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Fisica matematica. — *Action principles for electromagnetism.*
Nota di Carlo Morosi (*), presentata (**) dal Socio B. Finzi.

Riassunto. — La presente Nota riguarda un principio di azione per il campo elettromagnetico che differisce da quello usuale per il fatto che il potenziale elettromagnetico, in corrispondenza al quale il funzionale di azione è stazionario, coincide con la soluzione dell'equazione delle onde soddisfacente sia le condizioni al contorno che tutte quelle *iniziali* assegnate.

Infatti il funzionale d'azione classico è stazionario in corrispondenza ad un potenziale il quale soddisfa sì l'equazione delle onde e le condizioni al contorno, ma non tutte le condizioni iniziali date. Il progresso che si realizza con il presente metodo è possibile per l'uso di una forma bilineare convolutiva, che rende simmetrico l'operatore di campo. Come conseguenza di tale simmetria, si ottiene un teorema di reciprocità valido per campi variabili nel tempo, che contiene come caso particolare il teorema di reciprocità di Lorentz.

1. Introduction

The possibility of a variational formulation for linear initial value problems, discovered by Gurtin [1], has been recently discussed in detail by Tonti [2], who put in evidence the importance of a convolution bilinear form for the existence of such a variational formulation. In particular, using a convolution bilinear form, one can give a variational formulation for linear initial value problems in a way that greatly simplifies the Gurtin method [3]. The purpose of this paper is to apply the convolution bilinear form to the variational formulation of the boundary-initial value problem of the electromagnetic field. An analysis of the operator features of this problem is made, and a variational formulation is given, simpler than the one obtained with the Gurtin method [4]. Referring to quoted papers, and particularly to paper [2], for more details, in the next section we summarize the main characteristics of this procedure.

2. – Variational formulation for initial-value problems

A fundamental theorem in the calculus of variations states that in order that the *linear* problem (1)

$$\mathrm{L}u = f \tag{2.1}$$

admits a variational formulation the operator of the problem must be symmetric.

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(1) By this word we mean the set of the equation and of the initial and boundary conditions.

If the operator is non-linear, the corresponding necessary condition is that the Gateaux derivative of the operator be symmetric [5, pag. 56] (2). In the general case the functional that corresponds to the equation of the problem is

$$F[u] = F[u_0] + \int_0^1 \langle u - u_0 , L(u_0 + \lambda (u - u_0)) \rangle \, d\lambda - \langle f, u \rangle \tag{2.2}$$

where u_0 is any function that belongs to the domain D (L) [5, pag. 58]. In the case of linear operators, the functional (2.2) becomes

$$F[u] = \frac{1}{2} \langle u , Lu \rangle - \langle f , u \rangle . \tag{2.3}$$

As an example, let us consider the simplest initial value problem

$$\frac{d}{dt} u(t) = f(t) ; \quad u(0) = 0 ; \quad 0 \leq t \leq T ; \quad u(t) \in C^1 [0 , T] . \tag{2.4}$$

The operator of this problem will be denoted by L while the formal differential operator d/dt will be denoted by $\mathfrak{L}$ (3). The operator L is *not* symmetric with the usual bilinear form

$$\langle u , v \rangle = \int_0^T u(t) \, v(t) \, dt \tag{2.5}$$

but it is symmetric with the convolution bilinear form

$$\langle u , v \rangle_c = \int_0^T u(T - t) \, v(t) \, dt . \tag{2.6}$$

Then with the usual bilinear form we cannot have a variational formulation for initial-value problems, while using the convolution bilinear form that becomes possible.

If the problem is given by two sets of equations

$$Nu = f \quad ; \quad Mv = Cu \tag{2.7}$$

(2) We remember that the condition of symmetry of the Gateaux derivative of the operator becomes also sufficient if the domain of the operator is simply connected [5, pag. 32].

(3) Here and in the following we use capital Italic letters for the *formal operator* $\mathfrak{L}$ and capital letters for the operator L. To the domain D (L) of the operator belong space-time functions (with a specified functional class) that satisfy the boundary and initial conditions of the particular problem we intend to study.

