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On union curves and union curvature of a vector field. (From the standpoint of Cartan's Euclidean connection)

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Geometria differenziale. — *On union curves and union curvature of a vector field. (From the standpoint of Cartan's Euclidean connection).* Nota (*) di BAIJ NATH PRASAD, presentata dal Socio E. BOMPIANI.

RIASSUNTO. — Questo lavoro è dedicato allo studio delle curve d'unione e della curvatura d'unione di un campo vettoriale del sottospazio Finsler localmente euclideo. È stata ricavata l'espressione della curvatura associata del campo vettoriale rispetto alla sua curva d'unione.

1. INTRODUCTION

In the existing literature (Mishra and Singh [3], Singh [6] [7]) attempts have been made at studying the properties of union curve and union curvature in locally Minkowskian Finsler subspaces. Singh [8] has obtained the union curves and pseudogeodesics in Finsler spaces from the standpoint of Cartan's Euclidean connection and showed that the union curves and pseudogeodesics of the locally Euclidean Finsler subspace are identical with the corresponding curves of the locally Minkowskian Finsler subspace, if the element of support is tangential to the curve.

The present paper is devoted to the study of union curves and union curvature of a vector field of the locally Euclidean Finsler subspace. It has been shown that if the vector field is tangential to the curve, the union curve and union curvature are identical with the union curve and union curvature of the locally Minkowskian Finsler subspace.

We shall in the first instance outline some of the basic concepts and fundamental formulae which are mainly due to Davies [1] and Hombu [2].

Consider the subspace, F_m, given by $x^i = x^i(u^\alpha)$ $i = 1, \cdots n$; $\alpha = 1, \cdots, m$ be immersed in an n-dimensional Finsler space F_n.

Suppose that a vector $\dot{x}^i$ (or $\dot{u}^\alpha$), tangent to the subspace, is the element of support. We may write

$$\dot{x}^i = B^i_\alpha \dot{u}^\alpha \tag{1.1}$$

where
$$B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}.$$

The metric tensors g_{ij} and $g_{\alpha\beta}$ of F_n and F_m are related by

$$g_{\alpha\beta}(u, \dot{u}) = g_{ij}(x, \dot{x}) B^i_\alpha B^j_\beta . \tag{1.2}$$

There exist $(n - m)$ vectors $N^i_{(\mu)}(x, \dot{x})$ $(\mu = m + 1, \cdots n)$ called the normal vectors which satisfy the following conditions.

$$g_{ij}(x, \dot{x}) N^j_{(\mu)} B^i_\alpha = N_{(\mu)i} B^i_\alpha = 0 \tag{1.3}$$

$$g_{ij}(x, \dot{x}) N^i_{(\mu)} N^j_{(\nu)} = \delta_{(\mu\nu)} . \tag{1.4}$$

(*) Pervenuta all'Accademia il 12 ottobre 1971.

Suppose that dx^i is a displacement vector and $V^i(x, \dot{x})$ a vector field at a point P of the subspace. Both dx^i and $V^i(x, \dot{x})$ are tangent to the subspace and we have

$$dx^i = B^i_\alpha \, du^\alpha \tag{1.5}$$

$$V^i = B^i_\alpha \, V^\alpha . \tag{1.6}$$

It is further assumed that the vector field V^i is normalised by the condition

$$g_{\alpha\beta}(u, \dot{u}) \, V^\alpha V^\beta = 1 . \tag{1.7}$$

Let the Cartan differential of the vector V^i, for the displacement dx^i and for the element of support $\dot{x}^i$ be DV^i. The induced differential $\bar{D}V^\alpha$ is defined by

$$\bar{D}V^\alpha = B^\alpha_i \, DV^i$$

where

$$B^\alpha_i = g^{\alpha\varepsilon}(u, \dot{u}) \, g_{ij}(x, \dot{x}) \, B^j_\varepsilon .$$

