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Geometria differenziale. — *Hyperasymptotic and hypernormal congruences in a subspace of a Finsler space.* Nota (*) di CHANDRA MANI PRASAD, presentata dal Socio E. BOMPIANI.

RIASSUNTO. — Sono state definite da Mishra [2, 3] le curve ipersintotiche ed ipernormali di una ipersuperficie riemanniana, e da Singh [5, 6] quella di un sottospazio finsleriano. In questo lavoro, queste curve sono state generalizzate ulteriormente in modo di dare le congruenze ipersintotiche ed ipernormali del sottospazio finsleriano. Inoltre viene ricavata una condizione necessaria e sufficiente perché queste siano congruenze d'unione secondaria [7].

1. INTRODUCTION

Let a subspace F_m, $x^i = x^i(u^\alpha)$ $i = 1, 2, \cdots n$, $\alpha = 1, 2, \cdots, m$ be immersed in an n-dimensional Finsler space F_n. Consider a curve $C : x^i = x^i(s)$ of the subspace. The components $x'^i = dx^i/ds$ and $u'^\alpha = du^\alpha/ds$ of the unit tangent vector to C are related by $x'^i = B^i_\alpha u'^\alpha$, where $B^i_\alpha = \partial x^i/\partial u^\alpha$. A line-element (u^α, u'^α) is thus determined at a point of C. All the quantities in our discussion are considered for this line-element.

The metric tensors $g_{\alpha\beta}(u, u')$ and $g_{ij}(x, x')$ of F_m and F_n respectively are related by

$$g_{\alpha\beta}(u, u') = g_{ij}(x, x')\, B^i_\alpha B^j_\beta . \tag{1.1}$$

There exists a set of vectors $\overset{*}{n}{}^i_{(\sigma)}(x, x')$, $\sigma = m+1, \cdots, n$, normal to the subspace and are called secondary normals. These are given by the solutions [4]

$$\overset{*}{n}_{(\sigma)j}\, B^j_\alpha = g_{ij}(x, x')\, \overset{*}{n}{}^i_{(\sigma)}\, B^j_\alpha = 0 \tag{1.2}$$

$$g_{ij}(x, x')\, \overset{*}{n}{}^i_{(\sigma)}\, \overset{*}{n}{}^j_{(\nu)} = \delta^\nu_\sigma\, \psi_{(\nu)}, \qquad \text{[no summation on } \nu] \tag{1.3}$$

and

$$g_{ij}(x, \overset{*}{n}_{(\sigma)})\, \overset{*}{n}{}^i_{(\sigma)}\, \overset{*}{n}{}^j_{(\sigma)} = 1 . \tag{1.4}$$

Let a set of $n - m$ linearly independent vectors $\mu^i_{(\sigma)}(x, x')$, $\sigma = m+1, \cdots, n$ define $n - m$ congruences of curves which are such that exactly one curve of each congruence passes through each point of the space. At a point P of the subspace, we write

$$\mu^i_{(\sigma)} = l^\alpha_{(\sigma)}(u, u')\, B^i_\alpha + \sum_\nu \Gamma_{(\sigma\nu)}(u, u')\, \overset{*}{n}{}^i_{(\nu)} . \tag{1.5}$$

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Suppose that the vectors $\mu^i_{(\sigma)}$ with m linearly independent vectors of F_m form a set of n linearly independent vectors in F_n which is possible when $|\Gamma_{(\sigma\nu)}| \neq 0$.

