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On the solution of a non-linear mixed problem for the Navier-Stokes equations in a time dependent domain. Nota III

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Analisi matematica. — *On the solution of a non-linear mixed problem for the Navier-Stokes equations in a time dependent domain.* Nota III di GIOVANNI PROUSE, presentata (*) dal Corrisp. L. AMERIO.

RIASSUNTO. — Si dà la dimostrazione dei Teoremi 2, 3, 5 enunciati nella Nota I.

3. – *Proof of Theorem* 2. Let $\{\vec{v}_n(t)\}$ be a sequence such that

$$\|\vec{v}_n\|_{W_{T,\sigma}} = \tag{3.1}$$

$$= \left\{ \int_0^T (\|\vec{v}_n(t)\|^2_{V_{\sigma+1}(\Omega_t)} \, dt + \|\vec{v}'_n(t)\|^2_{V_{\sigma-1}(\Omega_t)}) \, dt + \sup_{0 \leq t \leq T} \|\vec{v}_n(t)\|_{V_\sigma(\Omega_t)} \right\}^{1/2} \leq M_2 .$$

We shall show that it is possible to select a subsequence (which will again be denoted by $\{\vec{v}_n(t)\}$) such that the sequence $\{\vec{u}_n(t)\}$, with $\vec{u}_n(t) = S(\vec{v}_n(t))$ converges strongly in $W_{T,\sigma}$.

Let $\vec{v}_m(t), \vec{v}_n(t)$ be any two functions of the sequence considered; by the definition of the transformation S, the corresponding functions $\vec{u}_m(t), \vec{u}_n(t)$ are such that, $\forall \vec{h}(t) \in L^2(0, T; V_{1-\sigma}(\Omega_t))$,

$$\int_0^T \{\langle \vec{u}'_m(t) - \vec{u}'_n(t), \vec{h}(t)\rangle + \mu \langle A\vec{u}_m(t) - A\vec{u}_n(t), \vec{h}(t)\rangle\} \, dt = \tag{3.2}$$

$$= - \int_0^T \Big\{ b(t, \vec{v}_m(t), \vec{v}_m(t), \vec{h}(t)) - b(t, \vec{v}_n(t), \vec{v}_n(t), \vec{h}(t)) -$$

$$- \frac{1}{2} \sum_{i=1}^2 \int_{\Gamma_i} (v^2_{m,1}(x, t) - v^2_{n,1}(x, t)) \vec{h}(x, t) \times \vec{\nu}_t \, d\Gamma_i +$$

$$+ \int_{\Gamma_{3,t}} \beta(x, t) \Big((\vec{v}_m(x, t) \times \vec{\nu}_t) |\vec{v}_m(x, t) \times \vec{\nu}_t| -$$

$$- (\vec{v}_n(x, t) \times \vec{\nu}_t) |\vec{v}_n(x, t) \times \vec{\nu}_t| \Big) \vec{h}(x, t) \times \vec{\nu}_t \, d\Gamma_{3,t} \Big\} \, dt$$

$$\vec{u}_m(0) - \vec{u}_n(0) = 0 . \tag{3.3}$$

(*) Nella seduta del 20 febbraio 1971.

Setting $\vec{z}_{mn}(t) = \vec{v}_m(t) - \vec{v}_n(t)$, $\vec{w}_{mn}(t) = \vec{u}_m(t) - \vec{u}_n(t)$, $\vec{h}(t) = A^\sigma \vec{w}_{mn}(t)$, when $0 \le t \le \bar{t} \le T$, $\vec{h}(t) = 0$ when $\bar{t} < t$, we obtain

$$(3.4) \qquad \int_0^{\bar{t}} \{\langle \vec{w}'_{mn}(t), A^\sigma \vec{w}_{mn}(t)\rangle + \mu \langle A\vec{w}_{mn}(t), A^\sigma \vec{w}_{mn}(t)\rangle\} \, dt =$$

$$= - \int_0^{\bar{t}} \Big\{ b(t, \vec{z}_{mn}(t), \vec{v}_m(t), A^\sigma \vec{w}_{mn}(t)) - b(t, \vec{v}_n(t), \vec{z}_{mn}(t), A^\sigma \vec{w}_{mn}(t)) -$$

$$- \frac{1}{2} \sum_{i=1}^{2} \int_{\Gamma_i} (v_{m,1}(x,t) + v_{n,1}(x,t)) \, z_{mn,1}(x,t) \, A^\sigma \vec{w}_{mn}(x,t) \times \vec{\nu}_t \, d\Gamma_i +$$

$$+ \int_{\Gamma_{3,t}} \beta(x,t) \Big((\vec{v}_m(x,t) \times \vec{\nu}_t) \, |\vec{v}_m(x,t) \times \vec{\nu}_t| -$$

$$- (\vec{v}_n(x,t) \times \vec{\nu}_t) \, |\vec{v}_n(x,t) \times \vec{\nu}_t| \Big) A^\sigma w_{mn}(x,t) \times \vec{\nu}_t \, d\Gamma_{3,t} \Big\} \, dt .$$

