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Convergence of solutions of perturbed non-linear differential equations

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Equazioni differenziali ordinarie. — *Convergence of solutions of perturbed non-linear differential equations.* Nota di B. S. LALLI e R. S. RAMBALLY, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si studiano i sistemi $y' = f(t, y)$, $y' = f(t, y) + g(t, y)$ e si trovano condizioni sufficienti per la convergenza delle soluzioni di tali sistemi. I risultati ottenuti generalizzano quelli di un altro Autore.

1. We shall consider the following systems of differential equations:

$$x' = f(t, x) \tag{1}$$

$$y' = f(t, y) + g(t, y), \tag{2}$$

where x, y, f and g are n–vectors. We assume that f and g are continuous from $R_+ \times R^n$ to R^n and that the Jacobian matrix f_x is continuous on $R_+ \times R^n$. Our main purpose is to generalize some of Hallam's results [1] through the use of his own techniques.

We denote by $x(t, t_0, x_0)$ the solution of (1) passing through $(t_0, x_0) \in R_+ \times R^n$. Recall that the variational system of (1) associated with the solution $x(t, t_0, x_0)$ is the system

$$z' = f_x(t, x(t, t_0, x_0))\, z. \tag{3}$$

We usually denote by $\Phi(t, t_0, x_0)$ the fundamental matrix of (3) such that $\Phi(t_0, t_0, x_0) = I$, the identity matrix. The non-linear variation of parameters formula, of which much use is made, can be written as

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s, t_0, y_0))\, g(s, y(s, t_0, y_0))\, ds. \tag{4}$$

2. Avramescu [2] introduced the following definitions concerning the convergent behavior of systems of differential equations:

(i) Equation (1) is *convergent* if $\lim_{t\to\infty} x(t, t_0, x_0) = \lambda(t_0, x_0)$ is defined for each $(t_0, x_0) \in R_+ \times R^n$.

(ii) Equation (1) is *equi-convergent* if it is convergent and for each $\varepsilon > 0$, $\alpha > 0$, $t_0 \geq 0$, there exists a function $T = T(t_0, \alpha, \varepsilon)$ such that $\|x(t, t_0, x_0) - \lambda(t_0, x_0)\| < \varepsilon$ whenever $t > t_0 + T$ and $\|x_0\| \leq \alpha$. Note that

(*) Nella seduta del 16 giugno 1972.

the symbol $\|\cdot\|$ denotes some convenient norm on R^n and a corresponding matrix norm.

(iii) Equation (1) is *equi-uniformly convergent* if it is equi-convergent and the T in the above definition is independent of t_0.

(iv) Equation (1) is called *coalescent* if it is convergent and if $\lambda(0, x_0)$ is a constant.

The above definitions are also applicable on a subset D of $R_+ \times R^n$. The following terminology will also be used in the sequel.

(v) Equation (1) is *equi-uniformly convergent in variation* if for each $\varepsilon > 0$ and each $\alpha > 0$ there exist a scalar function $T = T(\alpha, \varepsilon)$ and a matrix function $L = L(t_0, x_0)$, which is continuous on $R_+ \times R^n$ and bounded on R_+, such that

$$\|\Phi(t, t_0, x_0) - L(t_0, x_0)\| < \varepsilon$$

whenever $t > t_0 + T$, $\|x_0\| \leq \alpha$, $t_0 \in R_+$.

