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A theorem on fixed points

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Matematica. — *A theorem on fixed points.* Nota di TUDOR ZAMFIRESCU, presentata (*) dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — In questa Nota vengono ampliati alcuni teoremi sulla presenza di un unico elemento unito in trasformazioni di spazi metrici generalizzati in se stessi.

Let (X, d) be a complete metric space or a generalized complete metric space (i.e. a complete metric space in which the distance-function d may attain ∞) and consider the function $f: X \rightarrow X$ and the number $\alpha \in (0, 1)$. We list the following properties f may possess at some couple of distinct points x, y :

- 1) $d(f(x), f(y)) < \alpha d(x, y)$,
- 2) $d(f(x), f(y)) < \alpha d(x, f(x))$,
- 3) $d(f(x), f(y)) < \alpha d(y, f(y))$,
- 4) $d(f(x), f(y)) < \frac{\alpha}{2}(d(x, f(y)) + d(y, f(x)))$.

The function f is called an α -pseudocontraction if at each couple $x, y \in X$ with $0 < d(x, y) < \infty$, at least one of the conditions 1)–4) is satisfied; f is called an α -contraction if at each couple $x, y \in X$ with $0 < d(x, y) < \infty$, condition 1) is satisfied.

The following proposition is a generalization of a well-known result of Banach and also of theorems of R. Kannan [1], S. Reich [3], I. A. Rus [4], R. M. Tiberio Bianchini [6].

PROPOSITION 1 [7]. *In a complete metric space, each α -pseudocontraction ($\alpha \in (0, 1)$) has a unique fixed point.*

The next result is due to S. P. Singh and C. W. Norris, and constitutes an improvement of a theorem of W. A. J. Luxemburg [2].

PROPOSITION 2 [5]. *Suppose X is a generalized complete metric space, $f: X \rightarrow X$, $\alpha \in (0, 1)$ and $p \in \mathbb{N}$ (1). If*

- a) *for each point $x_0 \in X$, there exists $m \in \mathbb{N}$ such that $d(f^{pm}(x_0), f^{pn}(x_0)) < \infty$ for every $n \in \mathbb{N}$ with $n > m$,*
- b) *f^p is an α -contraction,*
- c) *the distance between every two fixed points of f^p is finite,*

then f has a unique fixed point in X .

(*) Nella seduta del 16 giugno 1972.

(1) \mathbb{N} denotes the set of natural numbers: $\{1, 2, 3, \dots\}$.

Both Propositions 1 and 2 will be generalized here. Before presenting the main result, we record the following

PROPOSITION 3 [7]: *Suppose M is a set in a complete metric space X, $f: M \rightarrow M$, $\alpha \in (0, 1)$, and for each couple of different points $x, y \in M$, at least one of the conditions 1)–4) is satisfied. Then, for $x_0 \in M$, the sequence $\{f^n(x_0)\}_{n=0}^\infty$ converges to a point in X independent on the choice of x_0 .*

THEOREM. *Suppose X is a generalized complete metric space, $f: X \rightarrow X$, $\alpha \in (0, 1)$ and $p \in \mathbb{N}$. If*

- a) *there exists a point $x_0 \in X$ and two distinct numbers $m, n \in \mathbb{N}$ such that $d(f^{pm}(x_0), f^{pn}(x_0)) < \infty$,*
- b) *$f^{p|n-m|}$ is an α -pseudocontraction,*
- c) *the distance between every two fixed points of $f^{p|n-m|}$ is finite,*

then f has a unique fixed point in X.

Proof. Suppose $m < n$. Consider the set $S \subset X$ consisting of the elements of the sequence $\sigma = \{f^{pm+sp(n-m)}(x_0)\}_{s=0}^\infty$, and the restriction $f^{p(n-m)}|S$. If $f^{pm}(x_0) = f^{pn}(x_0)$, then $f^{pm}(x_0)$ is a fixed point of $f^{p(n-m)}$, and σ converges to $f^{pm}(x_0)$. If $f^{pm}(x_0) \neq f^{pn}(x_0)$, then

$$0 < d(f^{pm}(x_0), f^{pm+p(n-m)}(x_0)) < \infty.$$

and since $f^{p(n-m)}$ is an α -pseudocontraction, it follows that

$$d(f^{pm+p(n-m)}(x_0), f^{m+2p(n-m)}(x_0)) < \infty.$$

Analogously, for each s ,

$$d(f^{pm+sp(n-m)}(x_0), f^{pm+(s+1)p(n-m)}(x_0)) < \infty.$$

Then, it easily follows that $d|S \times S$ does not attain ∞ ; moreover, $\{x \in X : d(x, f^{pm}(x_0)) < \infty\}$ is obviously a complete metric space (including the set S). Thus, we may use Proposition 3. It results that σ converges to some point $z \in X$. Because

$$\begin{aligned} d(f^{p(n-m)}(z), z) &\leq d(f^{p(n-m)}(z), f^{pm+(s+1)p(n-m)}(x_0)) \\ &\quad + d(f^{pm+(s+1)p(n-m)}(x_0), z) \\ &\leq \alpha d(z, f^{pm+sp(n-m)}(x_0)) + d(f^{pm+(s+1)p(n-m)}(x_0), z) \end{aligned}$$

for each $s \in \mathbb{N}$, we have $d(f^{p(n-m)}(z), z) = 0$, and therefore z is a fixed point of $f^{p(n-m)}$.

Suppose now z' is a fixed point of $f^{p(n-m)}$. Following condition (c), $d(z, z') < \infty$; then $d|S^* \times S^*$, where $S^* = S \cup \{z, z'\}$, takes only finite values. Also, one easily verifies that S^* is a complete subspace of X . Consider $f^{p(n-m)}|S^*$ and apply Proposition 1: it follows $z = z'$.

Thus, $f^{p(n-m)}$ has a unique fixed point. Consequently, f has a unique fixed point too. The proof is achieved.

We finally observe that: if $d(x, y) < \infty$ for every couple $x, y \in X$ and if $p = |n - m| = 1$, one obtains Proposition 1; if f^p is an α -contraction and if $|n - m| = 1$, one obtains an improvement of Proposition 2.

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