On the other hand, by (2.1), (2.4), (2.5), (2.6) and Hölder's inequality

$$(3.5) \qquad \left| \int_0^{\bar{t}} b(t, \vec{z}_{mn}(t), \vec{v}_m(t), A^\sigma \vec{w}_{mn}(t)) \, dt \right| \le$$

$$\le c_1 \|\vec{z}_{mn}(t)\|_{L^{2/\sigma}(0,\bar{t}; L^{2/(1-\sigma)}(\Omega_t))} \|\vec{v}_m(t)\|_{L^{2/(1-\sigma)}(0,\bar{t}; V_1(\Omega_t))} \|A^\sigma \vec{w}_{mn}(t)\|_{L^2(0,\bar{t}; L^{2/\sigma}(\Omega_t))} \le$$

$$\le c_2 \|\vec{z}_{mn}(t)\|_{H^{(1-\sigma)/2}(0,\bar{t}; V_\sigma(\Omega_t))} \|\vec{v}_m(t)\|_{H^{\sigma/2}(0,\bar{t}; V_1(\Omega_t))} \|\vec{w}_{mn}(t)\|_{L^2(0,\bar{t}; V_{\sigma+1}(\Omega_t))} ,$$

$$\left| \int_0^{\bar{t}} b(t, \vec{v}_n(t), \vec{z}_{mn}(t), A^\sigma \vec{w}_{mn}(t)) \, dt \right| \le$$

$$\le c_3 \|\vec{v}_n(t)\|_{L^\infty(0,\bar{t}; L^{2/(1-\sigma)}(\Omega_t))} \|\vec{z}_{mn}(t)\|_{L^2(0,\bar{t}; V_1(\Omega_t))} \|A^\sigma \vec{w}_{mn}(t)\|_{L^2(0,\bar{t}; L^{2/\sigma}(\Omega_t))} \le$$

$$\le c_4 \|\vec{v}_n(t)\|_{L^\infty(0,\bar{t}; V_\sigma(\Omega_t))} \|\vec{z}_{mn}(t)\|_{L^2(0,\bar{t}; V_1(\Omega_t))} \|\vec{w}_{mn}(t)\|_{L^2(0,\bar{t}; V_{\sigma+1}(\Omega_t))} ,$$

$$\left| \int_0^{\bar{t}} \Big\{ \frac{1}{2} \sum_{i=1}^{2} \int_{\Gamma_i} (v_{m,1}(x,t) + v_{n,1}(x,t)) \, z_{mn,1}(x,t) \, A^\sigma \vec{w}_{mn}(x,t) \times \vec{\nu}_t \, d\Gamma_i - \right.$$

$$-\int_{\Gamma_{3,t}} \beta(x,t)\Big((\vec{v}_m(x,t)\times\vec{\nu}_t)\,|v_m(x,t)\times\vec{\nu}_t| -$$

$$-(\vec{v}_n(x,t)\times\vec{\nu}_t)\,|\vec{v}_n(x,t)\times\vec{\nu}_t|\Big)\,A^\sigma\vec{w}_{mn}(x,t)\times\vec{\nu}_t\,d\Gamma_{3,t}\Big\}\,dt\,\Big| \leq$$

$$\leq c_5\int_0^{\bar t}(\|\gamma\vec{v}_m(t)\|_{L^4(\Gamma_t)}+\|\gamma\vec{v}_n(t)\|_{L^4(\Gamma_t)})\,\|\gamma\vec{z}_{mn}(t)\|_{L^4(\Gamma_t)}\,\|\gamma A^\sigma\vec{w}_{mn}(t)\|_{L^2(\Gamma_t)}\,dt \leq$$

$$\leq c_6\int_0^{\bar t}(\|\vec{v}_m(t)\|_{V_{3/4}(\Omega_t)}+\|\vec{v}_n(t)\|_{V_{3/4}(\Omega_t)})\,\|\vec{z}_{mn}(t)\|_{V_{3/4}(\Omega_t)}\,\|\vec{w}_{mn}(t)\|_{V_{(1/2)+2\sigma+\varepsilon}(\Omega_t)}\,dt \leq$$

