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Vasile I. Istrăţescu

On some classes of operators. I

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Analisi funzionale. — *On some classes of operators. I.* Nota di VASILE I. ISTRĂŢESCU, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si estende un teorema di J. Wermer sugli operatori normali di uno spazio di Hilbert agli operatori di uno spazio di Banach.

0. Our purpose in this Note is to prove that an operator which is quasi-similar to a unilateral weighted shift is irreducible and to prove Wermer's theorem for normal operators on Banach spaces.

1. Let H be a Hilbert space and $\{\varphi_n\}$ be an orthogonal basis of H and let $\{\alpha_n\}$ be a bounded sequence of complex scalars.

Then a unilateral weighted shift A with weights is the operator on H defined by

$$A\varphi_n = \alpha_n \varphi_{n+1} \qquad (n = 0, 1, 2, \cdots)$$

and its adjoint is given by

$$A^* \varphi_0 = 0 \quad \text{and} \quad A^* \varphi_n = \overline{\alpha_{n-1}} \varphi_{n-1} \qquad (n = 1, 2, 3, \cdots).$$

It is known that a unilateral shift is irreducible, i.e., it has no nontrivial reducing subspace. In [6] it is proved that every operator on a Hilbert space which is similar to a unilateral weighted shift with nonzero weights is irreducible.

Our aim in this section is to prove that this property is valid under quasi-similarity. We recall the definition [3].

DEFINITION. A_1 *and* A_2 *are said to be quasi-similar if there are bounded linear operators* $R : H_2 \to H_1$ *and* $S : H_1 \to H_2$ *which satisfy the following conditions*:

1) $SA_1 = A_2 R$ *and* $A_1 R = RA_2$

2) R *and* S *have zero kernels and dense ranges.*

THEOREM 1. *Every operator on a Hilbert space* H *which is quasi similar to a unilateral weighted shift with non zero weights is irreducible.*

Proof. Let R, S be the operators which invoke the quasi-similarity and A_0 an operator on H which is quasi-similar with the weighted shift A defined above.

Let m be the subspace which is reducing for B. Then

$$B^* m \subset m \quad , \quad B^* m^{\perp} \subset m^{\perp}$$

(*) Nella seduta del 16 giugno 1962.

and since $A^* B^* = S^* B^*$, $R^* A^* = B^* R^*$ we obtain that

$$A^* S^* m = S^* B^* m \subset S^* m$$
$$A^* S^* m^{\perp} = S^* B^* m^{\perp} \subset S^* m^{\perp}.$$

Thus $S^* m$ and $S^* m^{\perp}$ are invariant subspaces for A^*. It is known and follows from Theorem 10 [8, Ch. VI], that if A_1 and A_2 are quasi-similar then A_1^* and A_2^* are again quasi-similar. Since S has a null space equal to 0 and the range dense (in fact is equal to the whole space) we obtain that

$$H_1 = S^* m + S^* m^{\perp}$$

which is impossible because of the Lemma in [6]. The theorem is proved.

Remark. The structure theorem for spectral type operators suggests the consideration of the class of operators of the following type: $T = S + N$ where N is quasi-nilpotent and commuting with S and S is quasi-similar to a normal operator.

Some results about this class of operators will be given in [9].

In [7] J. Wermer has proved the following result: if N is a normal operators on a Hilbert space H and $\sigma(N)$ is of area zero then if m is an invariant subspace of dimension ≥ 2 then $\mathfrak{N}/m = T$ has a nontrivial invariant subspace.

For another proof see [2].

Our aim in this section is to prove the same result for operators on Banach space. First, we recall some definitions [4].

DEFINITION 1. *An operator* S *on the Banach space* $\mathfrak{X}$ *is called self conjugate if and only if for all* $t \in R$, $\|e^{its}\| = 1$.

DEFINITION 2. *If every element* T *of a closed commutative algebra* $\mathfrak{A} \subset \mathfrak{L}(\mathfrak{X})$ *can be written in the form* $T = R + iJ$ *with* R *and* J *self conjugate operators in* $\mathfrak{A}$ *then* $\mathfrak{A}$ *with the map* $T = R + iJ \to \bar{T} = R - iJ$ *will be called a commutative* V^**-algebra*.

DEFINITION 3. *An operator* N *is said to be normal if there exist self conjugate operators* R *and* J *such that*

(1) $N = R + iJ$

(2) $\mathfrak{U}_t = e^{itR}$, $V_t = e^{itJ}$ *are contained in a commutative* V^* *algebra for all* $t \in R^1$.

Our result is the following.

THEOREM 2. *If* N *is a normal operator on a Banach space* $\mathfrak{X}$ *and the following property holds: the area of the spectrum* $\sigma(N)$ *is zero and if* m *is an invariant subspace of* N *of dimension* ≥ 2, *then the operator* $T = N/m$ *has a nontrivial invariant subspace.*

Proof. Let K be the smallest subspace containing m and reducing for N (i.e. invariant for N and $\bar{N}$), since N/K is normal with the spectrum contained in $\sigma(N)$.

If $\sigma(T) \subset \sigma(N)$ then T is in fact normal. Indeed, we find a sequence of rational functions converging uniformly to $\bar{z}$ on $\sigma(N)$ and thus $r_n(N)$ converge uniformly to $\bar{N}$. Since m is invariant under N then m is clear invariant under $\bar{N}$ and thus N/m is a normal operator. Thus in this case the theorem is proved. If $\sigma(N) \subset \sigma(T)$ we remark that the point spectrum of N includes the spectrum of T and also the continuous spectrum of N includes that of T. But N has no residual spectrum (1). From this we obtain that $\sigma(T) - \sigma(N)$ is in the residual spectrum of T. If $\sigma(T) - \sigma(N) \neq \varnothing$ then T has a non trivial invariant subspace. If $\sigma(T) = \sigma(N)$ then as above, we can show that T is in fact normal.

Remark 1. In the case of Hilbert spaces we can prove that T is normal without use of Hartogs–Rosenthal theorem. Indeed, on every invariant subspace, $T = N/m$ is a hypernormal operator and as above, it remains to consider only the case when $\sigma(T)$ is of area zero. By a recent result of Putnam [5] T is normal.

Remark 2. In [9] we study a generalization of n–normal operators on Hilbert spaces to n–normal operators on Banach spaces which are defined in a natural way.

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(1) Let S be a normal operator on a Banach space. Thus $S = R + iJ$ where R, J are in a commutative V^*–algebra. But the Banach algebra generated by R, J is isometric and $*$–isomorphic with the $*$–algebra $\mathcal{A}^* = \{\bar{T}, T \in \mathcal{A}\} \subset \mathcal{L}(\mathfrak{X})$. If $\lambda \in \sigma_r(S)$ then there exists $x^* \in \mathfrak{X}^*$ such that $S^* x^* = \bar{\lambda} x^*$ and thus $E(\bar{\lambda}) \neq 0$. From the isometric $*$–isomorphism between $\mathcal{A}$ and $\mathcal{A}^*$ we obtain that $\lambda \in \sigma_p(S)$.