Consider the contravariant components $\lambda^i(x, x')$ of a congruence of curves which is not necessarily a member of the set of congruences defined by (1.5). At a point of the subspace, it may be expressed as

$$\lambda^i = t^\alpha B^i_\alpha + \sum_\nu C_{(\nu)} \overset{*}{n}{}^i_{(\nu)} \tag{1.6}$$

and it satisfies

$$g_{ij}(x, x') \lambda^i \lambda^j = \lambda_j \lambda^j = 1 . \tag{1.7}$$

The covariant derivative of (1.6) with respect to u^β in the direction of C is given by [7]

$$\frac{\delta\lambda^i}{\delta s} = W^\alpha B^i_\alpha + \sum_\nu D_{(\nu)} \overset{*}{n}{}^i_{(\nu)} \tag{1.8}$$

where

$$W^\alpha = \frac{\delta t^\alpha}{\delta s} + \sum_\nu C_{(\nu)} \overset{*}{A}{}^\alpha_{(\nu)\beta} u'^\beta \tag{1.9}$$

and

$$D_{(\nu)} = \overset{*}{\Omega}_{\alpha\beta}(u, u') t^\alpha u'^\beta + \frac{\delta C_{(\nu)}}{\delta s} + \sum_\sigma C_{(\sigma)} \overset{*}{N}{}^{(\sigma)}_{(\nu)\beta} u'^\beta . \tag{1.10}$$

The quantities $\overset{*}{\Omega}_{\alpha\beta}$ are called secondary second fundamental tensors and $\overset{*}{A}{}^\alpha_{(\nu)\beta}$ and $\overset{*}{N}_{(\nu)\beta}$ are defined in [7].

In the following sections, the hyperasymptotic and hypernormal congruences are introduced in F_m and their properties discussed.

2. HYPERASYMPTOTIC CONGRUENCES

In a Riemannian space V_n, the hyperasymptotic curves were defined as follows.

Let dx^i/ds, q^i and $b^i_{(\varphi)}$, $\varphi = 1, 2, \cdots, (n-2)$, be the tangent vector, the principal normal vector and $(n-2)$ binormal vectors (with respect to V_n) of a curve C of the subspace V_m.

DEFINITION (1). A curve of V_m is a hyperasymptotic curve with respect to the binormals $b^i_{(\varphi)}$ and relative to $\mu^i_{(\sigma)}(x)$ if the geodesic surface determined by dx^i/ds and $b^i_{(\varphi)}$ contains $\mu^i_{(\sigma)}$.

DEFINITION (2). The scalar

$$K_{(\sigma)} = g_{ij}(x) \mu^i_{(\sigma)} q^j$$

is called the hyperasymptotic curvature (of the curve) relative to $\mu^i_{(\sigma)}(x)$. If $K_{(\sigma)} = 0$ along a curve, it is a hyperasymptotic curve of V_m.

The first definition is due to Mishra [3] and the latter is due to Amur [1]. With the use of δ-differentiation [4], we are unable to ascertain the existence of $n-2$ binormals in the Finsler subspace. However, Singh [6] has introduced the existence of $n-2$ vectors in Finsler space which obey the conditions satisfied by the $n-2$ binormals of the Riemannian space and thereby defined the hyperasymptotic curves in the Finsler space.

The latter definition may be used to have a direct generalization of hyperasymptotic curve in Finsler space and, therefore, the hyperasymptotic congruence in Finsler subspace is introduced. It may be noted that the latter definition is a consequence of the former one.

DEFINITION (2.1). The scalar $K_{(\sigma)h}$ defined by

$$K_{(\sigma)h} \overset{\text{def}}{=} g_{ij}(x, x') \, \mu^i_{(\sigma)} \frac{\delta\lambda^j}{\delta s} \tag{2.1}$$

is called the hyperasymptotic curvature of the congruence λ^i on F_m. A congruence λ^i is said to be a hyperasymptotic congruence relative to the congruences $\mu^i_{(\sigma)}$ if at each point of a curve of F_m, $K_{(\sigma)h} = 0$. Its differential equation, therefore, is given by (simplifying (2.1) and equating to zero),

$$g_{\alpha\beta} \, l^\alpha_{(\sigma)} W^\beta + \sum_\nu \Gamma_{(\sigma\nu)} D_{(\nu)} \psi_{(\nu)} = 0 . \tag{2.2}$$

Particular Cases.

