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S. C. Srivastava, R. S. Sinha

Geodesic Congruences in a Finsler Space

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Geometria differenziale. — *Geodesic Congruences in a Finsler Space.* Nota di S. C. SRIVASTAVA e R. S. SINHA, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — In questa Nota vengono considerate le congruenze geodetiche di uno spazio di Finsler.

1. INTRODUCTION

In Riemannian space congruences and orthogonal ennuples were investigated and developed in detail by Levy, Ricci, Weatherburn, Eishenhart and others [4] (1). In the present paper we wish to study some properties of geodesic congruences.

In order to explain the notations and to clarify the concepts used below some basic formulae of the theory of Finsler spaces are briefly presented here.

Let us consider a Finsler space F_n of n–dimensions referred to a coordinate system x^i (henceforth all Latin indices run from 1 to n), whose metric function $F(x^i, \dot{x}^i)$ satisfies the conditions usually imposed upon such functions ([1], ch. I). The metric tensor of F_n is defined by

$$g_{ij}(x, \dot{x}) = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x}) \ ^{(2)},$$

and since $F(x^i, \dot{x}^i)$ is positively homogeneous of degree one in $\dot{x}^i$, the tensor $C_{ijk}(x, \dot{x}) = \frac{1}{2} \dot{\partial}_k g_{ij}(x, \dot{x})$ satisfies the identities

$$C_{ijk}(x, \dot{x}) \dot{x}^i = C_{ijk}(x, \dot{x}) \dot{x}^j = C_{ijk}(x, \dot{x}) \dot{x}^k = 0 \tag{1.1}$$

$$C^h_{ij|r} \dot{x}^i = 0 \tag{1.2}$$

where vertical bar denotes differentiation in the Cartan's sense. On multiplying (1.1) and (1.2) by F, and noting that $A_{ijk} = FC_{ijk}$ we have

$$A_{ijk} \dot{x}^i = A_{ijk} \dot{x}^j = A_{ijk} \dot{x}^k = 0 \tag{1.3}$$

$$A^h_{ij|r} \dot{x}^i = 0 . \tag{1.4}$$

(*) Nella seduta del 13 novembre 1971.

(1) The numbers in the square brackets refer to the References given at the end.

(2) $\partial_i = \partial/\partial x^i$, $\dot{\partial}_i = \partial/\partial \dot{x}^i$.

The connection coefficients of L. Berwald ([1], ch. III) are denoted by G^i_{jk} and are used to define a covariant derivative of a covariant tensor $T_{ij}(x, \dot{x})$ of degree 2 is given by

$$T_{ij(k)} = \partial_k T_{ij} - \dot{\partial}_r T_{ij} \dot{\partial}_k G^r - T_{rj} G^r_{ik} - T_{ir} G^r_{jk}.$$

The connection coefficent G^i_{hk} is homogeneous of degree zero in $\dot{x}^h$. As we know that the covariant derivative of g_{ij} in Berwald's sense is not zero, it is given by

$$g_{ij(k)} = -2 A_{ijk|h} l^h. \tag{1.5}$$

The geodesics in Finsler spaces

$$\frac{d^2 x^i}{ds^2} + 2 G^i \left(x^j, \frac{dx^j}{ds}\right) = 0 \tag{1.6}$$

are autoparallel curves; that is,

$$\frac{dx^j}{ds} \left(\frac{dx^i}{ds}\right)_{(j)} = 0. \tag{1.7}$$

2. Geodesic Curvature

DEFINITION 2.1. *Let* $C : x^i = x^i(s)$ *be a curve and* $\frac{dx^i}{ds}$ *be the components of the unit tangent vector to* C, *then the vector whose components* p^i *are given by*

$$p^i \overset{\text{def}}{=} \frac{dx^k}{ds} \left(\frac{dx^i}{ds}\right)_{(k)} \tag{2.1}$$

is called the geodesic curvature vector of C.

If we suppose

$$p^i = k_g n^i_g \tag{2.2}$$

where n^i_g is a unit vector, then k_g is called the geodesic curvature of C.

THEOREM 2.1. *The unit vector* n^i_g *is normal to the curve* C.

Proof. Since $\frac{dx^i}{ds}$ is a unit tangent vector we have

$$g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = 1. \tag{2.3}$$

Taking covariant derivative we get

$$g_{ij(k)} \frac{dx^i}{ds} \frac{dx^j}{ds} + 2 g_{ij} \left(\frac{dx^i}{ds}\right)_{(k)} \frac{dx^j}{ds} = 0. \tag{2.4}$$

Multiplying (2.4) by $\frac{dx^k}{ds}$ and by equations (2.1), (2.2) and (1.5) we have

$$k_g\, g_{ij}\, n_g^j \frac{dx^i}{ds} = -\, 2\, A_{ijk|h}\, l^h \frac{dx^i}{ds} \frac{dx^j}{ds} \frac{dx^k}{ds}\,.$$

Equation (1.4) reduces it to the form

$$k_g\, g_{ij}\, n_g^j \frac{dx^i}{ds} = 0$$

which proves the Theorem. The unit vector n_g^i is called the geodesic normal to C.