$$\leq c_7\int_0^{\bar t}\Big(\|\vec{v}_m(t)\|_{V_\sigma(\Omega_t)}^{\frac{1+4\sigma-4\varepsilon}{4(1-\varepsilon)}}\,\|\vec{v}_m(t)\|_{V_{\sigma+1-\varepsilon}(\Omega_t)}^{\frac{3-4\sigma}{4(1-\varepsilon)}}+\|\vec{v}_n(t)\|_{V_\sigma(\Omega_t)}^{\frac{1+4\sigma-4\varepsilon}{4(1-\varepsilon)}}\,\|\vec{v}_n(t)\|_{V_{\sigma+1-\varepsilon}(\Omega_t)}^{\frac{3-4\sigma}{4(1-\varepsilon)}}\Big)\cdot$$

$$\cdot\|\vec{z}_{mn}(t)\|_{V_\sigma(\Omega_t)}^{\frac{1+4\sigma-4\varepsilon}{4(1-\varepsilon)}}\,\|\vec{z}_{mn}(t)\|_{V_{\sigma+1-\varepsilon}(\Omega_t)}^{\frac{3-4\sigma}{4(1-\varepsilon)}}\,\|\vec{w}_{mn}(t)\|_{V_{\sigma+1}(\Omega_t)}\,dt \leq$$

$$\leq c_8\Big(\|\vec{v}_m(t)\|_{L^\infty(0,\bar t;V_\sigma(\Omega_t))}^{\frac{1+4\sigma-4\varepsilon}{4(1-\varepsilon)}}\,\|\vec{v}_m(t)\|_{L^{\frac{3-4\sigma}{1-\varepsilon}}(0,\bar t;V_{\sigma+1-\varepsilon}(\Omega_t))}^{\frac{3-4\sigma}{4(1-\varepsilon)}}+\|\vec{v}_n(t)\|_{L^\infty(0,\bar t;V_\sigma(\Omega_t))}^{\frac{1+4\sigma-4\varepsilon}{4(1-\varepsilon)}}\cdot$$

$$\cdot\|\vec{v}_n(t)\|_{L^{\frac{3-4\sigma}{1-\varepsilon}}(0,\bar t;V_{\sigma+1-\varepsilon}(\Omega_t))}^{\frac{3-4\sigma}{4(1-\varepsilon)}}\Big)\,\|\vec{z}_{mn}(t)\|_{L^\infty(0,\bar t;V_\sigma(\Omega_t))}^{\frac{1+4\sigma-4\varepsilon}{4(1-\varepsilon)}}\cdot$$

$$\cdot\|\vec{z}_{mn}(t)\|_{L^{\frac{3-4\sigma}{1-\varepsilon}}(0,\bar t;V_{\sigma+1-\varepsilon}(\Omega_t))}^{\frac{3-4\sigma}{4(1-\varepsilon)}}\,\|\vec{w}_{mn}(t)\|_{L^2(0,\bar t;V_{\sigma+1}(\Omega_t))}\,.$$

Moreover, bearing in mind (2.3), (2.26),

$$\int_0^{\bar t}\{\langle\vec{w}'_{mn}(t),A^\sigma\vec{w}_{mn}(t)\rangle+\mu\langle A\vec{w}_{mn}(t),A^\sigma\vec{w}_{mn}(t)\rangle\}\,dt \geq \tag{3.6}$$

$$\geq\int_0^{\bar t}\{\mu\|\vec{w}_{mn}(t)\|_{V_{\sigma+1}(\Omega_t)}^2-c_9\|\vec{w}_{mn}(t)\|_{V_1(\Omega_t)}^2\}\,dt+\frac{1}{2}\|\vec{w}_{mn}(\bar t)\|_{V_\sigma(\Omega_{\bar t})}^2\geq$$

$$\geq\frac{1}{2}\|\vec{w}_{mn}(\bar t)\|_{V_\sigma(\Omega_{\bar t})}^2+\int_0^{\bar t}\{\mu\|\vec{w}_{mn}(t)\|_{V_{\sigma+1}(\Omega_t)}^2-\frac{\mu}{2}\|\vec{w}_{mn}(t)\|_{V_{\sigma+1}(\Omega_t)}^2-$$

$$-c_{10}\|\vec{w}_{mn}(t)\|_{V_\sigma(\Omega_t)}^2\}\,dt\,.$$

Hence, by (3.4), (3.5), (3.6),

$$(3.7)\quad \frac{1}{2}\|\vec{w}_{mn}(\bar{t})\|^2_{V_\sigma(\Omega_{\bar{t}})}+\frac{\mu}{2}\|\vec{w}_{mn}(t)\|^2_{L^2(0,\bar{t};V_{\sigma+1}(\Omega_t))}\leq c_{10}\|\vec{w}_{mn}(t)\|^2_{L^2(0,\bar{t};V_\sigma(\Omega_t))}+$$