After writing

$$\bar{D}V^\alpha = dV^\alpha + C^\alpha_{\beta\gamma}(u, \dot{u}) \, V^\beta \, d\dot{u}^\gamma + \Gamma^\alpha_{\beta\gamma}(u, \dot{u}) \, V^\beta \, du^\gamma \tag{1.8}$$

we find that

$$C_{\alpha\beta\gamma} = g_{\beta\varepsilon} \, C^\varepsilon_{\alpha\gamma} = C_{ijk} \, B^i_\alpha \, B^j_\beta \, B^k_\gamma \tag{1.9}$$

and

$$\Gamma^\alpha_{\beta\gamma}(u, \dot{u}) = B^\alpha_i \, (B^i_{\beta\gamma} + C^i_{kh} \, B^k_{\varepsilon\gamma} \, \dot{u}^\varepsilon \, B^h_\beta + \Gamma^i_{hk} \, B^h_\beta \, B^k_\gamma) . \tag{1.10}$$

$\Gamma^\alpha_{\beta\gamma}(u, \dot{u})$ are called induced connection parameters and they satisfy the relations

$$\Gamma^\alpha_{\beta\gamma}(u, \dot{u}) \, \dot{u}^\beta = \Gamma^{*\alpha}_{\beta\gamma}(u, \dot{u}) \, \dot{u}^\beta \tag{1.11}$$

where

$$\Gamma^{*\alpha}_{\beta\gamma} = B^\alpha_i \, (B^i_{\beta\gamma} + \Gamma^{*i}_{hk} \, B^h_\beta \, B^k_\gamma) \tag{1.12}$$

and Γ^{*i}_{hk} are the connection parameters of the enveloping space F_n.

We quote the following results (refer Rund [5] pages 164–166) for reference in the later sections of this paper.

$$Dl^i = B^i_\alpha \, \bar{D}l^\alpha + H^i_\gamma(u, \dot{u}) \, du^\gamma , \tag{1.13}$$

$$DV^i = B^i_\alpha \, \bar{D}V^\alpha + H^i_{\gamma\beta} \, V^\beta \, du^\gamma + N^i_j \, FC^j_{hk} \, B^h_\beta \, B^k_\gamma \, V^\beta \, \bar{D}l^\gamma \tag{1.14}$$

where

$$\text{(1.15)}\qquad F H^i_\gamma = (B^i_{\beta\gamma} - B^i_\alpha \Gamma^\alpha_{\beta\gamma})\, \dot{u}^\beta + B^k_\gamma \Gamma^i_{hk} \dot{x}^h ,$$

$$\text{(1.16)}\qquad H^i_{\gamma\beta} = B^i_{\beta\gamma} - B^i_\alpha \bar{\Gamma}^{*\alpha}_{\beta\gamma} + \Gamma^{*i}_{hk} B^h_\beta B^k_\gamma + F C^i_{hk} B^h_\beta H^k_\gamma ,$$

$$\text{(1.17)}\qquad \bar{\Gamma}^{*\alpha}_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - C^\alpha_{\beta\delta} \Gamma^\delta_{\varepsilon\gamma} \dot{u}^\varepsilon ,$$

$$\text{(1.18)}\qquad N^i_j = \sum_\mu N^i_{(\mu)} N_{(\mu)j}$$

and

$$\text{(1.19)}\qquad F l^i = \dot{x}^i \quad , \quad F l^\alpha = \dot{u}^\alpha .$$

2. UNION CURVE OF A VECTOR FIELD

Let $C : u^\alpha = u^\alpha(s)$ be a curve of the subspace F_m and let the element of support be taken tangent to the curve C so that the components $l^i = \frac{dx^i}{ds} = \dot{x}^i$ and $l^\alpha = \frac{du^\alpha}{ds} = \dot{u}^\alpha$ are related by (1.1). Further $F(x, \dot{x}) = 1$.