THEOREM 1. *Suppose that equation* (1) *is convergent and equi-uniformly convergent in variation. For each* $\alpha > 0$, *let* $\omega_\alpha(t, \|y\|)$ *be such that*

$$\|g(t, y)\| \leq \omega_\alpha(t, \|y\|) \quad \textit{whenever} \quad \|y\| \leq \alpha, \tag{5}$$

where $\omega_\alpha(t, r)$ *is a continuous, monotone, non-decreasing function in* r *for each fixed* t *such that*

$$\int^{\infty} \omega_\alpha(t, c)\, dt < \infty, \qquad 0 \leq c \leq \alpha. \tag{6}$$

Then for any initial position y_0, *there exists a* $J = J(y_0)$ *such that if* $t_0 \geq J$ *the solution* $y(t, t_0, y_0)$ *of* (2) *is convergent.*

Proof. Brauer [3] has shown that if (1) is equi-uniformly convergent in variation, then it is uniformly stable in variation. Thus, for each $\alpha > 0$ there exists $M = M(\alpha)$ such that $\|x_0\| \leq \alpha$ implies that

$$\|\Phi(t, t_0, x_0)\| \leq M(\alpha), \qquad t \geq t_0 \geq 0.$$

Let y_0 be given and consider the solution $x(t, t_0, y_0)$. Since (1) is convergent, there exists a constant B such that

$$\|x(t, t_0, y_0)\| \leq B \quad \text{for} \quad t \geq t_0.$$

Let J be large enough so that

$$\int_J^{\infty} \omega_{2B}(t, 2B) < \frac{B}{M(2B)} \qquad \text{(i.e. let } \alpha \text{ in hypothesis be } 2B). \tag{7}$$

We claim that for $t \geq t_0 \geq J$, $\|y(t, t_0, y_0)\| < 2B$.

If this is not so, then consider the first t-value, call it t_1, such that $\|y(t_1, t_0, y_0)\| = 2B$. Then using (4), (5) and (7), we obtain the following inequalities.

$$2B = \|y(t_1, t_0, y_0)\| \leq \|x(t_1, t_0, y_0)\| + \int_{t_0}^{t_1} \|\Phi(t_1, s, y(s, t_0, y_0))\| \cdot$$

$$\cdot \|g(s, y(s, t_0, y_0))\| \, ds \leq B + M(2B) \int_{t_0}^{t_1} \omega_{2B}(s, 2B) \, ds < 2B.$$

Hence our claim is justified. [Note that on $[t_0, t_1]$, $\|y(t, t_0, y_0)\| \leq 2B$ so that $\|\Phi(t, s, y(s, t_0, y_0))\| \leq M(2B)$ on this interval].

In order to show the convergence of $y(t, t_0, y_0)$ we show

$$\lim_{t\to\infty} \int_{t_0}^{t} \Phi(t, s, y(s, t_0, y_0)) \, g(s, y(s, t_0, y_0)) \, ds \tag{8}$$

$$= \int_{t_0}^{\infty} L(s, y(s, t_0, y_0)) \, g(s, y(s, t_0, y_0)) \, ds.$$

[Note that the last integral is well defined since $L(s, y(s)) \leq \Phi(t, s, y(s, t_0, y_0))$ so that

$$\int_{t_0}^{\infty} L(s, y(s, t_0, y_0)) \, g(s, y(s, t_0, y_0)) \, ds$$

$$\leq \int_{t_0}^{\infty} \Phi(t, s, y(s, t_0, y_0)) \, g(s, y(s, t_0, y_0)) \, ds.$$

Thus this last integral is majorized by the integral

$$\int_{t_0}^{\infty} M(2B) \, \omega_{2B}(s, 2B) \, ds].$$

Now let $\varepsilon > 0$ be given. Let T_0 be large enough so that

$$\int_{T_0}^{\infty} \omega_{2B}(s, 2B) \, ds < \frac{\varepsilon}{3M(2B)}. \tag{9}$$