(where N, M, C are linear, and C is also symmetric), we can write the equations of the problem in matrix form

$$\begin{bmatrix} 0 & N \\ M & -C \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} \tag{2.8}$$

that, putting

$$L = \begin{bmatrix} 0 & N \\ M & -C \end{bmatrix} \quad ; \quad w = \begin{bmatrix} v \\ u \end{bmatrix} \quad ; \quad g = \begin{bmatrix} f \\ 0 \end{bmatrix} \tag{2.8 a}$$

becomes

$$Lw = g. \tag{2.9}$$

Remembering the condition stated above for the equation (2.1) we have that the necessary and sufficient condition for being able to deduce (2.7) from the stationarity of a functional $F[u, v]$ is that the operator L of (2.9) be symmetric, that is N and M must be one the adjoint of the other one: $N = \tilde{M}$. It's important to stress that the symmetry of the operator of the problem (2.1) and the condition $N = \tilde{M}$ for the problem (2.7) do not refer to formal operators only but to complete operators [10]. Then from (2.3) we obtain, using the bilinear form (2.6)

$$F[u, v] = \langle v, Mu \rangle_c - \frac{1}{2} \langle u, Cu \rangle_c - \langle v, f \rangle_c. \tag{2.10}$$

3. – The boundary-initial value problem for Maxwell equations

In every physical theory the variational principles rest upon a mathematical structure of field equations, and therefore they can be constructed in a rational way, testing their symmetry with respect to some bilinear form: when the symmetry is assured, the functional is given by (2.2).

Now we shall examine the structure of the equations of the electromagnetic field, and translate them into operator form. We consider two skew-symmetric tensors $F_{\alpha\beta}$ and $f^{\alpha\beta}$ with cartesian components (4)

$$f^{00} = 0 \quad ; \quad f^{0k} = -cD_k \quad ; \quad f^{hk} = \varepsilon^{hkm} H_m ; \tag{3.1}$$

$$F_{00} = 0 \quad ; \quad F_{0k} = -\frac{E_k}{c} \quad ; \quad F_{hk} = -\varepsilon_{hkm} B^m$$

and two vectors J^α and φ_α

$$J^\alpha = (c\rho ; \vec{J}) \qquad \varphi_\alpha = \left(\frac{2\Phi}{c} ; -2\vec{A}\right). \tag{3.2}$$

(4) From now on, Greek indices take values 0, 1, 2, 3 ($x^0 = ct$), Latin indices take values 1, 2, 3: letter $\boldsymbol{x}$ includes the set of "spatial" coordinates (x^1, x^2, x^3). A pseudo-euclidean metric with signature (1, −1, −1, −1) is used, and ∇_α is the covariant derivative operator with respect to the coordinate x^α. The tensor ε_{hkm} is the Ricci tensor.

With the introduction of the four-potential φ_α the two sets of Maxwell equations can be written [5] as

$$(3.3)\qquad \begin{cases} \nabla_\alpha f^{\alpha\beta} = J^\beta \\ f^{0i}(0\,;\boldsymbol{x}) = 0 \quad ; \quad f^{\alpha i}(x^0\,;\boldsymbol{x})\big|_{\boldsymbol{x}\in\Sigma'} = 0 \quad ; \quad n_i f^{\alpha i}(x^0\,;\boldsymbol{x})\big|_{\boldsymbol{x}\in\Sigma''} = 0 \end{cases}$$

where $\vec{n}$ is the outer normal vector of the boundary Σ of τ and $\Sigma' \cup \Sigma'' = \Sigma$, and

$$(3.4)\qquad \begin{cases} F_{\alpha\beta} = \frac{1}{2}\,[\nabla_\beta\,\varphi_\alpha - \nabla_\alpha\,\varphi_\beta] \\ \varphi_\alpha\,(0\,;\boldsymbol{x}) = 0\,. \end{cases}$$

We can write phenomenological equations in vacuo in the following form [6]

$$(3.5)\qquad \mu_0\,f_{\alpha\beta} = -\,F_{\alpha\beta}\,.$$

Then, from equations (3.3), (3.4), (3.5), one can show that the potential φ_α is solution of the equation

$$(3.6)\qquad [g^{\alpha\sigma}\,g^{\beta\delta}\,\nabla_\delta\,\nabla_\alpha - g^{\beta\sigma}\,g^{\alpha\gamma}\,\nabla_\gamma\,\nabla_\alpha]\,\varphi_\sigma = 2\,\mu_0\,J^\beta$$

where $g^{\alpha\beta}$ is the metric tensor.