(*i*) If the components of the vector λ^i are tangential to the subspace, we have

$$\begin{aligned} C_{(\nu)} &= 0 \quad , \quad W^\alpha = \frac{\delta t^\alpha}{\delta s} \\ D_{(\nu)} &= \Omega^*_{\beta\gamma} \frac{du^\beta}{ds} \cdot t^\gamma \end{aligned} \tag{2.3}$$

and, therefore, $\lambda^i = B^i_\alpha t^\alpha$ and the equation (2.2) takes the form

$$g_{\alpha\beta} \, l^\alpha_{(\sigma)} \frac{\delta t^\beta}{\delta s} + \sum_\nu \Gamma_{(\sigma\nu)} \Omega^*_{(\nu)\alpha\beta} \frac{du^\alpha}{ds} t^\beta \psi_{(\nu)} = 0 . \tag{2.4}$$

This is the equation of a hyperasymptotic (or hyperconjugate) curve of the vector-field λ^i tangential to the subspace (relative to $\mu^i_{(\sigma)}$).

(*ii*) If the components of the congruence λ^i are tangential to a curve $C : x^i = x^i(s)$ of the subspace, we have $\lambda^i = x'^i$ and therefore, the equation (2.2) reduces to

$$g_{\alpha\beta} \, l^\alpha_{(\sigma)} u'^\beta + \sum_\nu \Gamma_{(\sigma\nu)} \Omega^*_{(\nu)\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} \psi_{(\nu)} = 0 . \tag{2.5}$$

The equation (2.5) is the differential equation of the hyperasymptotic curves of the subspace F_m relative to $\mu^i_{(\sigma)}$ [6].

THEOREM (2.1). *If the congruences $\mu^i_{(\sigma)}$ are not normal to the subspace, the congruence λ^i is a hyperasymptotic congruence relative to $\mu^i_{(\sigma)}$ at the points of a λ–geodesic of F_m.*

Proof. Since $\Gamma_{(\sigma\nu)} = 0$ and $l^\alpha_{(\sigma)} \neq 0$ for all ν, the equation (2.2) reduces to

$$g_{\alpha\beta}\, l^\alpha_{(\sigma)}\, W^\beta = 0$$

which is identically satisfied for $W^\alpha = 0$ (λ–geodesic) [7].

The equation (2.2) may be written as

$$K_{(\sigma)h} = g_{\alpha\beta}\, l^\alpha_{(\sigma)}\, W^\beta + \sum_\nu \Gamma_{(\sigma\nu)}\, M_{(\nu)} \tag{2.6}$$

where

$$M_{(\nu)} = + D_{(\nu)}\, \psi_{(\nu)}\,. \tag{2.7}$$

The quantity $M_{(\nu)}$ is called the hyperasymptotic curvature of λ^i relative to the secondary normals $\overset{*}{n}{}^i_{(\nu)}$. Now the congruence λ^i is said to be a hyperasymptotic congruence relative to the vectors $\overset{*}{n}{}^i_{(\nu)}$ if

$$M_{(\nu)} = 0\,.$$

Consider a variety V generated by the secondary normals $\overset{*}{n}{}^i_{(\nu)}$ and suppose that all the $n - m$ congruences $\mu^i_{(\sigma)}$ lie in this variety. We, therefore, have $l^\alpha_{(\sigma)} = 0$ for all σ and the equation (2.2) reduces to

$$K_{(\sigma)h} = \sum_\nu \Gamma_{(\sigma\nu)}\, M_{(\nu)}\,. \tag{2.8}$$

Now we shall prove the following

THEOREM (2.2). *A necessary and sufficient condition that the congruence λ^i is a hyperasymptotic congruence relative to all $n - m$ congruences $\mu^i_{(\sigma)}$ of the variety V is that it is the hyperasymptotic congruence relative to each secondary normal.*