Since $H^i_{\gamma\beta}$ is normal to the subspace (Rund [5] page 168) we may write

$$\text{(2.1)}\qquad H^i_{\gamma\beta} = \sum_\mu H_{(\mu)\gamma\beta} N^i_{(\mu)} .$$

Considering the displacement du^γ in (1.14) along C and using (1.18) and (2.1) we get

$$\text{(2.2)}\qquad \frac{DV^i}{Ds} = B^i_\alpha \frac{\bar{D}V^\alpha}{Ds} + \sum_\mu \left(H_{(\mu)\gamma\beta} \frac{du^\gamma}{ds} + M_{(\mu)\gamma\beta} \frac{\bar{D}l^\gamma}{Ds} \right) V^\beta N^i_{(\mu)}$$

where

$$\text{(2.3)}\qquad M_{(\mu)\gamma\beta}(u, \dot{u}) = C_{ijk}(x, \dot{x})\, B^i_\gamma B^j_\beta N^k_{(\mu)} .$$

The normal curvature at P of the vector field V^i with respect to the curve C is defined by (Nagata [4])

$$\text{(2.4)}\qquad {}_v K^2_N = \sum_\mu \left(H_{(\mu)\gamma\beta} \frac{du^\gamma}{ds} + M_{(\mu)\gamma\beta} \frac{\bar{D}l^\gamma}{Ds} \right) \left(H_{(\mu)\alpha\delta} \frac{du^\alpha}{ds} + M_{(\mu)\alpha\delta} \frac{\bar{D}l^\alpha}{Ds} \right) V^\beta V^\delta .$$

Consider a set of $(n - m)$ congruences of curves (in F_n) given by the vector fields $\lambda^i_{(\mu)}$ $(\mu = m + 1, \cdots, n)$. At a point of the subspace we may write

$$\text{(2.5)}\qquad \lambda^i_{(\mu)}(x, \dot{x}) = t^\alpha_{(\mu)}(u, \dot{u})\, B^i_\alpha + \sum_\nu C_{(\mu\nu)}(x, \dot{x})\, N^i_{(\nu)}(x, \dot{x}) .$$

These vectors are normalised by the conditions

$$\text{(2.6)}\qquad g_{ij}(x, \dot{x})\, \lambda^i_{(\mu)} \lambda^j_{(\mu)} = 1 .$$

DEFINITION (2.1). The curve C is said to be a union curve of the vector field V^i relative to the congruence $\lambda^i_{(\mu)}$ if the geodesic surface determined by V^i and DV^i/DS contains the vector $\lambda^i_{(\mu)}$.

THEOREM (2.1). *The union curve of a vector field* V^i *relative to* $\lambda^i_{(\mu)}$ *is given by the equations*

$$\frac{\bar{D}V^\alpha}{Ds} = \frac{\left(H_{(\nu)\gamma\beta}\frac{du^\gamma}{ds} + M_{(\nu)\gamma\beta}\frac{\bar{D}l^\gamma}{Ds}\right)V^\beta}{C_{(\mu\nu)}}\left(t^\alpha_{(\mu)} - T_{(\mu)}\cos\alpha_{(\mu)}V^\alpha\right) \tag{2.7}$$

where the ratio

$$\frac{\left(H_{(\nu)\gamma\beta}\frac{du^\gamma}{ds} + M_{(\nu)\gamma\beta}\frac{\bar{D}l^\gamma}{Ds}\right)V^\beta}{C_{(\mu\nu)}} \tag{2.8}$$

is independent of ν *and*

$$T_{(\mu)} = \{g_{\alpha\beta}(u, \dot{u})\, t^\alpha_{(\mu)}\, t^\beta_{(\mu)}\}^{1/2}, \tag{2.9}$$

$$T_{(\mu)}\cos\alpha_{(\mu)} = g_{\alpha\beta}(u, \dot{u})\, t^\alpha_{(\mu)}V^\beta. \tag{2.10}$$

Proof. For the union curve of V^i we have

$$\lambda^i_{(\mu)} = A_{(\mu)}V^i + B_{(\mu)}\frac{DV^i}{Ds}. \tag{2.11}$$

Substituting from (2.2) and (2.5) and using the equation (1.6) we get

$$t^\alpha_{(\mu)} = A_{(\mu)}V^\alpha + B_{(\mu)}\frac{\bar{D}V^\alpha}{Ds} \tag{2.12}$$

and

$$C_{(\mu\nu)} = B_{(\mu)}\left(H_{(\nu)\gamma\beta}\frac{du^\gamma}{ds} + M_{(\nu)\gamma\beta}\frac{\bar{D}l^\gamma}{Ds}\right)V^\beta. \tag{2.13}$$

The covariant differentiation of the normalising condition (1.7) will yield

$$g_{\alpha\beta}(u, \dot{u})\frac{\bar{D}V^\alpha}{Ds}V^\beta = 0. \tag{2.14}$$

The equation (2.7) and the condition (2.8) follow immediately from (2.12), (2.13) and (2.14).