4. – Maxwell equations in operator notation

The problem (3.4) can be considered as describing a mapping between two function spaces as follows. Let us consider a first function vector space Φ whose elements φ are the set of four space-time functions $\varphi^\alpha \in C^2(\Omega)$ and the subset of this space formed by those elements that satisfy homogeneous initial conditions given in (3.4): such a subset is a linear manifold. Next we consider a second function space $\mathfrak{F}$ whose elements F are the skew-symmetric tensors $F_{\alpha\beta} \in C^1(\Omega)$. Likewise we introduce two more function spaces: we shall denote by $\mathfrak{J}$ the first one, whose elements J are the sets of four functions $J^\alpha \in C^1(\Omega)$, and by $\mathfrak{G}$ the second space, whose elements f are the skew-symmetric tensors $f_{\alpha\beta} \in C^1(\Omega)$. The problem (3.3) is a mapping between the two spaces $\mathfrak{J}$ and $\mathfrak{G}$ given by the linear differential operator N with a formal part defined by the differential equation (3.3) and a domain defined by the functional class specified above and by the conditions (3.3). The problem (3.4) is a mapping between the two spaces Φ and $\mathfrak{F}$ given by the differential linear operator M with formal part $\mathfrak{M}$ defined by (3.4) and with domain D(M) given by the linear manifold specified above.

(5) We consider the electromagnetic field in a bounded region $\tau \subset \mathbf{R}^3$ or, with space-time notation, in a cylindrical region Ω of V_4. We assume here initial and boundary conditions to be homogeneous, to work with a linear problem. An extension to the case of non-homogeneous conditions is easily done.

(6) Here and in the following we use the system $[\mathrm{M.K.S.Q.}]_{\mathrm{raz.}}$

Problems (3.3) and (3.4) with the equation (3.5) can thus be put in operator form

$$M\varphi = C^{-1}f \quad ; \quad Nf = J \qquad (f = CF). \tag{4.1}$$

According to what we said in § 2 in order to give a variational formulation to both problems we must verify whether $N = \tilde{M}$.

Now we consider the adjoint of the operator M corresponding to two possible bilinear forms. First we try with the usual bilinear forms

$$\langle \varphi , J \rangle = \iiiint\limits_{\Omega} \varphi^{\alpha}(x^0 ; \boldsymbol{x}) \, J_{\alpha}(x^0 ; \boldsymbol{x}) \, d\Omega ; \tag{4.2}$$

$$\langle F, f \rangle = \iiiint\limits_{\Omega} F_{\alpha\beta}(x^0 ; \boldsymbol{x}) \, f^{\alpha\beta}(x^0 ; \boldsymbol{x}) \, d\Omega \tag{4.3}$$

where Ω is the region of V_4 where we define the field functions. (See the footnote [5]). The function spaces Φ and $\mathfrak{J}$ are put in duality with the bilinear form (4.2), and the same is for the spaces $\mathfrak{F}$ and $\mathfrak{G}$: a natural request is that the duality be separating (i.e. for every $\varphi \neq \theta$, θ being the null element of the space, there must exist at least an element Ψ such that $\langle \varphi , \psi \rangle \neq 0$); with this request if $\langle \varphi , \psi \rangle = 0$ for every φ, then must be $\Psi = \theta$.

One can easily show that the duality between the function spaces Φ and $\mathfrak{J}$ is separating, and likewise for $\mathfrak{F}$ and $\mathfrak{G}$, as f and F are defined as skew-symmetric tensors.