Proof. If λ^i is hyperasymptotic relative to each secondary normal, we have

$$M_{(\nu)} = 0 \qquad \text{for all } \nu$$

and therefore,

$$K_{(\sigma)h} = 0 \qquad \text{for} \quad \sigma = m+1, \cdots, n\,.$$

Conversely, if $K_{(\sigma)h} = 0$ for every σ, the equation (2.8) give

$$\sum_\nu \Gamma_{(\sigma\nu)}\, M_{(\nu)} = 0\,.$$

As the vectors $\mu^i_{(\sigma)}$ are linearly independent, we have $|\Gamma_{(\sigma\nu)}| \neq 0$ and, therefore,

$$M_{(\nu)} = 0 .$$

COROLLARY (2.1). *If* λ^i *is a hyperasymptotic congruence relative to all* $n - m$ *congruences* $\mu^i_{(\sigma)}$ *of the variety* V, *then it is a hyperasymptotic congruence relative to any other congruence of the variety* V.

Proof. The proof follows from theorem (2.2) and equation (2.8).

THEOREM (2.3). *A necessary and sufficient condition that* λ^i *be a hyperasymptotic congruence relative to all* $\mu^i_{(\sigma)}$ *of the variety* V *at the points of a curve* C *of* F_m *is that the curve be an asymptotic line of the congruence* λ^i.

Proof. If λ^i is a hyperasymptotic congruence relative to $\mu^i_{(\sigma)}$ of the variety V, we have $K_{(\sigma)h} = 0$ and therefore,

$$\sum_{\nu} \Gamma_{(\sigma\nu)} D_{(\nu)} \psi_{(\nu)} = 0 \qquad \text{for} \quad \sigma = m + 1 , \cdots , n. \tag{2.9}$$

Since $\mu^i_{(\sigma)}$ are linearly independent, we have $|\Gamma_{(\sigma\nu)}| \neq 0$. The equation (2.9) then gives (since $\psi_{(\nu)} \neq 0$)

$$D_{(\nu)} = 0 \qquad \text{for all} \quad \nu , \tag{2.10}$$

which gives the asymptotic line of the congruence λ^i [8].

Conversely, if (2.10) are given, the equations (2.9) are identically satisfied.

3. HYPERNORMAL CONGRUENCES

Mishra [2] defined the hypernormal curve in a Riemannian hypersurface. A consequence of which has been adopted by Amur [1] where he defined that a curve of V_n is a hypernormal curve of a vector-field v if the congruence $\lambda^i(x)$ in V_{n+1} through a point P of V_n is orthogonal to the direction of vector-field at that point.

In view of the above definition, we introduce the concept of hypernormal congruence of a Finsler subspace.

DEFINITION (3.1). If a scalar $H_{(\sigma)}$ defined by

$$H_{(\sigma)} = g_{ij}(x , x') \mu^i_{(\sigma)} \lambda^j \tag{3.1}$$

vanishes identically at the points of a curve on F_m, the congruence λ^i is said to be a hypernormal congruence. Hence the equation of the hypernormal congruence is given by

$$g_{\alpha\beta}(u , u') t^{\alpha} l^{\beta}_{(\sigma)} + \sum_{\nu} C_{(\nu)} \Gamma_{(\sigma\nu)} \psi_{(\nu)} = 0 . \tag{3.2}$$

When we consider any general congruence of curves $\mu^i_{(\sigma)}$, the relation (3.2) is not satisfied at all points of F_m. However, there may exist certain curves on F_m at each point of which the relation is satisfied.

Particular Cases.

(*i*) Let the components of λ^i be tangential to the subspace, then we have $t^\alpha \neq 0$ and $C_{(\nu)} = 0$ and the equation (3.2) reduces to

$$g_{\alpha\beta}\, t^\alpha\, l^\beta_{(\sigma)} = 0 . \tag{3.3}$$

The equation (3.3) may identically be satisfied at each point of certain curves on F_m. These curves are called the hypernormal curves of the vector-field λ^i tangential to the subspace.

