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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 13 novembre 1971

Presiede il Presidente BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Properties of two cardinal topological invariants.* Nota di OFELIA TERESA ALAS, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si stabiliscono proprietà di due invarianti topologici riferiti, rispettivamente, alla intersezione di una collezione d'insiemi aperti ed ai ricoprimenti aperti localmente finiti di uno spazio.

We shall consider some properties of two cardinal topological invariants related, respectively, with the intersection of a collection of open sets and with the locally finite open coverings of a space.

In this Note all topological spaces are nonempty Hausdorff spaces. For every set Z, $|Z|$ denotes the cardinal number of Z.

Let X be a topological space.

DEFINITION 1. $p(X)$ is the least cardinal $p \geq \aleph_0$ such that every locally finite open covering of X has cardinality less than p.

DEFINITION 2. Suppose that X is nondiscrete. $m(X)$ is the least cardinal number m for which there is a collection (of cardinality m) of open subsets of X whose intersection is not an open set. $m(X)$ is called the index of X.

Examples: 1) Suppose that X is completely regular. X is pseudocompact if and only if $p(X) = \aleph_0$.

2) Suppose that X is a uniformly locally compact space. Then $p(X) = \aleph_0$ or $p(X)$ is the successor of an infinite cardinal p (in this case, X is the union of p compact subsets of X).

Let X and Y be topological spaces.

(*) Nella seduta del 13 novembre 1971.

THEOREM 1. *If* $f : X \to Y$ *is an onto continuous function, then* $p(Y) \leq p(X)$.

THEOREM 2. $p(X \times Y) \geq p(X)\, p(Y)$, *where* $X \times Y$ *is the product space.*

Proof. It follows from the fact that $p(X)\, p(Y)$ is the maximum of the set $\{p(X), p(Y)\}$.

THEOREM 3. *If* Y *is compact, then* $p(X \times Y) = p(X)\, p(Y)$.

Proof. Let C be a locally finite open covering of $X \times Y$. For each $x \in X$ there is an open neighbourhood U_x of x such that $U_x \times Y$ intersects only finitely many members of C. For each $x \in X$ put

$$C_x = \{W \in C \mid \{x\} \times Y \cap W \neq \emptyset\} \qquad \text{and} \quad A = \{C_x \mid x \in X\}.$$

Now, for each $B \in A$ put

$$T_B = \Big\{t \in X \mid \{t\} \times Y \subset \bigcup_{Z \in B} Z\Big\} \cap \bigcap_{Z \in B} pr\, Z,$$

where $pr : X \times Y \to X$ is the projection. $(T_B)_{B \in A}$ is a locally finite open covering of X; thus $|A| < p(X)$. But since B is finite for every $B \in A$ and $C = \bigcup_{B \in A} B$ we have that $|C| < p(X)$. The proof is completed by virtue of Theorem 2.

THEOREM 4. *If* $p(X) > \aleph_0$ *there is a paracompact space* Y *such that* $p(X \times Y) > p(X)\, p(Y)$.

Proof. Let (S_n) be a locally finite open covering of X such that $S_n - (S_1 \cup \cdots \cup S_{n-1})$ is nonempty for each natural number $n \geq 2$. Let m be the cardinal supremum of the set $\{y_1, \cdots, y_n, \cdots\}$, where $y_1 = 2^{p(X)}$, $y_{n+1} = 2^{y_n}$ for every $n \geq 1$. Let Y be a set of cardinality m and fix a point $b \in Y$. In Y we consider the topology defined below:

1) $\{z\}$ is open for every $z \in Y - \{b\}$;

2) $U \subset Y$ is a neighbourhood of b if and only if $b \in U$ and $|X - U| < m$.

For each natural $n \geq 1$ we choose a discrete open covering of Y, G_n, of cardinality y_n. The set $\bigcup_{n=1}^{\infty} \{S_n \times T \mid T \in G_n\}$ is a locally finite open covering of $X \times Y$ of cardinality $m\ (= p(Y))$.

COROLLARY. *Suppose that* X *is paracompact.* X *is compact if and only if* $p(X \times Y) = p(X)\, p(Y)$ *for every nonvoid paracompact space* Y.

THEOREM 5. *If* X *is regular,* $m(X) > \aleph_0$ *and every closed subset of* X *has a fundamental system of neighbourhoods of cardinality not greater than* $m(X)$, *then* X *is normal and* $m(X)$*-paracompact.*

Proof. The normality is a consequence of the facts that X is regular, every closed subset of X has a fundamental system of neighbourhoods of cardinality not greater than $m(X)$ and every subset of X of cardinality less than $m(X)$ is closed (has no accumulation point).

Since $m(X) > \aleph_0$ and X is normal, every open subset of X is the union of at most $m(X)$ open-closed subsets of X and, thus, the $m(X)$–paracompactness follows easily.

THEOREM 6. *If* X *is normal and* $m(X) = p(X) > \aleph_0$, *then every closed subset of* X *which is the intersection of at most* $m(X)$ *open subsets of* X *has a fundamental system of neighbourhoods of cardinality not greater than* $m(X)$.

Proof. Denote by I the set of all ordinal numbers smaller than the first ordinal number of cardinality $m(X)$. Let F be a closed subset of X which is the intersection of at most $m(X)$ open subsets of X. (If F is open the result is trivial). We can suppose that $F = \bigcap_{i \in I} A_i$, where each A_i is open-closed (because X is normal) and $A_i \subset A_j$ whenever $i > j$. Let W be an open-closed neighbourhood of F and consider the set $C = \{U_i - (U_{i'} \cup W) \mid i \in I - \{0\}\}$, where i' is the ordinal successor of i and $U_i = \bigcap_{j<i} A_j$. C is a discrete collection of open-closed subsets of X; so $|C| < m(X)$ and there is $k \in I$ such that $U_i - (U_{i'} \cup W) = \varnothing$ for every $i \geq k'$. It then follows that $U_{k'} \subset W$. Finally we have that $\{U_i \mid i \in I - \{0\}\}$ is a fundamental system of neighbourhoods of F.

THEOREM 7. *If every closed subset of* X *is the intersection of at most* $m(X)$ *closures of open subsets of* X *containing it and every subset of* X *of cardinality* $m(X)$ *has an accumulation point, then* X *is normal.*

Proof. Let I be the set of all ordinal numbers less than the first ordinal number of cardinality $m(X)$. Let F and K be two nonempty disjoint closed subsets of X. There are two families if open subsets of X, $(A_i)_{i \in I}$ and $(B_i)_{i \in I}$, such that:

1) $F \subset A_i \subset X - K$ and $K \subset B_i \subset X - F$ for every $i \in I$;

2) $F = \bigcap_{i \in I} \bar{A}_i$ and $K = \bigcap_{i \in I} \bar{B}_i$;

3) $A_i \subset A_j$ and $B_i \subset B_j$ whenever $i > j$.

It suffices to prove that for some $i \in I$, $|A_i \cap B_i| < m(X)$; then $A_i \cap B_i$ is closed and $F \subset A_i - B_i$ and $K \subset B_i - A_i$. On the contrary, let us suppose that for each $i \in I$ we choose (by induction) an element $c_i \in A_i \cap B_i - \{c_j \mid j < i\}$. The set $\{c_i \mid i \in I\}$ has cardinality $m(X)$ and does not admit an accumulation point (the accumulation point would belong to F and K), which is a contradiction.

A part of this material appeared in [1], which was supported by the Conselho Nacional de Pesquisas.

REFERENCE

[1] O. T. ALAS, *Sôbre uma extensão do conceito de compacidade e suas aplicações*, Thesis, 1968.