$$+c_{11}\|\vec{w}_{mn}(t)\|_{L^2(0,\bar{t};V_{\sigma+1}(\Omega_t))}\Big(\|\vec{z}_{mn}(t)\|_{H^{(1-\sigma)/2}(0,\bar{t};V_\sigma(\Omega_t))}\|\vec{v}_m(t)\|_{H^{\sigma/2}(0,\bar{t};V_1(\Omega_t))}+$$

$$+\|\vec{z}_{mn}(t)\|_{L^2(0,\bar{t};V_1(\Omega_t))}\|\vec{v}_n(t)\|_{L^\infty(0,\bar{t};V_\sigma(\Omega_t))}+$$

$$+\Big(\|\vec{v}_m(t)\|^{\frac{1+4\sigma-4\varepsilon}{4(1-\varepsilon)}}_{L^\infty(0,\bar{t};V_\sigma(\Omega_t))}\|\vec{v}_m(t)\|^{\frac{3-4\sigma}{4(1-\varepsilon)}}_{L^{\frac{3-4\sigma}{1-\varepsilon}}(0,\bar{t};V_{\sigma+1-\varepsilon}(\Omega_t))}+$$

$$+\|\vec{v}_n(t)\|^{\frac{1+4\sigma-4\varepsilon}{4(1-\varepsilon)}}_{L^\infty(0,\bar{t};V_\sigma(\Omega_t))}\|\vec{v}_n(t)\|^{\frac{3-4\sigma}{4(1-\varepsilon)}}_{L^{\frac{3-4\sigma}{1-\varepsilon}}(0,\bar{t};V_{\sigma+1-\varepsilon}(\Omega_t))}\Big)\cdot$$

$$\cdot\|\vec{z}_{mn}(t)\|^{\frac{1+4\sigma-4\varepsilon}{4(1-\varepsilon)}}_{L^\infty(0,\bar{t};V_\sigma(\Omega_t))}\|\vec{z}_{mn}(t)\|^{\frac{3-4\sigma}{4(1-\varepsilon)}}_{L^{\frac{3-4\sigma}{1-\varepsilon}}(0,\bar{t};V_{\sigma+1-\varepsilon}(\Omega_t))}\Big).$$

We observe now that, if $\sigma>\frac{1}{4}$, it is, for sufficiently small ε, $\frac{3-4\sigma}{1-\varepsilon}\leq 2$; moreover the embeddings of $H^{(1-\sigma)/2}(0,\bar{t};V_\sigma(\Omega_t))$ and $L^2(0,\bar{t};V_{\sigma+1-\varepsilon}(\Omega_t))$ in $L^2(0,\bar{t};V_{\sigma+1}(\Omega_t))\cap H^1(0,\bar{t};V_{\sigma-1}(\Omega_t))$ are completely continuous. In fact

$$L^2(0,\bar{t};V_{\sigma+1}(\Omega_t))\cap H^1(0,\bar{t};V_{\sigma-1}(\Omega_t))\subset$$

$$\subset[L^2(0,\bar{t};V_{\sigma+1}(\Omega_t)),H^1(0,\bar{t};V_{\sigma-1}(\Omega_t))]_\vartheta=H^\vartheta(0,\bar{t};V_{\sigma+1-2\vartheta}(\Omega_t))$$

and, consequently, choosing $\vartheta<\frac{\varepsilon}{2}$,

$$(3.8)\qquad H^\vartheta(0,\bar{t};V_{\sigma+1-2\vartheta}(\Omega_t))\subset L^2(0,\bar{t};V_{\sigma+1-\varepsilon}(\Omega_t)),$$

with completely continuous embedding, by property i), § 2.

Analogously, if $\vartheta<\frac{1}{2}$, $\sigma>1-2\vartheta$ (i.e. if $\sigma>0$)

$$H^\vartheta(0,\bar{t};V_{\sigma+1-2\vartheta}(\Omega_t))\subset H^\vartheta(0,\bar{t};V_\sigma(\Omega_t))\subset H^{(1-\sigma)/2}(0,\bar{t};V_\sigma(\Omega_t)),$$

the embedding again being completely continuous.