(*ii*) Again if $\lambda^i = dx^i/ds$, the equation (3.2) takes the form

$$g_{\alpha\beta}(u, u') \frac{du^\alpha}{ds}\, l^\beta_{(\sigma)} = 0 . \tag{3.4}$$

The equation (3.4) is the equation of a hypernormal curve of the subspace F_m relative to $\mu^i_{(\sigma)}$ defined by Singh [6]. Thus the equation (3.2) is a generalization of the hypernormal curve of the subspace F_m.

THEOREM (3.1). *If the vectors* $\mu^i_{(\sigma)}$, $\sigma = m + 1, \cdots, n$ *lie in a variety spanned by the secondary normals and the components of* λ^i *are tangential to* F_m, *the congruence* λ^i *is the hypernormal congruence relative to all* $n - m$ *congruences.*

Proof. Since $l^\alpha_{(\sigma)} = 0$ for all σ and $C_{(\nu)} = 0$ for $\nu = m + 1, \cdots, n$, the equation (3.2) is identically satisfied for all $l^\alpha_{(\sigma)}$ and $\sigma = m + 1, \cdots, n$.

4. UNION, HYPERASYMPTOTIC AND HYPERNORMAL CONGRUENCES

In this section, we shall consider the case when the hyperasymptotic and hypernormal congruences coincide with the union congruence.

In the Theorems given below we shall be using the following conditions:

(*i*) Let the vector λ^i be not parallel (along the curve C) in the enveloping space F_n and

(*ii*) that the projection of the first curvature vector of λ^i vanishes in its own direction.

THEOREM (4.1). *A necessary and sufficient condition that the union congruence relative to* $\mu^i_{(\sigma)}$ *is the hyperasymptotic congruence (relative to the same congruences) is that* $\mu^i_{(\sigma)}$ *lie in the direction of* λ^i.

Proof. The union congruence relative to $\mu^i_{(\sigma)}$ is given by [7],

$$\mu^i_{(\sigma)} = a_{(\sigma)}\, \lambda^i + b_{(\sigma)} \frac{\delta \lambda^i}{\delta s} . \tag{4.1}$$

Since $\delta\lambda^i/\delta s \neq 0$ and $(\delta\lambda^i/\delta s)\cdot\lambda_i = 0$, multiplying (4.1) by $g_{ij}(x, x')\,\delta\lambda^j/\delta s$, it reduces to

$$g_{ij}(x, x')\,\mu^i_{(\sigma)}\,\frac{\delta\lambda^j}{\delta s} = b_{(\sigma)}\,g_{ij}\left(\frac{\delta\lambda^i}{\delta s}\right)\left(\frac{\delta\lambda^j}{\delta s}\right). \tag{4.2}$$

From (4.2) and (2.1), it follows that $K_{(\sigma)h} = 0$ if and only if $b_{(\sigma)} = 0$ which completes the proof.

THEOREM (4.2). *A necessary and sufficient condition that the union congruence relative to* $\mu^i_{(\sigma)}$ *is the hypernormal congruence (relative to the same congruences) is that* $\mu^i_{(\sigma)}$ *lie along the first curvature vector of* λ^i *(with respect to* F_n*).*

Proof. The equation of the union congruence relative to $\mu^i_{(\sigma)}$ is given by the equation (4.1). This equation gives

$$g_{ij}(x, x')\,\mu^i_{(\sigma)}\,\lambda^j = a_{(\sigma)} \tag{4.3}$$

since $(\delta\lambda^i/\delta s)\,\lambda_i = 0$ by assumption. Now $H_{(\sigma)} = g_{ij}(x, x')\,\mu^i_{(\sigma)}\,\lambda^j = 0$ implies that $a_{(\sigma)} = 0$ so that $\mu^i_{(\sigma)}$ lies along $\delta\lambda^i/\delta s$.

Conversely, if $\mu^i_{(\sigma)}$ lies along $\delta\lambda^i/\delta s$, it implies that $a_{(\sigma)} = 0$ which from (4.3) implies that $H_{(\sigma)} = 0$. This proves the Theorem.

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