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Teoria dei controlli. — *Sufficient Conditions for Controllability of Nonlinear Systems.* Nota (*) di JERALD P. DAUER, presentata dal Socio G. SANSONE.

RIASSUNTO. — Con l'uso del Teorema del punto fisso di Schauder si stabiliscono condizioni sufficienti per la controllabilità e la totale controllabilità di un sistema nonlineare della forma $\dot{x} = A(t, x)x + B(t, x)u$.

1. INTRODUCTION

In a recent paper Davison and Kunze [1] used a fixed point approach to study global and local controllability of the nonlinear system

$$(1) \qquad \dot{x} = A(t, x)x + B(t, x)u \qquad (\dot{x} = dx/dt)$$

on $I = [t_0, t_1]$. For global controllability it was assumed that A and B are uniformly bounded on $I \times E^n$, E^n is Euclidean n-space. In this paper we modify the Davison-Kunze approach to examine the (null) controllability of system (1) under somewhat less restrictive assumptions on A and B. In particular, we assume only local conditions on A and B in place of the constrictive global conditions used in [1]. However, we shall assume an additional condition on the behaviuor of $B(t, x)$ near $x = 0$; namely, $|B(t, x)| \leq c|x|$ locally in x.

In Section 2 we obtain sufficient conditions for controllability of system (1) by examining the controllability of the linear system $\dot{x} = A(t, z)x + B(t, z)u$ for bounded sets of continuous functions z. We use this result in Section 3 to consider total controllability of system (1). Our result there, under the additional hypothesis on B, improves the results on total controllability obtained by Davison and Kunze [1]. In this section we also consider ε–approximate controllability using piecewise constant controls. This type of controllability is interesting in a number of applications.

2. CONTROLLABILITY

We shall assume that A and B are $n \times n$ and $n \times m$ matrix functions, respectively, that are continuous in x for fixed t and piecewise continuous in t for fixed x. System (1) is said to be *controllable* if given any $x_1 \in E^n$ there is a piecewise continuous (control) function $u : I \to E^m$ such that the solution of the initial value problem

$$\dot{x} = A(t, x)x + B(t, x)u(t)$$
$$x(t_0) = 0$$

satisfies $x(t_1) = x_1$.

(*) Pervenuta all'Accademia il 13 settembre 1971.

Let C [I] be the set of continuous E^n–valued functions defined on I. Then C [I] is a Banach space with the norm $\|z\| = \max_{t \in I} |z(t)|$. For positive constants N and d we define

$$C_N[I] = \{z \in C[I] : \|z\| \leq N\},$$
$$\|z\|_d = \max_{t \in I} e^{-d(t-t_0)} |z(t)|,$$
$$C_N^d[I] = \{z \in C[I] : \|z\|_d \leq N\}.$$

For each $z \in C[I]$ let $\Phi(t, z)$ denote the fundamental matrix solution of $\dot{x} = A(t, z(t))\, x$ such that $\Phi(t_0, z)$ is the identity matrix and let

$$W_z[t, t'] = \int_t^{t'} \Phi^{-1}(s, z)\, B(s, z(s))\, B(s, z(s))^T\, \Phi^{-1}(s, z)^T\, ds.$$

Denote $W_z[t_0, t_1]$ by W_z.

If $z \in C[I]$ is such that the determinant of W_z, det W_z, is nonzero, then define the control function $u_{zx_1} : I \to E^m$ by

$$u_{zx_1}(t) = B(t, z(t))^T\, \Phi^{-1}(t, z)^T\, W_z^{-1}\, \Phi^{-1}(t_1, z)\, x_1. \tag{2}$$

For such z the solution, denoted by $P(z)$, of the linear initial value problem

$$\dot{x} = A(t, z(t))\, x + B(t, z(t))\, u_{zx_1}(t)$$
$$x(t_0) = 0$$

satisfies $x(t_1) = x_1$, (cf. [2]). In fact

$$P(z)(t) = \Phi(t, z) \int_{t_0}^{t} \Phi^{-1}(s, z)\, B(s, z(s))\, u_{zx_1}(s)\, ds. \tag{3}$$

THEOREM 1. *System* (1) *is controllable if the following two conditions hold*:

i) *For each* $N > 0$ *there exists a constant* $k = k(N)$ *which satisfies*

$$|B(t, x)| \leq k |x|$$

for all (t, x) *such that* $t \in I$ *and* $|x| \leq N$.

ii) *For each* $N > 0$ *there exists a constant* $c = c(N) > 0$ *such that*

$$\inf_{z \in C_N[I]} \det W_z \geq c.$$

Proof. Fix $x_1 \in E^n$ and choose $N \geq |x_1|$. Define the continuous operator $P : C[I] \to C[I]$ by equation (3). Since $A(t, z(t))$ and $B(t, z(t))$ are bounded (on I) uniformly in $z \in C_N[I]$ it follows that $\Phi(t, z)$, $\Phi^{-1}(t, z)$ and W_z are bounded uniformly in $z \in C_N[I]$. By condition (ii) we therefore have that W_z^{-1}, and hence $u_{zx_1}(t)$ (see equation (2)), is bounded uniformly in $z \in C_N [I]$. Hence, using condition (i), there exists a constant $d > 0$ which depends only on N and x_1 such that

$$|P(z)(t)| \leq d \int_{t_0}^{t} |z(s)| \, ds$$

for all $t \in I$ and each $z \in C_N [I]$. Thus for each $z \in C_N [I]$ we have

$$e^{-d(t-t_0)} |P(z)(t)| \leq d \int_{t_0}^{t} e^{-d(t-t_0)} |z(s)| \, ds$$
$$\leq \|z\|_d$$

for all $t \in I$.