THEOREM 2.2. *A geodesic in a F_n is a curve whose geodesic curvature relative to F_n vanishes at every point.*

Proof. The proof at once follows from (1.7), (2.1) and (2.2) i.e. $k_g = 0$.

THEOREM 2.3. *Any vector which undergoes a parallel displacement along a geodesic is inclined at a constant angle to the curve.*

Proof. Let λ be a vector field and $C : x^i = x^i(s)$ be the geodesic whose differential equation is given by (1.6). If the vector λ undergoes a parallel displacement along the curve C, then we have

$$\frac{dx^j}{ds} \lambda^i_{(j)} = 0\,. \tag{2.5}$$

If θ is the angle between λ and the tangent vector to C then

$$\operatorname{Cos} \theta = g_{ij}\, \lambda^i \frac{dx^j}{ds}\,.$$

Differentiating covariantly we have

$$(\operatorname{Cos} \theta)_{(k)} = g_{ij(k)}\, \lambda^i \frac{dx^j}{ds} + g_{ij}\, \lambda^i_{(k)} \frac{dx^j}{ds} + g_{ij}\, \lambda^i \left(\frac{dx^j}{ds}\right)_{(k)}\,.$$

Multiplying by $\frac{dx^k}{ds}$ and by equations (1.5), (1.7) and (2.5) we can get

$$\frac{d}{ds} \operatorname{Cos} \theta = 0\,.$$

Or

$$\operatorname{Cos} \theta = \text{constant}.$$

Hence the Theorem.

3. Ricci's Coefficient of Rotation

Let $P(x^i)$ be an arbitrary point of F_n. Relative to a direction through P we construct orthonormal n-ennuple e^i_A $(A, B, C = 1 \cdots n)$, where A, B, C will not stand for contravariant or covariant index;

$$g_{ij}(x, \dot{x})\, e^i_A\, e^j_B = \delta_{AB}\,. \tag{3.1}$$

These orthonormal unit vectors satisfy the identity

$$g^{jk}(x\,,\dot{x}) = e^j_{\mathrm{A}}\, e^k_{\mathrm{A}} \tag{3.2}$$

where the repeated index stands for summation. From (3.2) we can deduce

$$g_{jk} = e_{\mathrm{A}j}\, e_{\mathrm{A}k}\,. \tag{3.3}$$

If e^{A}_i is the inverse of e^i_{A} then we also have

$$e^i_{\mathrm{A}}\, e^{\mathrm{A}}_j = \delta^i_j \tag{3.4 a}$$

$$e^i_{\mathrm{A}}\, e^{\mathrm{B}}_i = \delta^{\mathrm{B}}_{\mathrm{A}}\,. \tag{3.4 b}$$

Now we define the quantities γ_{ABC} by the relation [3]

$$\gamma_{\mathrm{ABC}} = g_{ij}\, e^i_{\mathrm{A}(k)}\, e^j_{\mathrm{B}}\, e^k_{\mathrm{C}} \tag{3.5}$$

which can be considered as the generalisation of Ricci's coefficient of rotation in the Finsler space with respect to Berwald's connection.

Differentiating (3.1) and multiplying with e^k_{C} we get

$$g_{ij(k)}\, e^i_{\mathrm{A}}\, e^j_{\mathrm{B}}\, e^k_{\mathrm{C}} + g_{ij}\, e^i_{\mathrm{A}(k)}\, e^j_{\mathrm{B}}\, e^k_{\mathrm{C}} + g_{ij}\, e^i_{\mathrm{A}}\, e^j_{\mathrm{B}(k)}\, e^k_{\mathrm{C}} = 0\,.$$

From (1.5) and (3.5) it becomes

$$\gamma_{\mathrm{ABC}} + \gamma_{\mathrm{BAC}} = 2\,\mathrm{A}_{ijk|h}\, l^h\, e^i_{\mathrm{A}}\, e^j_{\mathrm{B}}\, e^k_{\mathrm{C}}\,. \tag{3.6}$$

In particular, if e^i_{B} coincides with the direction of $\dot{x}^i$ then (3.6) reduces to

$$\gamma_{(\mathrm{AB})\mathrm{C}} = 0\,. \tag{3.7}$$

Multiplying (3.5) by e^{C}_l we have from (3.4 *a*)

$$\gamma_{\mathrm{ABC}}\, e^{\mathrm{C}}_l = e_{\mathrm{B}i}\, e^i_{\mathrm{A}(l)}\,. \tag{3.8}$$