We can therefore assume, by (3.1), that

$$(3.9)\qquad \lim_{m,n\to\infty}\|\vec{z}_{mn}(t)\|_{L^2(0,T;V_{\sigma+1-\varepsilon}(\Omega_t))}=\lim_{m,n\to\infty}\|\vec{z}_{mn}(t)\|_{H^{(1-\sigma)/2}(0,T;V_\sigma(\Omega_t))}=0.$$

In exactly the same way it can be shown that, by (3.1),

$$(3.10)\qquad \|\vec{v}_n(t)\|_{H^{\sigma/2}(0,T;V_1(\Omega_t))\cap L^\infty(0,T;V_\sigma(\Omega_t))\cap L^{\frac{3+4\varepsilon}{1-\varepsilon}}(0,T;V_{\sigma+1-\varepsilon}(\Omega_t))}\leq M_3.$$

It follows then from (3.7) that

$$\lim_{m,n\to\infty}\left(\frac{1}{2}\,\|\vec{w}_{mn}(\bar{t})\|^2_{V_\sigma(\Omega_{\bar{t}})}+\frac{\mu}{2}\,\|\vec{w}_{mn}(t)\|^2_{L^2(0,\bar{t};V_{\sigma+1}(\Omega_t))}\right)=0$$

and consequently, since $\bar{t}$ is an arbitrary point $\in[0,T]$, the sequence $\{\vec{u}_n(t)\}$ converges strongly in $L^2(0,T;V_{\sigma+1}(\Omega_t))\cap L^\infty(0,T;V_\sigma(\Omega_t))$.

We have, on the other hand, analogously to (3.5),

$$\left|\int_0^T\langle\vec{w}'_{mn}(t),\vec{h}(t)\rangle\,dt\right|=\left|\int_0^T\Big\{\mu\,\langle A\vec{w}_{mn}(t),\vec{h}(t)\rangle+\right.$$

$$+\,b(t,\vec{z}_{mn}(t),\vec{v}_m(t),\vec{h}(t))-b(t,\vec{v}_n(t),\vec{z}_{mn}(t),\vec{h}(t))-$$

$$-\frac{1}{2}\sum_{i=1}^{2}\int_{\Gamma_i}(v^2_{m,1}(x,t)-v^2_{n,1}(x,t))\,\vec{h}(x,t)\times\vec{\nu}_t\,d\Gamma_i+$$

$$+\int_{\Gamma_{3,t}}\beta(x,t)\Big((\vec{v}_m(x,t)\times\vec{\nu}_t)\,|\vec{v}_m(x,t)\times\vec{\nu}_t|-$$

$$\left.-(\vec{v}_n(x,t)\times\vec{\nu}_t)\,|\vec{v}_n(x,t)\times\vec{\nu}_t|\Big)\,\vec{h}(x,t)\times\vec{\nu}_t\,d\Gamma_{3,t}\Big\}\,dt\right|\le$$

$$\le\Big(\mu\,\|\vec{w}_{mn}(t)\|_{L^2(0,T;V_{\sigma+1}(\Omega_t))}+\|\vec{z}_{mn}(t)\|_{H^{(1-\sigma)/2}(0,T;V_\sigma(\Omega_t))}\,\|\vec{v}_m(t)\|_{H^{\sigma/2}(0,T;V_1(\Omega_t))}+$$

$$+\,\|\vec{z}_{mn}(t)\|_{L^2(0,T;V_1(\Omega_t))}\,\|\vec{v}_n(t)\|_{L^\infty(0,T;V_\sigma(\Omega_t))}+$$

$$+\Big(\|\vec{v}_m(t)\|^{\frac{1+4\sigma-4\varepsilon}{4(1-\varepsilon)}}_{L^\infty(0,T;V_\sigma(\Omega_t))}\,\|\vec{v}_m(t)\|^{\frac{3-4\sigma}{4(1-\varepsilon)}}_{L^{\frac{3-4\sigma}{1-\varepsilon}}(0,T;V_{\sigma+1-\varepsilon}(\Omega_t))}+$$

$$+\,\|\vec{v}_n(t)\|^{\frac{1+4\sigma-4\varepsilon}{4(1-\varepsilon)}}_{L^\infty(0,T;V_\sigma(\Omega_t))}\,\|\vec{v}_n(t)\|^{\frac{3-4\sigma}{4(1-\varepsilon)}}_{L^{\frac{3-4\sigma}{1-\varepsilon}}(0,T;V_{\sigma+1-\varepsilon}(\Omega_t))}\Big)\cdot$$

$$\cdot\,\|\vec{z}_{mn}(t)\|^{\frac{1+4\sigma-4\varepsilon}{4(1-\varepsilon)}}_{L^\infty(0,T;V_\sigma(\Omega_t))}\,\|\vec{z}_{mn}(t)\|^{\frac{3-4\sigma}{4(1-\varepsilon)}}_{L^{\frac{3-4\sigma}{1-\varepsilon}}(0,T;V_{\sigma+1-\varepsilon}(\Omega_t))}\Big)\,\|\vec{h}(t)\|_{L^2(0,T;V_{1-\sigma}(\Omega_t))}.$$

Hence, as above, $\forall\vec{h}(t)\in L^2(0,T;V_{1-\sigma}(\Omega_t))$,

$$\left|\int_0^T\langle\vec{w}'_{mn}(t),\vec{h}(t)\rangle\,dt\right|\le\chi_{mn}\,\|\vec{h}(t)\|_{L^2(0,T;V_{1-\sigma}(\Omega_t))},$$

with $\lim\limits_{m,n\to\infty}\chi_{mn}=0$.