Let $M = Ne^{-d(t_1-t_0)}$. Then $C_M^d [I]$ is a subset of $C_N [I]$ and thus $\Omega = \{P(z) : z \in C_M^d [I]\}$ is a subset of $C_M^d [I]$. By the Arzelà-Ascoli Theorem [3] the closure of the image set Ω is compact. Hence by Schauder's fixed point theorem [3], the operator P has a fixed point $\bar{z} \in C_M^d [I]$. The function $\bar{z}$ is clearly a solution of system (1) corresponding to a control function of the form (2), $\bar{z}(t_0) = 0$ and $\bar{z}(t_1) = x_1$. This completes the proof.

Remark. As was pointed out in [1], a difficulty in the application of Theorem 1 is in showing that condition (ii) is satisfied. A computable criterion for this condition based on the controllability matrix of Silverman and Meadows [4] can be adapted from [1, Theorem 3].

3. TOTAL AND ε–APPROXIMATE CONTROLLABILITY

System (1) is said to be *totally controllable* if given any $x_0, x_1 \in E^n$ and any $t_f \in (t_0, t_1]$ there is a piecewise continuous function $u : [t_0, t_f] \to E^m$ such that the solution of the initial value problem

$$\dot{x} = A(t, x) x + B(t, x) u(t)$$
$$x(t_0) = x_0$$

satisfies $x(t_f) = x_1$.

THEOREM 2. *System* (1) *is totally controllable if the following two conditions hold*:

i) *For each* $N > 0$ *there exists a constant* $k = k(N)$ *which satisfies*

$$|B(t, x)| \leq k |x|$$

for all (t, x) *such that* $t \in I$ *and* $|x| \leq N$.

ii)′ *For each* $N > 0$ *there exists a constant* $c = c(N) > 0$ *such that*

$$\inf_{z \in C_N[I]} \det W_z [t, t'] \geq c$$

for all $t, t' \in I$.

Proof. Let x_0, x_1 and t_f be given and choose $t_2 \in (t_0, t_f)$. Define the operator $P' : C[I] \to C[I]$ by

$$P'(z)(t) = -\int_{t_0}^{t} \Phi^{-1}(s, z)\, B(s, z(s))\, u'_{zx_0}(s)\, ds,$$

where $u'_{zx_0}(t) = B(t, z(t))^T \Phi^{-1}(t, z)^T (W_z[t_0, t_2])^{-1} x_0$. As in the proof of Theorem 1, the operator P' has a fixed point z_1. The function z_1 is a solution of system (1) corresponding to the control function $u'_{z_1x_0}$, $z_1(t_0) = x_0$ and $z_1(t_2) = 0$. Also as in the proof of Theorem 1, there is a function $z_2 \in C[I]$ which is a solution of system (1) corresponding to the control function

$$u''_{z_2x_1}(t) = B(t, z_2(t))^T \Phi^{-1}(t, z_2)^T (W_{z_2}[t_2 t_f])^{-1} \Phi^{-1}(t_f, z_2)\, x_1,$$

$z_2(t_2) = 0$ and $z_2(t_f) = x_1$. Define the control function $u : I \to E^m$ by

$$u(t) = \begin{cases} u'_{z_1x_0}(t) & \text{for } t \in [t_0, t_2] \\ u''_{z_2x_1}(t) & \text{for } t \in (t_2, t_f]. \end{cases}$$

Then the solution of

$$\dot{x} = A(t, x)\, x + B(t, x)\, u(t)$$
$$x(t_0) = x_0$$

satisfies $x(t_f) = x_1$. This completes the proof.

The following result is on approximate controllability. Its proof follows directly from Theorem 1 using continuous dependence of solutions on parameters (cf. [5, p. 18]). We say that system (1) is *ε–approximately controllable using piecewise constant controls* if given any $x_1 \in E^n$ there is a piecewise constant function $u : I \to E^m$ such that the solution of the initial value problem

$$\dot{x} = A(t, x)\, x + B(t, x)\, u(t)$$
$$x(t_0) = 0$$

satisfies $|x(t_1) - x_1| < \varepsilon$.

THEOREM 3. *Suppose* A *and* B *are continuous on* $I \times E^n$. *If conditions i) and ii) of Theorem 1 hold, then system* (1) *is ε–approximately controllable using piecewise constant controls for every* $\varepsilon > 0$.

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