The adjoint operator $\tilde{M}$ of the operator M is defined by the identity

$$\langle f, M\varphi \rangle \equiv \langle \tilde{M}f, \varphi \rangle \tag{4.4}$$

for every $\varphi \in D(M)$ and $f \in D(\tilde{M})$. Even if the formal operator $\mathfrak{N}$ is the formal adjoint of $\mathfrak{M}$, $D(\tilde{M}) \neq D(N)$, on account of initial conditions: therefore $N \neq \tilde{M}$. So we can say that *a variational formulation of the problems* (4.1) *with usual bilinear forms is not possible.*

The customary trick introduced in field theory to overcome this difficulty is to ignore one physical initial condition and to add an "ad hoc" *unphysical* final condition. So in our problem we obtain another operator M′ defined by the same formal part $\mathfrak{M}$ defined in (3.4) but with another condition on the time variable

$$\varphi_{\alpha}(0 ; \boldsymbol{x}) = 0 \quad ; \quad \varphi_{\alpha}(X^0 ; \boldsymbol{x}) = 0 \tag{4.5}$$

while we can eliminate the initial condition of the problem (3.3) thus obtaining a new operator N′. Now, we have $N' = \tilde{M}'$ and, according to (2.10), we deduce the new problem from the stationarity of the functional

$$G[f, \varphi] = \langle f, M' \varphi \rangle - \frac{1}{2} \langle f, C^{-1}f \rangle - \langle \varphi , J \rangle . \tag{4.6}$$

The functional (4.6) is one of the various but equivalent forms of the "classical" variational formulation (for example see [6], [7], [8, ch. 24]). Now if we want to maintain our physically meaningful initial conditions, we can overcome the difficulty stated above by introducing a new bilinear form, precisely a convolution bilinear form defined as follows:

$$\langle \mathrm{J}, \varphi \rangle_c = \iiiint\limits_{\Omega} \mathrm{J}^{\alpha}(\mathrm{X}^0 - x^0 ; \boldsymbol{x})\, \varphi_{\alpha}(x^0 ; \boldsymbol{x})\, d\Omega \tag{4.7}$$

$$\langle \mathrm{F}, f \rangle_c = \iiiint\limits_{\Omega} \mathrm{F}_{\alpha\beta}(\mathrm{X}^0 - x^0 ; \boldsymbol{x})\, f^{\alpha\beta}(x^0 ; \boldsymbol{x})\, d\Omega . \tag{4.8}$$

With the bilinear form (4.7), (4.8) one can verify that $\mathrm{N} = \tilde{\mathrm{M}}$ and our problems can be deduced from the stationarity of a functional, as we shall see in the next sections.

5. – VARIATIONAL FORMULATION FOR THE WAVE PROBLEM

The fundamental problem of the electromagnetic field is to find the electromagnetic potential given by a charge-current distribution. The electromagnetic potential φ^{α} is solution of the equation (3.6) with the initial condition (3.4): in operator notation we have

$$\tilde{\mathrm{M}}\mathrm{C}\mathrm{M}\varphi = \mathrm{J} . \tag{5.1}$$

The operator $\tilde{\mathrm{M}}\mathrm{C}\mathrm{M}$ is symmetric as can be easily seen:

$$\langle \psi , \tilde{\mathrm{M}}\mathrm{C}\mathrm{M}\varphi \rangle_c \equiv \langle \mathrm{M}\psi , \mathrm{C}\mathrm{M}\varphi \rangle_c \equiv \langle \tilde{\mathrm{M}}\mathrm{C}\mathrm{M}\psi , \varphi \rangle_c . \tag{5.2}$$