It follows that

$$\lim_{m,n\to\infty} \|\vec{w}'_{mn}(t)\|_{L^2(0,T;V_{\sigma-1}(\Omega_t))} = 0$$

and the sequence $\{\vec{u}_n(t)\}$ converges therefore strongly in $H^1(0, T; V_{\sigma-1}(\Omega_t))$.

The theorem is then completely proved.

4. – *Proof of Theorem* 3. Let $\vec{u}(t)$ be a solution of the equation $\vec{u}(t) = S(\vec{u}(t), \lambda)$ in $[0, T]$; $\vec{u}(t)$ satisfies then (1.8), (1.9).

Setting $\vec{h}(t) = A^\sigma \vec{u}(t)$ when $0 \le t \le \bar{t} \le T$, $\vec{h}(t) = 0$ when $t > \bar{t}$, we obtain then

$$(4.1)\qquad \int_0^{\bar{t}} \{\langle \vec{u}'(t), A^\sigma \vec{u}(t)\rangle + \mu\langle A\vec{u}(t), A^\sigma \vec{u}(t)\rangle - \lambda\langle \vec{f}(t), A^\sigma \vec{u}(t)\rangle\}\, dt =$$

$$= -\lambda \int_0^{\bar{t}} \Big\{ b(t, \vec{u}(t), \vec{u}(t), A^\sigma \vec{u}(t)) +$$

$$+ \sum_{i=1}^{2} \int_{\Gamma_i} \Big(\alpha_i(x,t) - \frac{1}{2} u_1^2(x,t)\Big) A^\sigma \vec{u}(x,t) \times \vec{\nu}_t \, d\Gamma_i +$$

$$+ \int_{\Gamma_{3,t}} \beta(x,t)\, (\vec{u}(x,t) \times \vec{\nu}_t)\, |\vec{u}(x,t) \times \vec{\nu}_t|\, A^\sigma \vec{u}(x,t) \times \vec{\nu}_t \, d\Gamma_{3,t} \Big\}\, dt.$$

Proceeding in exactly the same way as in the preceding §, we have

$$(4.2)\qquad \frac{1}{2}\|\vec{u}(\bar{t})\|^2_{V_\sigma(\Omega_{\bar{t}})} + \int_0^{\bar{t}} \mu \|\vec{u}(t)\|^2_{V_{\sigma+1}(\Omega_t)}\, dt \le \frac{\lambda}{2}\|\vec{u}_0\|^2_{V_\sigma(\Omega_0)} +$$

$$+ \lambda \int_0^{\bar{t}} \Big\{ \|\vec{f}(t)\|_{V_{\sigma-1}(\Omega_t)} \|\vec{u}(t)\|_{V_{\sigma+1}(\Omega_t)} + c_1 \|\vec{u}(t)\|_{V_\sigma(\Omega_t)} \|\vec{u}(t)\|_{V_1(\Omega_t)} \|\vec{u}(t)\|_{V_{\sigma+1}(\Omega_t)} +$$

$$+ c_2 \|\vec{u}(t)\|^2_{V_{3/4}(\Omega_t)} \|\vec{u}(t)\|_{V_{(1/2)+2\sigma+\varepsilon}(\Omega_t)} + c_4 \|\vec{u}(t)\|_{V_{(1/2)+2\sigma+\varepsilon}(\Omega_t)} \sum_{i=1}^{2} \|\alpha_i(t)\|_{L^2(\Gamma_i)} \Big\} dt +$$

$$+ c_3 \int_0^{\bar{t}} \|\vec{u}(t)\|^2_{V_1(\Omega_t)}\, dt \le \frac{\lambda}{2}\|\vec{u}_0\|^2_{V_\sigma(\Omega_0)} + c_5 \lambda \int_0^{\bar{t}} \{ \|\vec{u}(t)\|_{V_{\sigma+1}(\Omega_t)} +$$

$$+ \|\vec{u}(t)\|^{1+\sigma}_{V_\sigma(\Omega_t)} \|\vec{u}(t)\|^{2-\sigma}_{V_{\sigma+1}(\Omega_t)} + \|\vec{u}(t)\|^{2\sigma+(1/2)}_{V_\sigma(\Omega_t)} \|\vec{u}(t)\|^{(5/2)-2\sigma}_{V_{\sigma+1}(\Omega_t)} \}\, dt +$$

$$+ \int_0^{\bar{t}} \Big(\frac{\mu}{4} \|\vec{u}(t)\|^2_{V_{\sigma+1}(\Omega_t)} + c_6 \|\vec{u}(t)\|^2_{V_\sigma(\Omega_t)} \Big)\, dt.$$