3. UNION CURVATURE OF A VECTOR FIELD

The vector $\eta^\alpha_{(\mu)}(u, \dot{u})$ defined by

$$\eta^\alpha_{(\mu)} = \frac{\bar{D}V^\alpha}{Ds} - \frac{\left[\sum_\nu\left\{\left(H_{(\nu)\gamma\beta}\frac{du^\gamma}{ds} + M_{(\nu)\gamma\beta}\frac{\bar{D}l^\gamma}{Ds}\right)V^\beta\right\}^2\right]^{1/2}}{\left\{\sum_\nu C^2_{(\mu\nu)}\right\}^{1/2}}\cdot\left(t^\alpha_{(\mu)} - T_{(\mu)}\cos\alpha_{(\mu)}V^\alpha\right) \tag{3.1}$$

is called the union curvature vector (relative to $\lambda^i_{(\mu)}$) of the vector field V^i with respect to the curve $u^\alpha = u^\alpha(s)$. The scalar ${}_v K_{(\mu)}(u, \dot{u})$ defined by

$$(3.2) \qquad {}_v K^2_{(\mu)} = g_{\alpha\beta}(u, \dot{u})\, \eta^\alpha_{(\mu)}\, \eta^\beta_{(\mu)}$$

is called the union curvature of the vector field.

It is obvious from equations (2.7), (3.1) and the condition (2.8) that the union curvature of a vector field with respect to its union curve is zero. From (2.2) $\frac{DV^i}{Ds} = 0$ implies that $\frac{\overline{D}V^\alpha}{Ds} = 0$ and

$$H_{(\nu)\gamma\beta} \frac{du^\gamma}{ds} + M_{(\nu)\gamma\beta} \frac{\overline{D} l^\gamma}{Ds} = 0 \qquad \text{for} \quad \nu = m+1, \cdots, n.$$

Hence by (3.1) $\eta^\alpha_{(\mu)} = 0$. We have, therefore, the following proposition.

THEOREM (3.1). *When the vector field is parallel, in the enveloping space, along a curve* C, *then* C *is a union curve of the vector field relative to any congruence.*

A direct calculation based on the equations (2.4), (3.1), (3.2), (2.9) and (2.10) will yield the following.

THEOREM (3.2). *The union curvature of a vector field* V^i *is given by*

$$(3.3) \qquad {}_v K^2_{(\mu)} = K^2 - 2K T_{(\mu)}\, {}_v K_N \cos\beta_{(\mu)} + {}_v K^2_N T^2_{(\mu)} \sin^2 \alpha_{(\mu)}$$

where

$$(3.4) \qquad K^2 = g_{\alpha\beta}(u, \dot{u}) \frac{\overline{D}V^\alpha}{Ds} \frac{\overline{D}V^\beta}{Ds}$$

$$(3.5) \qquad T_{(\mu)} K \cos\beta_{(\mu)} = g_{\alpha\beta}(u, \dot{u})\, t^\alpha_{(\mu)} \frac{\overline{D}V^\beta}{Ds}$$

and

$$(3.6) \qquad \sin^2 \alpha_{(\mu)} = 1 - \cos^2 \alpha_{(\mu)}.$$

The following corollary is an immediate consequence of the above Theorem.

COROLLARY (3.1). *A necessary and sufficient condition that the union curvature of a vector field be expressed in the form*

$$(3.7) \qquad {}_v K_{(\mu)} = K - {}_v K_N T_{(\mu)} \cos\beta_{(\mu)}$$

is that

$$(3.8) \qquad \cos\beta_{(\mu)} = \pm \sin\alpha_{(\mu)}.$$

COROLLARY (3.2). *The associate curvature of the vector field with respect to its union curve is given by*

$$(3.9) \qquad K = {}_v K_N T_{(\mu)} \cos\beta_{(\mu)}.$$

Proof. For the union curve of V^i, we have from (2.12), (2.14), (2.9), (3.4) and (1.7)