Further we can also deduce with the help of (3.4 *a*) and (1.5) by multiplying (3.5) by $g^{hl}\, e^{\mathrm{B}}_l$ and e^{B}_j respectively that

$$\gamma_{\mathrm{ABC}}\, e^{\mathrm{B}h} = e^k_{\mathrm{C}}\, e^h_{\mathrm{A}(k)} \tag{3.9}$$

and

$$\gamma_{\mathrm{ABC}}\, e^{\mathrm{B}}_j = e^k_{\mathrm{C}}\, e_{\mathrm{A}j(k)} + 2\,\mathrm{A}_{ijk|h}\, l^h\, e^i_{\mathrm{A}}\, e^k_{\mathrm{C}}\,.$$

4. GEODESIC CONGRUENCE

A congruence is said to be geodesic if all its curves are geodesics.

We now have the following Theorem.

THEOREM 4.1. *The necessary and sufficient condition that the curves of a congruence, whose unit tangents e^i_{A} are geodesics, are given by the equations*

$$\gamma_{\mathrm{ABC}} = 0 \quad , \quad \mathrm{A} = \mathrm{C}\,. \tag{4.1}$$

Proof. If k_g is the geodesic curvature and n_g^i is the geodesic normal to the curve of the congruence whose unit tangent is e_A^i then

$$k_g\, n_g^i = e^i_{A(j)}\, e^j_A \qquad \text{(A not summed).}$$

From (3.9) we have

$$k_g\, n_g^i = \gamma_{ABA}\, e^{Bi} \qquad \text{(A not summed).} \tag{4.2}$$

From Theorem (2.2) $k_g = 0$ if the curve of the congruence is geodesic.

Now we have

$$\gamma_{ABA}\, e^{Bi} = 0 \qquad \text{(A not summed)}$$

for arbitrary e^{Bi}. Hence

$$\gamma_{ABA} = 0 \qquad \text{(A not summed)}$$

is the necessary condition.

Conversely, if $\gamma_{ABA} = 0$ (A not summed), then from (4.2) $k_g = 0$ i.e. the condition is sufficient also.

Hence the Theorem.

If we define the tendency of the vector λ at a point in the direction of the unit vector μ at the same point by the expression

$$g_{ij}\, \mu^j\, \mu^k\, \lambda^i_{(k)}$$

then we have the following Theorems:

THEOREM 4.2. *The divergence of a vector λ in a F_n is the sum of the tendencies of λ for n–mutually orthogonal directions in F_n.*

Proof. If e_A^i are the unit tangents to the curves of an orthogonal ennuple, then the tendency of λ in the direction of e_A^i is

$$g_{ij}\, e^i_A\, e^k_A\, \lambda^j_{(k)} \qquad \text{(A not summed).}$$

Therefore the sum of the tendencies in n–mutually orthogonal direction e_A^i is given by

$$g_{ij}\, e^i_A\, e^k_A\, \lambda^j_{(k)}\,.$$

From (3.2) we have

$$g_{ij}\, e^i_A\, e^k_A\, \lambda^j_{(k)} = g_{ij}\, g^{ik}\, \lambda^j_{(k)} = \lambda^j_{(j)}$$

which is the divergence of λ.

THEOREM 4.3. A *congruence e_A^i of an orthogonal ennuple be a geodesic congruence if the tendencies of all the other congruences of the ennuple in the direction of e_A^i is given by*

$$\gamma_{BAA} = 2\, A_{ijk|h}\, l^h\, e^i_A\, e^j_B\, e^k_A\,. \tag{4.3}$$

Proof. From (3.5)

$$\gamma_{BAA} = g_{ij}\, e^i_{B(k)}\, e^j_A\, e^k_A \qquad \text{(A not summed)}$$

which is the tendency of e^i_B in the direction of e^i_A.

From (3.6) we have

$$\gamma_{ABA} + \gamma_{BAA} = 2\, A_{ijk|h}\, l^h\, e^i_A\, e^j_B\, e^k_A . \tag{4.4}$$

From Theorem (4.1) the condition for C to be a geodesic congruence is $\gamma_{ABA} = 0$, hence (4.4) becomes

$$\gamma_{BAA} = 2\, A_{ijk|h}\, l^h\, e^i_A\, e^j_B\, e^k_A .$$

Hence the Theorem.

When e^i_B coincides with the direction of $\dot{x}^i$ then the right hand side of (4.3) vanishes so we have the following corollary.

COROLLARY 4.1. *A congruence e^i_A of an orthogonal ennuple is a geodesic congruence if the tendencies of $\dot{x}^i$ in the direction of e^i_A vanish identically.*

In case we define Ricci's coefficient of rotation by the help of Cartan's covariant derivative then the results obtained are the same as in the Riemannian space.

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