Since $\sigma > \frac{1}{4}$, setting $\eta = \sigma - \frac{1}{4}$, it follows from (4.2), by (2.2), that

$$(4.3) \qquad \frac{1}{2}\|\vec{u}(\bar{t})\|^2_{V_\sigma(\Omega_{\bar{t}})} + \int_0^{\bar{t}} \mu \|\vec{u}(t)\|^2_{V_{\sigma+1}(\Omega_t)}\, dt \le$$

$$\le \frac{\lambda}{2}\|\vec{u}_0\|^2_{V_\sigma(\Omega_0)} + c_5\lambda \int_0^{\bar{t}} \Big\{ \|\vec{u}(t)\|_{V_{\sigma+1}(\Omega_t)} + \frac{\mu}{4c_4}\|\vec{u}(t)\|^2_{V_{\sigma+1}(\Omega_t)} +$$

$$+ c_7 \|\vec{u}(t)\|^{\frac{2(1+\sigma)}{\sigma}}_{V_\sigma(\Omega_t)} + \frac{\mu}{4c_5}\|\vec{u}(t)\|^2_{V_{\sigma+1}(\Omega_t)} + c_8\|\vec{u}(t)\|^{\frac{4\sigma+1}{2\eta}}_{V_\sigma(\Omega_t)}\Big\}\, dt +$$

$$+ \int_0^{\bar{t}} \Big\{ \frac{\mu}{4}\|\vec{u}(t)\|^2_{V_{\sigma+1}(\Omega_t)} + c_6\|\vec{u}(t)\|^2_{V_\sigma(\Omega_t)}\Big\}\, dt .$$

Hence

$$(4.4) \qquad \frac{1}{2}\|\vec{u}(\bar{t})\|^2_{V_\sigma(\Omega_{\bar{t}})} + \int_0^{\bar{t}} \frac{\mu}{4}\|\vec{u}(t)\|^2_{V_{\sigma+1}(\Omega_t)}\, dt \le \frac{1}{2}\|\vec{u}_0\|^2_{V_\sigma(\Omega_0)} +$$

$$+ \int_0^{\bar{t}} \Big\{ c_5\|\vec{u}(t)\|_{V_{\sigma+1}(\Omega_t)} + c_6\|\vec{u}(t)\|^2_{V_\sigma(\Omega_t)} + c_7\|\vec{u}(t)\|^{\frac{2(1+\sigma)}{\sigma}}_{V_\sigma(\Omega_t)} + c_8\|\vec{u}(t)\|^{\frac{4\sigma+1}{2\eta}}_{V_\sigma(\Omega_t)}\Big\}\, dt .$$

From (4.4) it follows that, if T is sufficiently small, then

$$(4.5) \qquad \|\vec{u}(t)\|_{L^2(0,T;V_{\sigma+1}(\Omega_t))\cap L^\infty(0,T;V_\sigma(\Omega_t))} \le M_3 ,$$

where M_3 does not depend on λ.

Repeating, without any modification, the procedure given in Theorem 2, we can prove that

$$(4.6) \qquad \|\vec{u}'(t)\|_{L^2(0,T;V_{\sigma-1}(\Omega_t))} \le M_4$$

i.e. that

$$\|\vec{u}\|_{W_{T,\sigma}} \le M_1 ,$$

M_1 being independent of λ.

From (4.3) we obtain, finally, for $\lambda = 0$,

$$(4.7) \qquad \frac{1}{2}\|\vec{u}(\bar{t})\|^2_{V_\sigma(\Omega_{\bar{t}})} + \int_0^{\bar{t}} \frac{3}{4}\mu\|\vec{u}(t)\|^2_{V_{\sigma+1}(\Omega_t)}\, dt \le c_6 \int_0^{\bar{t}} \|\vec{u}(t)\|^2_{V_\sigma(\Omega_t)}\, dt .$$

Consequently, $\vec{u}(t) = 0$, which means that the only solution of our problem corresponding to $\lambda = 0$ is the trivial one $\vec{u} = 0$.

The theorem is therefore proved.