Then from (2.3) the functional of the problem (5.1) is

$$\mathrm{F}[\varphi] = \frac{1}{2} \langle \varphi , \tilde{\mathrm{M}}\mathrm{C}\mathrm{M}\varphi \rangle_c - \langle \varphi , \mathrm{J} \rangle_c \tag{5.3}$$

that is

$$\mathrm{F}[\varphi] = -\frac{1}{4\mu_0} \iiiint\limits_{\Omega} [\varphi_{\beta}(\mathrm{X}^0 - x^0 ; \boldsymbol{x})\,(g^{\beta\gamma}\varphi^{\sigma}{}_{/\sigma\gamma}(x^0 ; \boldsymbol{x}) - g^{\alpha\gamma}\varphi^{\beta}{}_{/\alpha\gamma}(x^0 ; \boldsymbol{x})) - \varphi^{\alpha}(\mathrm{X}^0 - x^0 ; \boldsymbol{x})\, \mathrm{J}_{\alpha}(x^0 ; \boldsymbol{x})]\, d\Omega . \tag{5.3 a}$$

We stress the fact that the domain D(F) of the functional is exactly the domain of the given problem.

From its variation and the properties of the convolution we have

$$\delta\mathrm{F}[\varphi] = \langle \delta\varphi , \tilde{\mathrm{M}}\mathrm{C}\mathrm{M}\varphi - \mathrm{J} \rangle_c \quad ; \quad \varphi \in \mathrm{D}(\tilde{\mathrm{M}}\mathrm{C}\mathrm{M}) \tag{5.4}$$

from which, if we require that $\delta F = 0$ the fundamental problem follows. Then we can state the following

THEOREM I: *The electromagnetic four-potential* φ^α, *solution of the boundary-initial value wave problem* (5.1) *makes stationary the functional given by* (5.3), *and viceversa.*

What about the minimum character of this functional?

Let us consider the functional (5.3): if we calculate its second variation putting $\varphi = \varphi_0 + \varepsilon\overline{\varphi}$ (when φ_0 is solution of the problem, and therefore makes F stationary) we have

$$\delta^2 F = \frac{\varepsilon^2}{2} \langle \overline{\varphi} , \tilde{M}CM\overline{\varphi} \rangle_c \tag{5.5}$$

and we can easily verify that $\delta^2 F$ is not positive or negative definite. The absence of minimum is not typical of the convolution bilinear form, and also with the usual bilinear form we cannot speak of " minimum " property of the functionals, even if in the literature the minimum character of the functional is sometimes stated.

6. – Canonical form of the variational Theorem I

The boundary-initial value problem for the equations (3.3) and (3.4) can be written in the following operator form (see § 4)

$$\tilde{M} f = J \qquad M\varphi = C^{-1} f. \tag{6.1}$$

Therefore the functional of the problem is (see the functional (2.10))

$$F[f, \varphi] = \langle f, M\varphi\rangle_c - \langle \varphi , J\rangle_c - \frac{1}{2}\langle f, C^{-1} f\rangle_c \tag{6.2}$$

that is

$$\iiiint_\Omega \Big[f^{\alpha\beta}(X^0 - x^0 ; \boldsymbol{x}) (\varphi_{\beta/\alpha}(x^0 ; \boldsymbol{x}) - \varphi_{\alpha/\beta}(x^0 ; \boldsymbol{x})) - \tag{6.2 a}$$

$$- \varphi_\alpha (X^0 - x^0 ; \boldsymbol{x}) J^\alpha (x^0 ; \boldsymbol{x}) + \frac{1}{2} \mu_0 f^{\alpha\beta} (X^0 - x^0 ; \boldsymbol{x}) f_{\alpha\beta} (x^0 ; \boldsymbol{x}) \Big] d\Omega .$$

The elements f and φ of the domain D(F) of the functional (6.2) are the elements of the domains D(N) and D(M) respectively. Then we have the following

THEOREM II: *The electromagnetic four-potential* φ^α *and the electromagnetic field tensor* $f^{\alpha\beta}$, *solutions of the boundary-initial value problems* (3.3) *and* (3.4), *that is of the Maxwell equations in canonical form, make stationary the functional* (6.2), *and viceversa.*

Again we stress the fact that no final condition on the field functions φ^α and $f^{\alpha\beta}$ is needed.

