |y|$.

The condition (1.4) implies the existence of finite constants $x_0 > 0$, $D_2 > 0$ such that

(2.12) $$|x| \geq x_0 \Rightarrow g(x) \operatorname{sgn} x - 2\gamma(x) - 2M > D_2 .$$

Let

(2.13) $$x_1 = \max\{1, x_0, \delta_*^{-1}\} .$$

We assert that, for some finite constant $D_3 > 0$,

(2.14) $$\dot{V}^+ \leq -D_3 \quad \text{if} \quad |x| \geq x_1 .$$

Indeed, if $|y| \leq |x|$ so that $\dot{V}^+$ satisfies (2.10) and, if $|y| \geq 1$, then by (1.2) and (1.5)

$$\begin{aligned}\dot{V}^+ &\leq -g(x) \operatorname{sgn} x - |y| f(x, y) (|y| - 1) + (1 + |y|) |p(t, x, y)| \\ &\leq -g(x) \operatorname{sgn} x - \delta(|y| - 1) + M(1 + |y|) \\ &\leq -g(x) \operatorname{sgn} x + 2M .\end{aligned}$$

Thus

(2.15) $$\dot{V}^+ \leq -D_2 \quad \text{if} \quad |x| \geq x_1,$$

by (2.12) and (2.13). Suppose however that $|y| \leq 1$. Then on using (1.2) and (1.5) in (2.10), we obtain

$$\dot{V}^+ \leq -g(x) \operatorname{sgn} x + 2\gamma(x) + 2M ,$$

so that by (2.12), (2.15) still holds in this case.

We are now left with the case: $|x| \leq |y|$ for which $\dot{V}^+$ satisfies (2.11). If we note that $|x| \geq x_1$ implies that $|y| \geq x_1$, with x_1 fixed by (2.13), we have that

$$\begin{aligned}\dot{V}^+ &\leq -\delta|y| + (1 + M)|y| \\ &= -\delta_*|y|\end{aligned}$$

by (1.2) and (1.5). Hence $\dot{V}^+ \leq -1$ since $|y| \geq x_1$; that is,

$$|x| \geq x_1 \Rightarrow \dot{V}^+ \leq -1 .$$

This together with (2.15) show that (2.14) holds with $D_3 = \max(1, D_2)$.

To complete the proof of (2.6), suppose, on the contrary, that $|x| \leq x_1$ and assume for a start that $|y| \geq x_1$. Then $|y| \geq |x|$ and so $\dot{V}^+$ satisfies (2.11). If we recall the definition (2.13), we get, in the same way as before,

(2.16) $$\dot{V}^+ \leq -1 \quad \text{if} \quad |y| \geq x_1 .$$

The results (2.14) and (2.16) show clearly that

$$\dot{V}^+ \leq -D_3 \quad \text{if} \quad x^2 + y^2 \geq 2x_1 ,$$

$D_3 = \max(1, D_2)$. This completes the proof of (2.6) and, as remarked earlier, the theorem now follows.

3. *Proof of Theorem 2*. The procedure here is the same as that used for Theorem 1. For reasons which have been carefully outlined in § 2 the proof of Theorem 2 will be immediate as soon as we show that the properties (2.5) and (2.6) of the function $V = V(x, y)$ hold under the conditions of Theorem 2.

The verification of (2.5) given in § 2 carries over with obvious modifications.

In order to verify (2.6), our starting point will be the estimates (2.10) and (2.11) which are still valid in this case. In view of (1.9) there are constants $x_0 > 0$, $D_4 > 0$ such that

$$|x| \geq x_0 \Rightarrow (g(x) \operatorname{sgn} x - 2\gamma(x) - 2M_*) \geq D_4 . \tag{3.1}$$

Suppose also that $y_0 > 0$ is a constant such that

$$|y| \geq y_0 \Rightarrow -\delta^* y^2 + M|y| \leq -1 \tag{3.2}$$

and set

$$x_1 = \max\{1, x_0, y_0\} . \tag{3.3}$$

First we show that for some constant $D_5 > 0$

$$|x| \geq x_1 \Rightarrow \dot{V}^+ \leq -D_5 . \tag{3.4}$$

As before we consider the two cases $|y| \leq |x|$, $|x| \leq |y|$ separately. Let $|y| \leq |x|$ and suppose that $|y| \geq 1$. Then on using (1.7) and (1.11) in (2.10), one shows readily that

$$\dot{V}^+ \leq -g(x) \operatorname{sgn} x + \frac{\delta + M}{4\delta_*} .$$

If, however, $|y| \leq 1$, (1.7), (1.11) together with (2.10) yield

$$\dot{V}^+ \leq -g(x) \operatorname{sgn} x + 2\gamma(x) + 2M .$$

By (3.1), (1.10) and (3.3) it is clear that in either case (3.4) holds.

Suppose now that $|x| \leq |y|$. Then $|x| \geq x_1 \Rightarrow |y| > x_1 \geq y_0$ by (3.3). Hence, (2.11) yields

$$\begin{aligned} \dot{V}^+ &\leq -\delta_* y^2 + M|y| \\ &\leq -1 \end{aligned}$$

since $|x| \geq x_1$. Therefore (3.4) holds always with $D_5 = \max(1, D_4)$.

Suppose on the contrary that $|x| \leq x_1$ and assume that $|y| \geq x_1$. Then $|y| \geq |x|$ and so, by (2.11),

$$\begin{aligned} \dot{V}^+ &\leq -\delta_* y^2 + M|y| \\ &\leq -1 \end{aligned}$$

since $|y| \geq x_1 \geq y_0$. This together with (3.4) shows that

$$\dot{V}^+ \leq -D_5 \quad \text{if} \quad x^2 + y^2 \geq 2x_1 ,$$

which verifies (2.6).

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