5. – *Proof of Theorem* 5. Let $\vec{u}(t), \vec{v}(t)$ be two solutions satisfying the same initial and boundary conditions; then $\vec{w}(t) = \vec{u}(t) - \vec{v}(t) \in W_{T,\sigma}$ and satisfies the equation

$$(5.1) \qquad \int_0^T \{\langle \vec{w}'(t), \vec{h}(t)\rangle + \mu \langle A\vec{w}(t), \vec{h}(t)\rangle +$$

$$+ b(t, \vec{u}(t), \vec{u}(t), \vec{h}(t)) - b(t, \vec{v}(t), \vec{v}(t), \vec{h}(t))\} \, dt =$$

$$= -\int_0^T \Big\{ \sum_{i=1}^2 \int_{\Gamma_i} \frac{1}{2} (v_1^2(x,t) - u_1^2(x,t)) \vec{h}(x,t) \times \vec{\nu}_t \, d\Gamma_i +$$

$$+ \int_{\Gamma_{3,t}} \beta(x,t) \Big((\vec{u}(x,t) \times \vec{\nu}_t) \, |\vec{u}(x,t) \times \vec{\nu}_t| - (\vec{v}(x,t) \times \vec{\nu}_t) \, |\vec{v}(x,t) \times \vec{\nu}_t| \Big) \cdot$$

$$\cdot \vec{h}(x,t) \times \vec{\nu}_t \, d\Gamma_{3,t} \Big\} \, dt$$

$\forall \vec{h}(t) \in L^2(0, T; V_{1-\sigma}(\Omega_t))$, with $\vec{w}(0) = 0$.

Setting $\vec{h}(t) = A^\sigma \vec{w}(t)$, we obtain, in exactly the same way as (3.7),

$$(5.2) \qquad \frac{1}{2} \|\vec{w}(T)\|^2_{V_\sigma(\Omega_T)} + \frac{\mu}{2} \|\vec{w}(t)\|^2_{L^2(0,T;V_{\sigma+1}(\Omega_t))} \leq$$

$$\leq c_{10} \|\vec{w}(t)\|^2_{L^2(0,T;V_\sigma(\Omega_t))} + c_{11} \|\vec{w}(t)\|_{L^2(0,T;V_{\sigma+1}(\Omega_t))} \cdot$$

$$\cdot \Big(\|\vec{w}(t)\|_{H^{(1-\sigma)/2}(0,T;V_\sigma(\Omega_t))} \|\vec{u}(t)\|_{H^{\sigma/2}(0,T;V_1(\Omega_t))} +$$

$$+ \|\vec{w}(t)\|_{L^2(0,T;V_1(\Omega_t))} \|\vec{v}(t)\|_{L^\infty(0,T;V_\sigma(\Omega_t))} \Big) +$$

$$+ \Big(\|\vec{u}(t)\|^{\frac{1+4\sigma-4\varepsilon}{4(1-\varepsilon)}}_{L^\infty(0,T;V_\sigma(\Omega_t))} \|\vec{u}(t)\|^{\frac{3-4\sigma}{4(1-\varepsilon)}}_{L^{\frac{3-4\sigma}{1-\varepsilon}}(0,T;V_{\sigma+1-\varepsilon}(\Omega_t))} + \|\vec{v}(t)\|^{\frac{1+4\sigma-4\varepsilon}{4(1-\varepsilon)}}_{L^\infty(0,T;V_\sigma(\Omega_t))} \cdot$$

$$\cdot \|\vec{v}(t)\|^{\frac{3-4\sigma}{4(1-\varepsilon)}}_{L^{\frac{3-4\sigma}{1-\varepsilon}}(0,T;V_{\sigma+1-\varepsilon}(\Omega_t))} \Big) \|\vec{w}(t)\|^{\frac{1+4\sigma-4\varepsilon}{4(1-\varepsilon)}}_{L^\infty(0,T;V_\sigma(\Omega_t))} \|\vec{w}(t)\|^{\frac{3-4\sigma}{4(1-\varepsilon)}}_{L^{\frac{3-4\sigma}{1-\varepsilon}}(0,T;V_{\sigma+1-\varepsilon}(\Omega_t))}$$

Hence, by (3.8), (3.10), since $\vec{u}(t), \vec{v}(t) \in W_{T,\sigma}$,

$$\frac{1}{2} \|\vec{w}(T)\|^2_{V_\sigma(\Omega_T)} + \frac{\mu}{2} \|\vec{w}(t)\|^2_{L^2(0,T;V_{\sigma+1}(\Omega_t))} \leq$$

$$\leq c_{10} \|\vec{w}(t)\|^2_{L^2(0,T;V_\sigma(\Omega_t))} + c_{12} \|\vec{w}(t)\|_{L^2(0,T;V_{\sigma+1}(\Omega_t))},$$

from which follows that $\vec{w}(t) = 0$.