$$T^2_{(\mu)} = A^2_{(\mu)} + B^2_{(\mu)} K^2 . \tag{3.10}$$

Multiplying (2.12) with $g_{\alpha\beta}(u, \dot{u}) V^\beta$ and $g_{\alpha\beta}(u, \dot{u}) \frac{\bar{D}V^\beta}{Ds}$ respectively and using (1.7), (2.14), (3.4), (3.5) and (2.10) we get

$$T_{(\mu)} \cos \alpha_{(\mu)} = A_{(\mu)} \quad \text{and} \quad T_{(\mu)} \cos \beta_{(\mu)} = KB_{(\mu)}$$

from which we have

$$T^2_{(\mu)} (\cos^2 \alpha_{(\mu)} + \cos^2 \beta_{(\mu)}) = A^2_{(\mu)} + B^2_{(\mu)} K^2 . \tag{3.11}$$

The equations (3.10) and (3.11) reveal that the condition (3.8) is satisfied for a union curve of vector field. Corollary (3.2) will, therefore, follow from equation (3.7) and the fact that the union curvature of a vector field with respect to its union curve is zero.

4. A particular case

Let the vector field V^i is tangent to the curve C, i.e., $\dot{x}^i = \frac{dx^i}{ds} = V^i$. From the equations (1.16) and (1.17) we have

$$H^i_{\gamma\beta} \frac{du^\gamma}{ds} \frac{du^\beta}{ds} = I^i_{\gamma\beta} \frac{du^\gamma}{ds} \frac{du^\beta}{ds} \tag{4.1}$$

where

$$I^i_{\gamma\beta} = B^i_{\beta\gamma} - B^i_\alpha \Gamma^\alpha_{\beta\gamma} + \Gamma^{*i}_{hk} B^h_\beta B^k_\gamma . \tag{4.2}$$

Since $I^i_{\gamma\beta}$ is also normal to the subspace we may write

$$I^i_{\gamma\beta} = \sum_\mu \bar{\Omega}_{(\mu)\gamma\beta} N^i_{(\mu)} . \tag{4.3}$$

The equations (2.1), (4.1) and (4.3) will yield

$$H_{(\nu)\gamma\beta}(u, \dot{u}) \frac{du^\gamma}{ds} \frac{du^\beta}{ds} = \bar{\Omega}_{(\nu)\gamma\beta}(u, \dot{u}) \frac{du^\gamma}{ds} \frac{du^\beta}{ds} . \tag{4.4}$$

In this particular case, the expressions for $\bar{D}V^\alpha$ and induced δ–derivative, δV^α, defined by (1.8) and

$$\delta V^\alpha = dV^\alpha + \Gamma^{*\alpha}_{\beta\gamma} V^\beta du^\gamma \tag{4.5}$$

respectively give

$$\frac{\bar{D}\dot{u}^\alpha}{Ds} = \frac{\delta \dot{u}^\alpha}{\delta s} \tag{4.6}$$

where we have used the relation (1.1). Using the equations (4.4), (4.6) and the fact that $M_{(\nu)\beta\gamma} \frac{du^\gamma}{ds} = 0$ we observe that (2.7) reduces to the form

$$\frac{\delta \dot{u}^\alpha}{\delta s} = \frac{\overline{\Omega}_{(\nu)\beta\gamma}(u, \dot{u})\, \dot{u}^\beta \dot{u}^\gamma}{C_{(\mu\nu)}} \left(t^\alpha_{(\mu)} - T_{(\mu)} \cos \theta_{(\mu)}\, \dot{u}^\alpha\right) \tag{4.7}$$

where

$$\frac{\overline{\Omega}_{(\nu)\beta\gamma}(u, \dot{u})\, \dot{u}^\beta \dot{u}^\gamma}{C_{(\mu\nu)}} \tag{4.8}$$

is independent of ν and

$$T_{(\mu)} \cos \theta_{(\mu)} = g_{\alpha\beta}(u, \dot{u})\, t^\alpha_{(\mu)} \frac{du^\beta}{ds} . \tag{4.9}$$

These represent the union curve of the locally Minkowskian Finsler spaces (Mishra and Singh [3]).

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