Since (1) is equi-uniformly convergent in variation, there exists T_1, $T_1 \geq T_0$ such that if $t \geq T_1$, then

$$\|\Phi(t, s, y(s, t_0, y_0)) - L(s, y(s, t_0, y_0))\| < \frac{\varepsilon}{3T_0 K} \tag{10}$$

where $K = \max \|g(t, y)\|$ over

$$\|y\| \leq 2B \quad , \quad t_0 \leq t \leq T_0 .$$

Then for $t \geq T_1$ we have, using (9) and (10)

$$\begin{aligned}
(11) \quad & \left\| \int_{t_0}^{t} \Phi(t, s, y(s, t_0, y_0))\, g(s, y(s, t_0, y_0))\, ds \right. \\
& \left. - \int_{t_0}^{\infty} L(s, y(s, t_0, y_0))\, g(s, y(s, t_0, y_0))\, ds \right\| \\
& \leq \int_{t_0}^{T_0} \|\Phi(t, s, y(s, t_0, y_0)) - L(s, y(s, t_0, y_0))\| \, \|g(s, y(s, t_0, y_0))\|\, ds \\
& + \int_{T_0}^{\infty} \|\Phi(t, s, y(s, t_0, y_0))\, g(s, y(s, t_0, y_0))\|\, ds \\
& + \int_{T_0}^{\infty} \|L(s, y(s, t_0, y_0))\, g(s, y(s, t_0, y_0))\|\, ds \\
& \leq \frac{\varepsilon}{3 T_0 K} K (T_0 - t_0) + 2 \int_{T_0}^{\infty} M(2B)\, \omega_{2B}(s, 2B)\, ds \\
& < \frac{\varepsilon}{3} + 2\left(\frac{\varepsilon}{3}\right) = \varepsilon .
\end{aligned}$$

Thus

$$\lim_{t \to \infty} \int_{t_0}^{t} \Phi(t, s, y(s, t_0, y_0))\, g(s, y(s, t_0, y_0))\, ds$$

$$= \int_{t_0}^{\infty} L(s, y(s, t_0, y_0))\, g(s, y(s, t_0, y_0))\, ds .$$

Let $\lambda_y(t_0, y_0)$ and $\lambda_x(t_0, y_0)$ be the limits $\lim_{t \to \infty} y(t, t_0, y_0)$ and $\lim_{t \to \infty} x(t, t_0, y_0)$ respectively. Now

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s, t_0, y_0))\, g(s, y(s, t_0, y_0))\, ds .$$

Thus

$$\lambda_y(t_0, y_0) = \lambda_x(t_0, y_0) + \lim_{t \to \infty} \int_{t_0}^{t} \Phi(t, s, y(s, t_0, y_0))\, g(s, y(s, t_0, y_0))\, ds .$$

Using (8) we get

$$\lambda_y(t_0, y_0) = \lambda_x(t_0, y_0) + \int_{t_0}^{\infty} L(s, y(s, t_0, y_0))\, g(s, y(s, t_0, y_0))\, ds. \tag{12}$$

Hence $y(t, t_0, y_0)$ is convergent and the theorem is proved.

We now consider a fixed $y_0 \in R^n$. Let $J_0(y_0)$ be the infimum of all those $J(y_0)$ for which Theorem 1 is satisfied. Let $D = \{(t, y) \mid y \in R^n, t > J_0(y_0)\}$. In the next two theorems, we consider convergence on this set D.

THEOREM 2. *Assume that equation* (1) *is equi-convergent and equi-uniformly convergent in variation. Assume that* (5) *and* (6) *hold. Then equation* (2) *is equi-convergent on* D.

Proof. We first note that all hypotheses of Theorem 1 are satisfied and so all equations in its proof are applicable here. In particular, we shall make use of equation (12).

Since equation (1) is equi-convergent, we have, for $(t_0, y_0) \in D$ with $\|y_0\| \leq \alpha$, $B = B(\alpha)$ such that $\|y(t, t_0, y_0)\| \leq B$. Using equations (4) and (12), we obtain

$$\begin{aligned} y(t, t_0, y_0) - \lambda_y(t_0, y_0) &= x(t, t_0, y_0) - \lambda_x(t_0, y_0) \\ &+ \int_{t_0}^{t} \Phi(t, s, y(s, t_0, y_0))\, g(s, y(s, t_0, y_0))\, ds \\ &- \int_{t_0}^{\infty} L(s, y(s, t_0, y_0))\, g(s, y(s, t_0, y_0))\, ds. \end{aligned} \tag{13}$$