7. – RECIPROCITY THEOREM

On account of the symmetry of the operator of the wave problem, we can easily deduce a formulation of the reciprocity Theorem; let us consider two different charge distributions J_1, J_2: be φ_1 and φ_2 the corresponding potentials.

From the symmetry of $\tilde{M}CM$ we obtain

$$\langle J_1, \varphi_2\rangle_c \equiv \langle \tilde{M}CM\varphi_1, \varphi_2\rangle_c \equiv \langle \varphi_1, \tilde{M}CM\varphi_2\rangle_c \equiv \langle \varphi_1, J_2\rangle_c \tag{7.1}$$

that is

$$\iiiint\limits_{\Omega} J_{1\alpha}(X^0 - x^0; \boldsymbol{x})\, \varphi_2^\alpha(x^0; \boldsymbol{x})\, d\Omega \equiv \iiiint\limits_{\Omega} \varphi_1^\alpha(X^0 - x^0; \boldsymbol{x})\, J_{2\alpha}(x^0; \boldsymbol{x})\, d\Omega. \tag{7.2}$$

We stress the fact that this formulation depends only upon the symmetry of the operator $\tilde{M}CM$.

We consider three particular cases, the electrostatic field, the magnetostatic field and the electromagnetic stationary field. In the first case, we have from (7.1), (3.1) and (3.2)

$$\iiint\limits_{V} \rho_2(P)\ \Phi_1(P)\, dV \equiv \iiint\limits_{V} \rho_1(P)\ \Phi_2(P)\, dV \tag{7.3}$$

that gives the electrostatic formulation of the reciprocity Theorem. Likewise we obtain the magnetostatic formulation of this Theorem, given by:

$$\iiint\limits_{V} \vec{J}_2(P)\cdot\vec{A}_1(P)\, dV \equiv \iiint\limits_{V} \vec{J}_1(P)\cdot\vec{A}_2(P)\, dV. \tag{7.4}$$

Now let us consider a stationary electromagnetic field, that is we suppose that $\varphi = \bar{\varphi}\cdot\exp(i\omega t)$ and $J = \bar{J}\cdot\exp(i\omega t)$: then from (7.2) we have

$$\begin{aligned}\iiint\limits_{V} [\bar{\rho}_2(P)\cdot\bar{\Phi}_1(P) - \vec{\bar{J}}_2\cdot\vec{\bar{A}}_1(P)]\, dV \equiv \\ \equiv \iiint\limits_{V} [\bar{\rho}_1(P)\,\bar{\Phi}_2(P) - \vec{\bar{J}}_1(P)\cdot\vec{\bar{A}}_2(P)]\, dV.\end{aligned} \tag{7.5}$$

If we remember that for stationary fields and charges distributions the continuity equation (that is a necessary condition of the wave equation) begins

$$\operatorname{div}\vec{\bar{J}} + i\omega\bar{\rho} = 0 \tag{7.6}$$

and that the electric field is

$$\vec{\bar{E}} = -[\operatorname{grad}\bar{\Phi} + i\omega\vec{\bar{A}}] \tag{7.7}$$

from (7.5) we have, on account of the divergence Theorem and boundary conditions:

$$\iiint_V \vec{J}_2 \cdot \vec{E}_1 \, dV = \iiint_V \vec{J}_1 \cdot \vec{E}_2 \, dV \tag{7.8}$$

that is the Lorentz reciprocity Theorem for stationary fields. The various formulations of the reciprocity Theorem (see ref. [8, ch. 3] and [9, ch. 10]) are thus derivable from the properties of structure of the operator $\tilde{\mathrm{M}}\mathrm{CM}$ and are all particular cases of the general reciprocity Theorem given by (7.2).

8. – Conclusion

The paper deals with a variational formulation of the complete boundary-*initial* value problem of electromagnetism.

Contrary to the usual practice, initial value conditions are taken into account without recourse to artful final conditions. This is possible by using a convolution bilinear form that makes the field operator symmetric: moreover, as a consequence of such symmetry, a reciprocity Theorem valid for time-varying electromagnetic fields is obtained, that contains the Lorentz reciprocity Theorem as a particular case.

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