Exactly as was done in Theorem 1, we can find T_1 such that for $t \geq T_1$ we have

$$\left\| \int_{t_0}^{t} \Phi(t, s, y(s, t_0, y_0))\, g(s, y(s, t_0, y_0))\, ds - \int_{t_0}^{\infty} L(s, y(s, t_0, y_0))\, g(s, y(s, t_0, y_0))\, ds \right\| < \frac{\varepsilon}{2}$$

for any $\varepsilon > 0$. By the equi-convergence of equation (1) we can also find T_2 such that

$$\|x(t, t_0, y_0) - \lambda_x(t_0, y_0)\| < \frac{\varepsilon}{2} \quad , \quad t \geq T_2.$$

Let $t > \max\{T_1, T_2\}$. For any such t, (13) gives

$$\|y(t, t_0, y_0) - \lambda_y(t_0, y_0)\| < \varepsilon.$$

This gives us the equi-convergence of equation (2) on D.

By a very similar argument we can obtain:

THEOREM 3. *Assume that equation* (1) *is equi-uniformly convergent and equi-uniformly convergent in variation. Assume that equations* (5) *and* (6) *are satisfied. Then equation* (2) *is equi-uniformly convergent on* D.

Finally we give a condition for convergent solutions of (2) to be coalescent.

THEOREM 4. *Assume that equation* (1) *is coalescent to* x_∞ *and that*

$$\|g(t, y)\| \leq \omega_\alpha(t, \|y\|), \qquad \text{any } \alpha > 0, \quad \|y\| \leq \alpha$$

where $\omega_\alpha(t, r)$ *is a continuous, non-decreasing function in* r *for each fixed* t. *Also suppose that for each* $\alpha > 0$ *there exists a continuous function* $\eta_\alpha : R_+ \times R_+ \to R_+$ *such that*

$$\|\Phi(t, t_0, x_0)\| \leq \eta_\alpha(t, t_0) \qquad \text{for} \quad t \geq t_0 \geq 0, \quad \|x_0\| \leq \alpha.$$

Suppose

$$\lim_t \int_0^t \eta_\alpha(t, s)\, \omega_\alpha(s, c)\, ds = 0, \qquad 0 \leq c \leq \alpha.$$

Then all solutions of (2) *coalesce to* x_∞.

Proof. Since equation (1) is coalescent, it is convergent. Let $y(t, t_0, y_0)$ be a convergent solution of (2). Then

$$\lambda_y(t_0, y_0) = \lambda_x(t_0, y_0) + \lim_t \int_{t_0}^t \Phi(t, s, y(s, t_0, y_0))\, g(s, y(s, t_0, y_0))\, ds.$$

By convergence of $y(t, t_0, y_0)$, there exists a constant $B > 0$ such that $\|y(t, t_0, y_0)\| \leq B$, $t \geq t_0$. From the hypotheses on Φ and g, we obtain

$$\left\| \int_{t_0}^t \Phi(t, s, y(s, t_0, y_0))\, g(s, y(s, t_0, y_0))\, ds \right\| \leq \int_0^t \eta_B(t, s)\, \omega_B(s, B)\, ds.$$

Thus

$$\lambda_y(t_0, y_0) = \lambda_x(t_0, y_0).$$

Hence equation (2) is coalescent to x_∞.

REFERENCES

[1] T. G. HALLAM, *Convergence of solutions of perturbed non-linear differential equations*, « Ann. Mat. Pura Appl. » (to appear).

[2] C. AVRAMESCU, *Sur l'existence des solutions convergentes des systèmes d'equations differentielles non linéaires*, « Ann. Mat. Pura Appl. », *4* (81), 147–168 (1969).

[3] F. BRAUER, *Perturbations of non-linear systems (III)*, « J. Math. Anal. Appl. », *31*, 37–48 (1970).